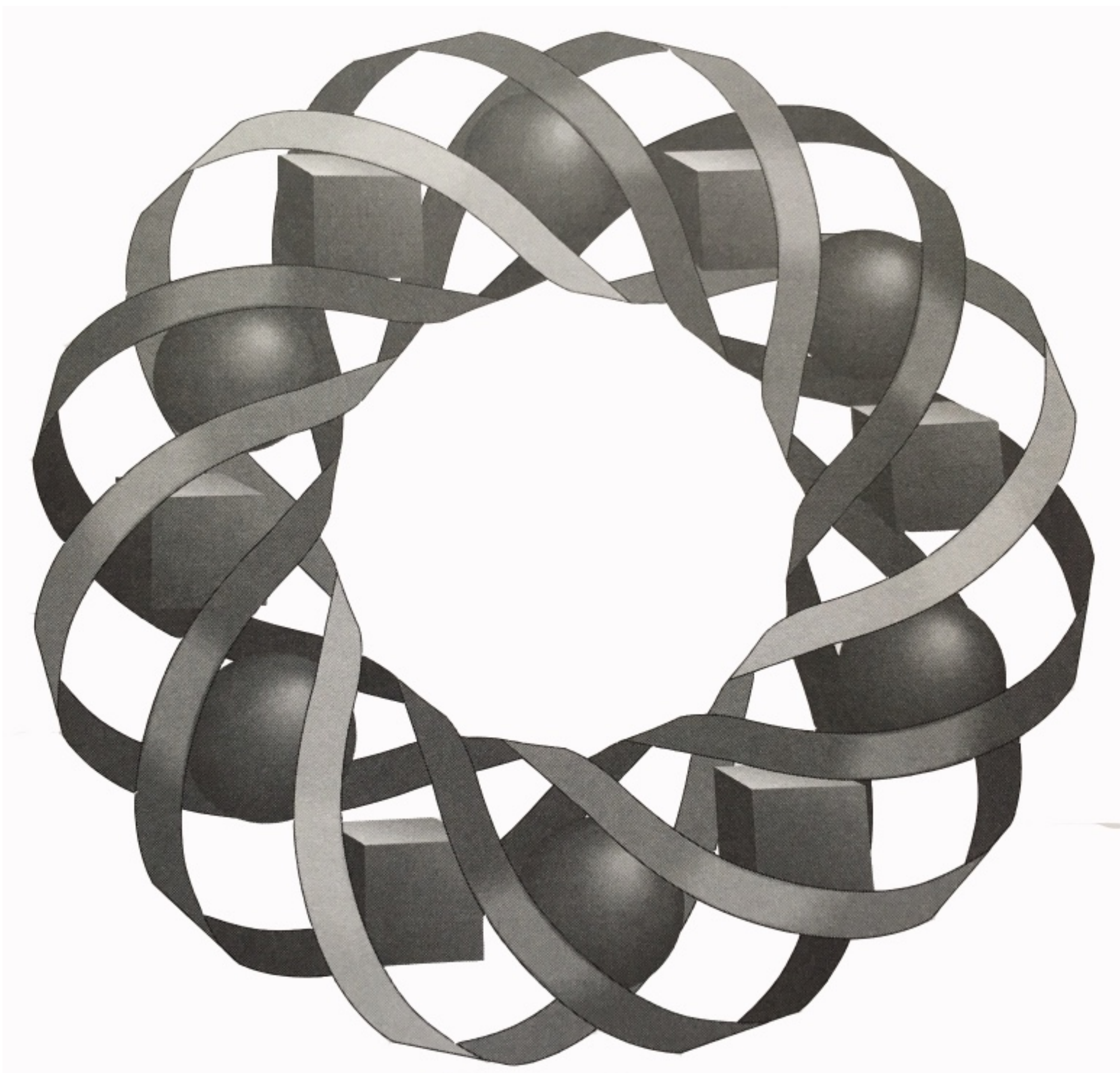
Is It Really True?



In this module, you continue your investigations of logical reasoning.

Terri Dahl • Laurie Paladichuk • Peter Rasmussen

Is It Really True?

Introduction

The August sun beats down on Colter's aching back. He has been working for Mykelti Construction on this job for the past 2 weeks, 7 days a week, 12 hours a day. When Colter asked his supervisor for a day off, she replied, "If you finish the foundation, then you're off for the weekend."

It is now Friday afternoon. Colter is sure that he won't finish the foundation by the end of the day. He considers the responsibilities of his job and of keeping the entire construction project on schedule. A dull pain throbs inside his head. Then he imagines two days of rest, an afternoon by the pond, and dinner with his family.

