What Shape Is Your World?

Where in the world are you? Where in the world are you going? When answering either question, it may help to use a two-dimensional projection of the earth's three-dimensional surface.

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Introduction

Cartography, the science of mapmaking, dates back at least to the time of the ancient Greeks. Humans are travelers, and travelers have always needed directions. Historically, most maps have been drawn on a flat surface, despite the approximately spherical shape of the earth. Over short distances, this projection of a three-dimensional surface onto two dimensions is relatively accurate. However, when mapping large regions covering thousands of square kilometers, some distortion is inevitable.

For example, each map of Greenland shown below was created using a different mapping technique. The map in Figure **1a** was made using a **stereographic projection**, a projection whose center is at one of the earth's poles. The map in Figure **1b** was made using a cylindrical projection, a projection with its center at the earth's center.

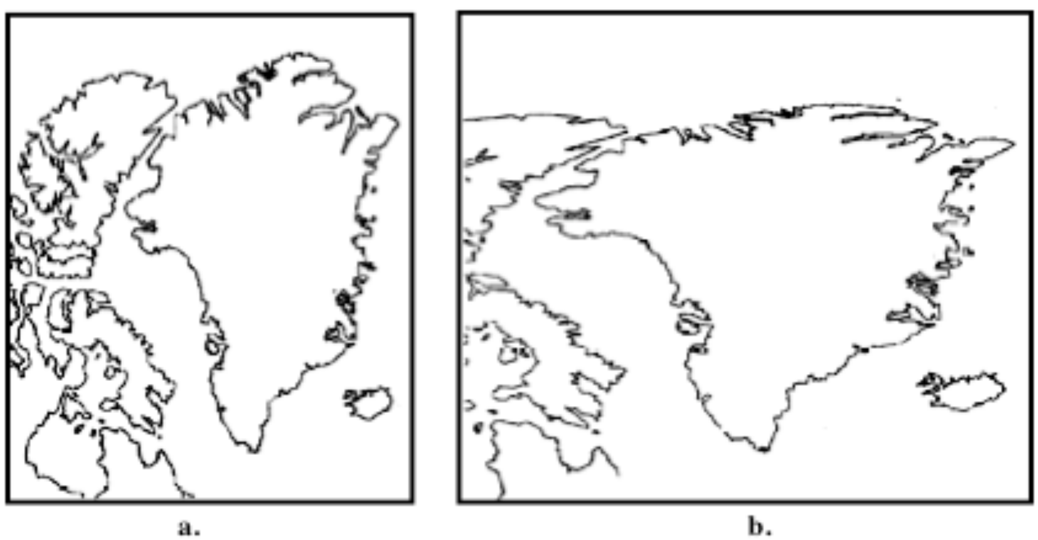

Figure 1: Two maps of Greenland made using different projections

While distortions can make flat paper maps somewhat misleading, such maps have been quite useful in the past, and will certainly remain so in the future. In the following activities, you investigate some of the inconsistencies and distortions that may result from representing a three-dimensional surface on a flat sheet of paper.

Exploration

a. Obtain two sheets of centimeter graph paper from your teacher. Also get the templates of the two maps in Figure **1**. Use each of the templates to make an estimate of Greenland's area, in square kilometers.

b.
1. Compare your two estimates with those of your classmates.
2. Use the class estimates to find a mean value for Greenland's area on each template.

c. Determine which template of Greenland more closely resembles the shape and proportional size of Greenland on a globe.

Discussion

a. What differences did you observe in the two maps of Greenland?

b.
1. The actual area of Greenland is about $2{,}175{,}600 \text{ km}^2$. How does this compare to the two values you obtained in Part **b** of the exploration?
2. What might account for any differences between your estimates and the actual area?

c. How do you think that geographers determined the actual area of Greenland?

d. Despite the distortions in flat maps of Greenland, what useful information may still be obtained from them?

e. Do you think a map of Greenland on a globe is more accurate than the maps in Figure **1**? Explain your response.

Activity 1

One method of giving directions from a starting place to a destination involves specifying a distance east or west, followed by a distance north or south. In other words, the destination is described as a pair of distances along perpendicular lines beginning from a point of origin. This task can be accomplished using a rectangular (Cartesian) coordinate system.

Another method of giving directions involves specifying the distance "as the crow flies" from the starting point to the destination as well as an angle measured from a fixed ray. This task can be accomplished using a **polar coordinate system**. In this activity, you use polar coordinate systems to identify locations on different types of maps.

Mathematics Note

A **polar coordinate system** describes the location of a point P in a plane using an ordered pair consisting of a **radius** r and a **polar angle** θ.

The plane containing a polar coordinate system is the **polar plane**. The polar angle is an angle measured from a fixed ray, called the **polar axis**. The endpoint of the polar axis is the **pole**. The distance from the pole to point P, measured in the polar plane, is r.

To establish a polar coordinate system, a point in the plane is designated as the pole O. Any ray with endpoint O can be designated as the polar axis for the plane. In a standard polar coordinate system, the pole corresponds to the origin of a rectangular coordinate system, while the polar axis corresponds to the positive x-axis, as shown in Figure **2**.

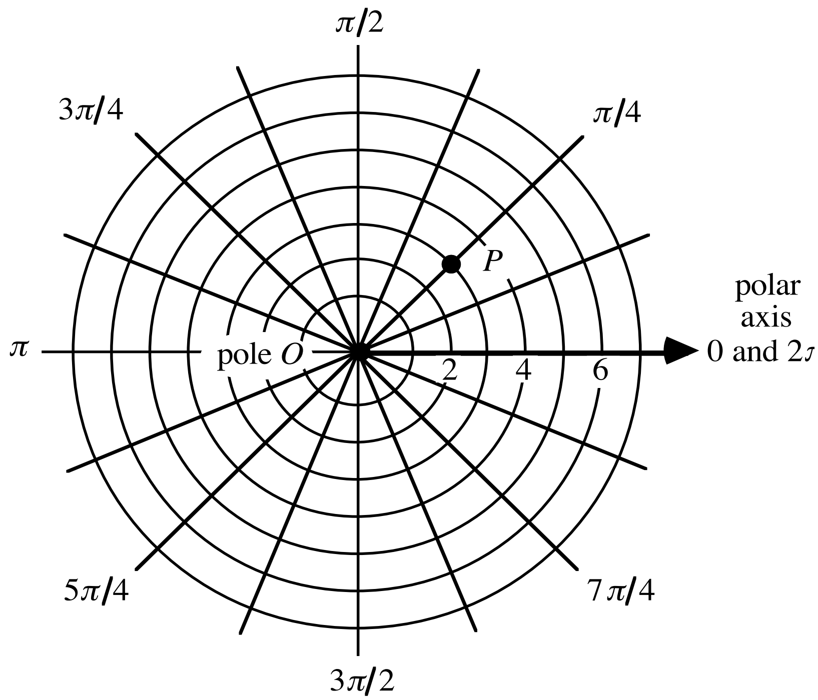

Figure 2: A polar coordinate system

In this system, any point P in the plane may be represented as an ordered pair (r, θ). The variable r represents the distance between O and P, while θ represents the directed measure of the angle formed by the polar axis and $\overrightarrow{OP}$. The measure of the polar angle may be given in either radians or degrees.

In Figure **2**, for example, the coordinates of point P may be given as $(3, 45^\circ)$ using degrees or as $(3, \pi/4) \approx (3, 0.79)$ using radians.

Exploration 1

Figure **3** shows a map of a popular hiking area. In this exploration, you use a polar coordinate system to describe some of the destinations on the map.