Discussion

a. How should Colter expect his supervisor to react if he takes the weekend off without finishing the foundation? Justify your response.

b. How should Colter expect his supervisor to react if he takes the weekend off after finishing the foundation? Justify your response.

Activity 1

Colter's company won a bid for a construction project. The project involves remodeling a dentist's office in a large office complex. The following guidelines are included in the bid.

- All materials used by Mykelti Construction will be purchased locally.
- Some of the wood used by Mykelti Construction is oak.
- No employees will smoke on the job site.

Exploration

To determine how closely the company follows these guidelines, Mykelti Construction decides to conduct a study. Model this study by completing the following exploration using two sets of chips, each a different color.

a. Recall that a **statement** is a sentence that is either true or false, but not both. The truth or falseness of a statement is its **truth value**.

Consider the statement "All materials used by Mykelti Construction will be purchased locally." This statement implies that two subsets of materials could exist: one consists of materials that are purchased locally, and the other consists of materials that are not purchased locally.

1. The company plans to examine four recent purchases. Let one set of colored chips represent materials purchased locally, and the other set of chips represent materials not purchased locally. Use the different colored chips to represent all possible combinations of purchases.
2. Identify every combination of four chips that correctly represents the true statement "All materials used by Mykelti Construction will be purchased locally."

b. In previous modules, you have seen that a statement can be shown to be false by finding at least one counterexample. Identify every combination of four chips that represents a counterexample to the statement "All materials used by Mykelti Construction will be purchased locally."

c. A **quantifier** is a word or phrase that indicates quantity in a statement. Some commonly used quantifiers are *some*, *at least one*, *all*, *every*, and *none*. Use a quantifier to write a single statement that correctly describes all the counterexamples identified in Part **b**.

Mathematics Note

Two general statements are **logically equivalent** if one statement is true (or false) exactly when the other is true (or false). In other words, the truth of one statement implies the truth of the other in all cases, and vice versa. Logical equivalence is denoted by the symbol $\equiv$.

For example, consider the general statements "all *p* are *q*" and "no *p* are not *q*." These are logically equivalent because they are either both true or both false for all possible selections for *p* and *q*. Symbolically, this equivalence can be denoted as follows:

$$\text{all } p \text{ are } q \equiv \text{no } p \text{ are not } q$$

For instance, suppose that *p* represents squares and *q* represents rectangles. In this case, "all *p* are *q*" indicates "All squares are rectangles." The statement "no *p* are not *q*" represents "No squares are not rectangles." Both of these statements are true.

It also is possible to consider logical equivalence using sets. For example, Figure **1** shows a Venn diagram for two sets A and B. In this case, the statement "no A are not B" is logically equivalent to "all A are B," because they are either both true or both false, no matter what sets A and B represent.

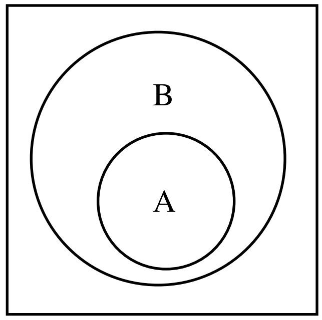


Figure 1: Venn diagram for sets A and B

d. Use a different quantifier to rewrite your statement in Part **c** without changing its truth value.

e. The company plans to use two different woods, oak and pine, in the remodeling project. Let one set of chips represent oak and the other represent pine. Identify every combination of four chips that correctly represents the true statement "Some of the wood used by Mykelti Construction is oak." Then repeat Parts **b–d** for this statement.

f. Four different employees will work on the remodeling project. Let one set of chips represent smokers and the other represent nonsmokers. Identify every combination of four chips that correctly represents the true statement "No employees will smoke on the job site." Repeat Parts **b–d** for this statement.

Mathematics Note

The **negation** of a statement p has the opposite truth value of p. In general, the negation of p is written as "not p," symbolized as "$\sim p$."

For example, consider the statement p, "All lawyers are accountants." In this case, $\sim p$ is "Not all lawyers are accountants."

In order to write clear and grammatically correct negations, it is often helpful to use logically equivalent statements. Table **1** shows three general statements, their negations, and some statements that are logically equivalent to each negation.

Table 1: Three general statements and their negations

Statement	Negation (and some Logical Equivalents)
All A are B.	Not all A are B. It is not the case that all A are B. Some A are not B. At least one A is not B.
Some A are B.	It is not the case that some A are B. No A are B. All A are not B.
No A are B.	It is not the case that no A are B. Some A are B. At least one A is B.

For example, suppose that A is the set of squares and B is the set of rectangles. In this case, the statement "all A are B" would represent "all squares are rectangles." This statement is true. Its negation is "not all squares are rectangles." This statement is false. Using the logically equivalent statements listed in Table **1**, the negation has the same truth value (false) as each of the following:

- It is not the case that all squares are rectangles. (false)
- Some squares are not rectangles. (false)
- At least one square is not a rectangle. (false)

g. Write the negation of each of the following statements.

1. All materials used by Mykelti Construction will be purchased locally.

2. Some of the wood used by Mykelti Construction is oak.

3. No employees will smoke on the job site.

Discussion

a. Is the statement you wrote in Part **c** of the exploration the negation of the original statement? Explain your response.

b. Consider the false statement "All houses are painted white."

1. Describe the negation of this statement.

2. Explain why the following true statement is not the negation of the original statement: "Some houses are painted white."

c. Use logically equivalent statements to express the negation of each of the following in two different ways.

1. All Saint Bernards are dogs.
2. Some presidents of the United States have been women.
3. None of my friends have been to Los Angeles.
4. Some puppies are canines.

d. Give an example of a sentence that doesn't have a negation and explain why this is the case.

Assignment

1.1 **a.** Use logically equivalent statements to rewrite each of the following with a different quantifier.

1. Some people are old.
2. No one can dance.
3. All buildings have elevators.

b. Write the negation of each statement in Part **a**.

1.2 **a.** Write the negation to the statement "All horses are not black."

b. Use a logical equivalent to rewrite the statement "All horses are not black."

1.3 Consider the statement "All police officers are single." Do any of the statements in Parts **a–f** represent the negation of this statement? Justify your response.

a. No police officers are single.

b. Some police officers are not single.

c. All police officers are not single.

d. Some police officers are divorced.

e. Some police officers are not married.

f. Not all police officers are single.

1.4 Use logically equivalent statements to rewrite each of the following.

a. All Mykelti construction workers are men.

b. Some college students major in engineering.

c. No professional football team includes female players.

1.5 The diagram below shows three houses, all of which are painted white. Use similar diagrams to demonstrate the difference between the statements “some houses are painted white” and “some houses are not painted white.”

* * * * *

1.6 Describe how you could determine the truth value for each of the following statements.

a. Some people in your math class are on the football team.

b. All people in your math class are taking a science class this year.

c. No students in your math class ride the bus.

1.7 An advertisement for a department store states, “All sale items are 40% off.” One of the sale items in the store, normally $25.00, is priced at $20.00. Is the advertisement true? Explain your response.

1.8 Write the negation of each of the following statements and identify its truth value.

a. All integers are real numbers.

b. Some integers are whole numbers.

c. No irrational numbers are rational numbers.

* * * * * * * * * * *

Activity 2

Mykelti Construction is currently accepting applications for several new supervisory positions. In this activity, you investigate how different criteria for the jobs affect potential candidates.

Exploration

Mykelti Construction plans to promote one of its current employees to supervising electrician. The personnel director recommends that candidates be previously trained for a supervisory position and be nonsmokers. Her assistant, however, worries that these restrictions may result in too few potential candidates. He recommends that the person be previously trained for a supervisory position or be a nonsmoker. In this exploration, you examine the differences between these two recommendations.

a. Draw a Venn diagram to represent all the employees of the company. This diagram should show employees who have received supervisory training, those who are nonsmokers, those that are both trained and are nonsmokers, and those who are neither trained nor nonsmokers.

b.

1. Shade the region that represents those employees who satisfy the personnel director's recommendation, in other words, those who have received supervisory training and are nonsmokers.
2. Identify the regions in the Venn diagram that represent employees who do not satisfy the director's recommendation.
3. Describe the employees who correspond with the regions identified in Step **2**.
4. Given the two requirements in Step **1**, each employee fits into one of four types. The suitability of each of these types for the supervisor's job can be analyzed using a table.

   Create a table with headings like those in Table **2** below, showing each of the four types of employees. Use T to indicate that the column heading is true for a candidate and F to indicate that it is false. This type of table is referred to as a **truth table**.

   Table 2: Employees to hire for supervising electrician

Nonsmoker	Trained	Possible Candidate

5. Using the personnel director's recommendation, the employees who are not qualified for the supervisor's job may be described by the following sentence: "These employees are not both trained and nonsmokers."

   Recall that two statements can be joined into a **compound statement** using the connectives *and* and *or*. Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, a compound statement like the one above can be represented in general as follows: $\sim(p$ and $q)$.

   Complete a truth table with headings like those in Table **3** for $\sim(p$ and $q)$.

   Table 3: Truth table for $\sim(p$ and $q)$

p	q	p and q	$\sim(p$ and $q)$

6. Another way to describe the employees who are not qualified for the supervisor's job is the following sentence: "These employees-are not trained or are smokers."

Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, the sentence above may be represented in general as follows: $\sim p$ or $\sim q$.

Add the necessary columns to Table **3** for this statement, then complete the truth table. **Note:** In mathematics, the connective *or* is **inclusive**. The inclusive *or* is interpreted as meaning one statement or the other statement, or both statements.

c. **1.** Make another copy of your Venn diagram from Part **a**. Shade the region that represents those employees who satisfy the assistant's recommendation, in other words, those who have received supervisory training or are nonsmokers.

2. Identify the regions in the Venn diagram that represent employees who do not satisfy the assistant's recommendation.

3. Describe the employees who correspond with the regions in Step **2**.

4. As in Part **b**, each employee fits into one of four types and the suitability of each type for the supervisor's job can be analyzed using a table. Complete a table with headings like those in Table **4** for each of the four types of employees.

Table 4: Employees to hire for supervising electrician

Nonsmoker	Trained	Possible Candidate

5. Using the assistant's recommendation, the employees who are not qualified for the supervisor's job may be described by the following sentence: "These employees are not both trained or nonsmokers."

Let p represent employees who are nonsmokers and q represent employees who are trained. Using these symbols, the sentence above may be represented in general as follows: $\sim(p$ or $q)$. Complete Table **5** for this statement.

Table 5: Truth table for $\sim$(*p* or *q*)

p	q	p or q	$\sim(p$ or $q)$

6. Another way to describe the employees who are not qualified for the supervisor's job is the following sentence: "These employees are not trained and are smokers."

Let *p* represent employees who are nonsmokers and *q* represent employees who are trained. Using these symbols, the sentence above may be represented in general as follows:
~*p* and ~*q*.

Add the necessary columns to Table **5** for this statement, then complete the truth table.

Discussion

a. Given your results in the exploration, explain how you could use truth tables to identify when two compound statements are logically equivalent.

b. Describe two different ways of expressing the negation of "*p* and *q*," where *p* and *q* represent statements.

c. Describe two different ways of expressing the negation of "*p* or *q*," where *p* and *q* represent statements.

Mathematics Note

De Morgan's laws, named in honor of British mathematician Augustus De Morgan (1806–1871), apply to negating statements that contain the connectives *and* and *or*.

First, the negation of the compound statement "*p* and *q*" is the compound statement "not (*p* and *q*)." This is logically equivalent to the compound statement "not *p* or not *q*."

For example, consider the compound statement "Joan is tall and she is a basketball player." One form of the negation of this statement is "It is not true that Joan is both tall and a basketball player." This has the same truth value as the statement "Joan is not tall or she is not a basketball player."

Second, the negation of the compound statement "*p* or *q*" is the compound statement "not (*p* or *q*)." This is logically equivalent to the compound statement "not *p* and not *q*."

For example, consider the statement "They want to go to the baseball game on Wednesday or Thursday night." One form of the negation of this statement is "It is not true that they want to go to the baseball game on Wednesday or Thursday night." This has the same truth value as the statement "They do not want to go to the baseball game on Wednesday and they do not want go to the baseball game on Thursday."

d. Describe how to use De Morgan's laws to rewrite the following compound statement in an equivalent form: "It is not the case that they are overworked and underpaid."

Assignment

2.1 When *or* is used as a connector in casual English, it is sometimes interpreted as one or the other, but not both. This is an **exclusive** *or*. For example, consider an intersection where the driver must turn left or right. In this case, turning left or turning right means you choose one or the other, but not both. Computer scientists use the letters *xor* to indicate that the exclusive *or* is desired.

a. Construct a truth table for the exclusive *or*.

b. How does this truth table differ from the one for the inclusive *or*?

2.2 **a.** Suppose you are in a restaurant. A waiter asks "Would you like coffee or tea?" Which interpretation of the connective *or* does this situation imply? Explain your response.

b. When you answer, "Coffee," the waiter then asks, "Cream or sugar?" Which interpretation of the connective *or* does this situation imply? Explain your response.

c. List two examples of questions, statements, or phrases that use the exclusive *or*.

d. Give two examples of questions, statements, or phrases that use the inclusive *or*.

2.3 In most two-door cars, the interior light comes on when at least one of the doors is open.

a. Create a truth table that describes when the interior light is on using the following two statements: "The driver's door is open" and "The passenger's door is open."

b. Which interpretation of the connective *or* does this situation imply? Explain your response.

2.4 Consider the statement "It is not the case that Julia's age is less than or equal to 5 yr."

a. Rewrite this statement using mathematical notation.

b. Use De Morgan's laws to rewrite the statement with the connective *and*.

2.5 Eileen is of average height. A newspaper writes the following sentence describing Eileen: "It is not true that she is too tall or too short."

a. Rewrite this statement using mathematical notation.

b. Use De Morgan's laws to rewrite the statement in an equivalent form.

c. Which sentence do you think is easier to interpret, the original or the one written in Part **b**?

* * * * *

2.6 Determine the truth value of each of the following statements. Then use De Morgan's laws to write the negation of the statement.

a. The number 3 is an even integer and the number 6 is an odd integer.

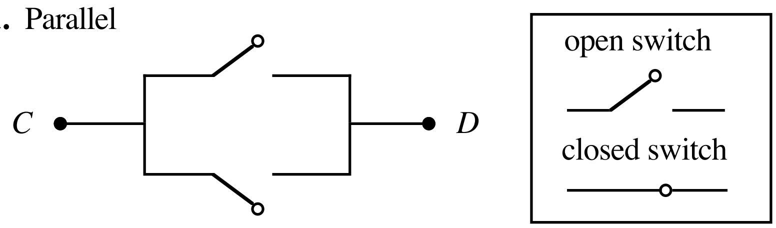
b. The Pacific is a river or the Pacific is an ocean.

c. $3 = 5$ or $4 \neq 5$

2.7 The diagrams in Parts **a** and **b** below show portions of two different electrical circuits. In both circuits, electricity flows from point C to point D. Both have switches that allow electricity to pass when they are closed, but do not allow electricity to flow when they are open. The switches in Part **a** are parallel, while the switches in Part **b** are in series.

Which circuit illustrates the connective *or*? Which illustrates the connective *and*? Explain your responses.

a. Parallel



b. Series

* * * * * * * * * *

Research Project

George Boole was instrumental in the development of symbolic logic. Find out more about Boole's life and describe the connection between Boolean algebra, symbolic logic, and the fields of electronics and engineering.

Activity 3

It's lunch time at the Mykelti construction site. Two workers are arguing about the relative merits of their dogs. One is the proud owner of a Chesapeake Bay retriever. The other has a Labrador retriever.

The Chesapeake's owner states, "No Labrador can retrieve better than my dog."

The Lab's owner replies, "If the dog is smart, then it's a Labrador."

In this activity, you explore the meaning of conditional statements and practice rewriting them in logically equivalent forms.

Exploration

In this exploration, you use Venn diagrams to illustrate the statements made by the two workers, then use logically equivalent statements to rewrite them.

a. Recall that a **conditional statement** may be written in the form "$p \rightarrow q$," where p represents the hypothesis and q represents the conclusion. Identify the hypothesis and the conclusion in the statement, " If the dog is smart, then it is a Labrador."

b. Figure **2** shows a Venn diagram that depicts the statement, "If the dog is smart, then it is a Labrador." Using the quantifiers *all, some*, and *no*, write a statement describing the dogs represented by each of the three regions of the Venn diagram. Do not use the words *if* or *then* in your three statements.

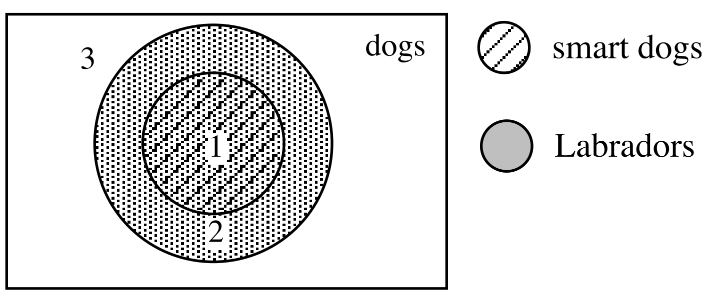


Figure 2: Venn diagram for a conditional statement

c. To disprove the Labrador owner's statement "If the dog is smart, then it is a Labrador," the Chesapeake owner considers some different types of dogs. If the worker finds a smart Labrador, then—for that dog—the hypothesis is true and the conclusion is true. In this case, there is no conflict with the Venn diagram in Figure **2**, since such a dog could be found in region 1. Therefore, the original statement is true.

Consider the other three types of dogs that the Chesapeake owner could find:

- a smart dog that is not a Labrador—(true hypothesis, but false conclusion)
- a dog that is not smart and is a Labrador—(false hypothesis, but true conclusion)
- a dog that is not smart and is not a Labrador—(false hypothesis and false conclusion)

To see if there are any types of dogs that the Chesapeake owner could use to dispute the original statement, copy and complete a table with headings like those in Table **6**.

Table 6: Possible cases of the Labrador owner's statement

Hypothesis	Conclusion	Conflict?	Statement
T	T	no	T
T	F		
F	T		
F	F		

d. Consider the truth table for the compound statement "p or q" shown in Table **7** below. Although this truth table and the one in Part **c** are not identical, they both have three cases where the statement is true and one case where the statement is false. This may suggest a relationship between a conditional statement "$p \rightarrow q$" and a compound statement with a connective *or*.

Table 7: Truth table for *p* or *q*

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

Consider three other cases of a statement with a connective *or*: "$\sim p$ or q," "p or $\sim q$," and "$\sim p$ or $\sim q$." Are any of these statements logically equivalent to the conditional "$p \rightarrow q$"? Use truth tables to verify your response.

Discussion

a. **1.** Consider the conditional statement "If the dog is smart, then it is a Labrador." If you find a Labrador that is not smart, is this statement necessarily false? Explain your response.

2. Is this conditional statement necessarily false if you find a smart dog that is not a Labrador? Explain your response.

b. Describe all circumstances that would make the conditional statement "if *p*, then *q*" false.

c. Explain how to rewrite the following statement using the connective *or*: "If a dog is smart, then it is a Labrador."

d. After the Labrador's owner finished speaking, the Chesapeake's owner retorted, "If the moon is made of green cheese, then all smart dogs are Labs." What can you conclude about this statement?

Assignment

3.1 Consider the conditional statement, "If the tree is an oak, then it is tall."

a. Construct a Venn diagram that models this statement.

b. Using the quantifiers *all*, *some*, and *no*, write three statements that describe the sets of trees in the Venn diagram. Do not use the words *if* or *then* in your three statements.

3.2 **a.** Identify the hypothesis and conclusion in the conditional statement "If the house was built by Mykelti Construction, then its materials were purchased locally."

b. When is this conditional statement false?

c. Use a quantifier such as *all*, *some*, or *no* to rewrite the statement in Part **a** in a logically equivalent form. Do not use the words *if* or *then* in your statement.

d. Use the connective *or* to rewrite the conditional statement in Part **a** in a logically equivalent form.

3.3 In the introduction, Colter's supervisor told him "If you finish the foundation, then you're off for the weekend."

a. Identify the hypothesis and conclusion of this conditional statement.

b. Describe the four possibilities Colter should consider when interpreting this statement.

c. If Colter doesn't finish the foundation, can he justify taking the weekend off? Explain your response.

3.4 Use the connective *or* to rewrite each of the following conditional statements in a logically equivalent form.

a. If it is fall, then the leaves are falling.

b. If the worker is not a carpenter, then she is an electrician.

c. If the person is happy, then he is not young.

d. If it is not snowing, then it is not winter.

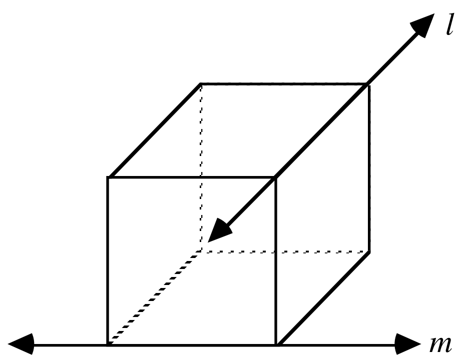
3.5 **a.** Write the following statement as a conditional: "All prime numbers greater than 2 are odd."

b. Is this statement true? Explain your response.

3.6 Consider the statement "If three angles of one triangle are congruent to three angles of another triangle, then the triangles are congruent." Is this statement true? Explain your response.

* * * * *

3.7 **Skew lines** are lines in space that lie in different planes. For example lines *l* and *m* contain the edges of a cube that lie in different planes. Although skew lines are not parallel, they do not intersect.



a. Construct a Venn diagram that models the conditional statement, "If two lines are parallel, then they never intersect."

b. Using the quantifiers *all*, *some*, and *no*, write three statements that describe the sets of lines in the Venn diagram. Do not use the words *if* or *then* in your three statements.

c. **1.** Where do skew lines fit in your Venn diagram from Part **a**?

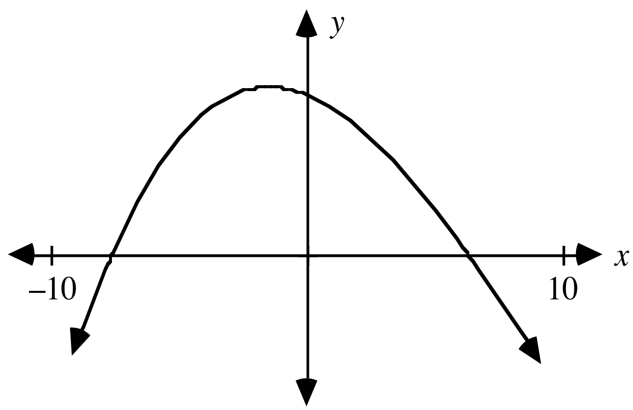
2. Do skew lines serve as a counterexample to the statement in Part **a**? Explain your response.

3.8 Consider the conditional statement "If the polygon is a triangle, then $a^2 + b^2 = c^2$, where a, b, and c are the lengths of the sides of the triangle and c is the length of the longest side."

a. Identify the hypothesis and the conclusion of this conditional.

b. Is this statement true? Explain your response.

3.9 The following graph shows a portion of a polynomial function.



a. One of your classmates makes the claim: "All the roots are between –10 and 10." Write this statement as a conditional.

b. Another classmate makes the claim "Some root is not between –10 and 10." Restate this claim using the connective *or*.

* * * * * * * * * *

Activity 4

Back at the Mykelti Construction site, the two dog owners are still arguing. While the rest of the crew listens, the Labrador owner repeats his claim: "If a dog is smart, then it's a Labrador."

One of his coworkers then asks, "Do you mean that if a dog is a Labrador, then it's smart?"

"No," a second worker chimes in. "He means that if a dog isn't smart, then it's not a Labrador."

"That's not it," cries a third. "What he really wants to say is if a dog's not a Labrador, then it's not smart."

In this activity, you determine which of these three statements has the same truth value as the owner's original claim.

Mathematics Note

The **converse** of a conditional statement is formed by interchanging the hypothesis and the conclusion. The converse of $p \rightarrow q$ can be represented by $q \rightarrow p$ or "if q, then p."

The **inverse** of a conditional statement is formed by negating the hypothesis and the conclusion. The inverse of $p \rightarrow q$ can be represented by $\sim p \rightarrow \sim q$ or "if $\sim p$, then $\sim q$."

The **contrapositive** of a conditional statement is formed by interchanging the hypothesis and the conclusion and negating both of them. The converse of $p \rightarrow q$ can be represented by $\sim q \rightarrow \sim p$ or "if $\sim q$, then $\sim p$."

For example, consider the conditional statement "If today is July 4, then it is summer in the northern hemisphere." Its converse is the conditional "If it is summer in the Northern hemisphere, then today is July 4."

Its inverse is the conditional "If today is not July 4, then it is not summer in the Northern hemisphere."

Its contrapositive is the conditional "If it is not summer, then today is not July 4 in the Northern hemisphere."

Exploration

In this exploration, you look for patterns in the truth tables for the converse, inverse, and contrapositive of the conditional $p \rightarrow q$.

a. Create a truth table for $p \rightarrow q$ and its converse, inverse, and contrapositive.

b. Describe any patterns or relationships you observe in the table.

Discussion

a. When a conditional statement is true, is its converse also true? Explain your response.

b. When a conditional statement is true, is its inverse also true? Explain your response.

c. When a conditional statement is true, is its contrapositive also true? Explain your response.

d. Which worker's statement best describes the Labrador owner's original claim? Justify your response.

Assignment

4.1 Consider the statement, "If a polygon is a square, then it is a quadrilateral."

a. Write the converse of this statement and explain whether or not the converse is true.

b. Write the inverse of the statement and explain whether or not the inverse is true.

c. Write the contrapositive of the statement and explain whether or not the contrapositive is true.

4.2 **a.** Write an example of a conditional statement in mathematics whose converse is true.

b. Write an example of a conditional statement in mathematics whose converse is false.

4.3 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw? Explain your responses.

a. If a house was built by Mykelti Construction, then its materials were purchased locally. The materials used in the house were not purchased locally.

b. If a house was built by Mykelti Construction, then its materials were purchased locally. The house was not built by Mykelti Construction.

4.4 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw?

a. If it's raining, then the sidewalk is wet. It's raining.

b. If it's raining, then the sidewalk is wet. The sidewalk is wet.

c. If it's raining, then the sidewalk is wet. The sidewalk isn't wet.

d. If it's raining, then the sidewalk is wet. It's not raining.

4.5 Consider the statement, "If the diagonals of a quadrilateral are congruent, then the quadrilateral is a square."

a. Explain whether or not the converse of this statement is true.

b. Explain whether or not the inverse of this statement is true.

c. Explain whether or not the contrapositive of this statement is true.

* * * * *

4.6 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw?

a. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The polygon is a triangle.

b. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The sum of the measures of its exterior angles is 360˚.

c. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The polygon is not a triangle.

d. If the polygon is a triangle, then the sum of the measures of its exterior angles is 360˚. The sum of the measures of its exterior angles is not 360˚.

4.7 A **tautology** is a statement that is always true. Use truth tables to determine which of the following statements are tautologies.

a. $\left[\left(p \rightarrow q\right) \text{ and } p\right] \rightarrow q$

b. p or $\sim p$

c. $\left(p \text{ and } \sim q\right)$ and $\left(\sim p \text{ or } q\right)$

d. $\left(p \text{ and } q\right)$ or $\left(\sim p\right)$

4.8 Given that each of the following pairs of statements is true, what conclusions, if any, can you draw?

a. If it is sunny, then I'll go for a bike ride. I'll go for a bike ride.

b. If it is sunny, then I'll go for a bike ride. I will not go for a bike ride.

c. If it is sunny, then I'll go for a bike ride. It is sunny.

d. If it is sunny, then I'll go for a bike ride. It is not sunny.

* * * * * * * * * *

Summary Assessment

1. The home of a wealthy businesswoman has been burglarized. Because of your expert knowledge of the rules governing logic, you have been called in to assist local detectives. So far, they have collected the following statements from various individuals.

a. The niece and not the butler committed the crime.

b. No silver was taken.

c. If the crime happened on Monday, then the woman's banker or lawyer committed the crime.

d. Some of the fingerprints at the crime are those of the woman's sister.

e. If no jewelry was taken, then the woman's husband committed the crime.

f. The thief took silver and no jewelry.

g. If the butler committed the crime, then no silver was taken.

h. If the crime happened on Monday, then the woman's sister committed the crime.

i. If the woman's husband did not commit the crime, then it was committed by her lawyer.

As a result of their investigations, the detectives believe that statements **a**, **b**, **d**, **f**, and **h** are false. Assuming they are correct, identify both the thief and the time of the theft. Justify your conclusions.

2. Hood Wink the magician has a deck of eight cards. Each card has a whole number printed on one side and a letter of the alphabet on the other side. Hood Wink claims, "In my deck of cards, if a card has an even number on one side, then it has a vowel on the other side." Hood Wink then deals the cards as shown in the diagram below.

8	A	D	17	P	3	10	E

Which card(s) would you need to turn over to make sure that Hood Wink's statement is true? Explain your response.

Module Summary

- A **statement** is a sentence that can be determined to be either true or false, but not both. The truth or falseness of a statement is its **truth value**.
- A **quantifier** is a word or phrase that indicates quantity in a statement. Some commonly used quantifiers are *some*, *at least one*, *all*, *every*, and *none*.
- Two general statements are **logically equivalent** if one statement is true (or false) exactly when the other is true (or false). In other words, the truth of one statement implies the truth of the other in all cases, and vice versa. Logical equivalence is denoted by the symbol $\equiv$.
- The **negation** of a statement p has the opposite truth value of p. In general, the negation of p is written as "not p," symbolized as "$\sim p$."
- **De Morgan's laws** apply to negating statements that contain the connectives *and* and *or*. First, the negation of the compound statement "p and q" is the compound statement "not (p and q)." This is logically equivalent to the compound statement "not p or not q."

  Second, the negation of the compound statement "p or q" is the compound statement "not (p or q)." This is logically equivalent to the compound statement "not p and not q."
- In mathematics, the connective *or* is **inclusive**. The inclusive *or* is interpreted as meaning one or the other or both. Sometimes when *or* is used as a connector in casual English, it is interpreted as one or the other, but not both. This is an **exclusive** *or*.
- A **conditional statement** consists of two parts: a **hypothesis** and a **conclusion**. It is usually written in the form "if p, then q," where p is the hypothesis and q is the conclusion. This can be represented as "$p \rightarrow q$."
- A conditional statement "if p, then q" is false only when the hypothesis (p) is true and the conclusion (q) is false.
- The **inverse** of a conditional statement is formed by negating the hypothesis and the conclusion. The inverse of $p \rightarrow q$ can be represented by $\sim p \rightarrow \sim q$ or "if $\sim p$, then $\sim q$."
- The **converse** of a conditional statement is formed by interchanging the hypothesis and the conclusion. The converse of $p \rightarrow q$ can be represented by $q \rightarrow p$ or "if q, then p."
- The **contrapositive** of a conditional statement is formed by interchanging the hypothesis and the conclusion and negating both of them. The converse of $p \rightarrow q$ can be represented by $\sim q \rightarrow \sim p$ or "if $\sim q$, then $\sim p$."
- A conditional statement and its contrapositive are logically equivalent.

Selected References

Boyer, C. B. *A History of Mathematics*. Second Edition. New York: John Wiley & Sons, 1991.

Consortium for Mathematics and Its Applications (COMAP). *Principles and Practice of Mathematics, Volume 2*. Arlington, MA: COMAP, 1993.

O'Regan, P. J. "Intuition and Logic." *Mathematics Teacher* 81(November 1988): 664–668.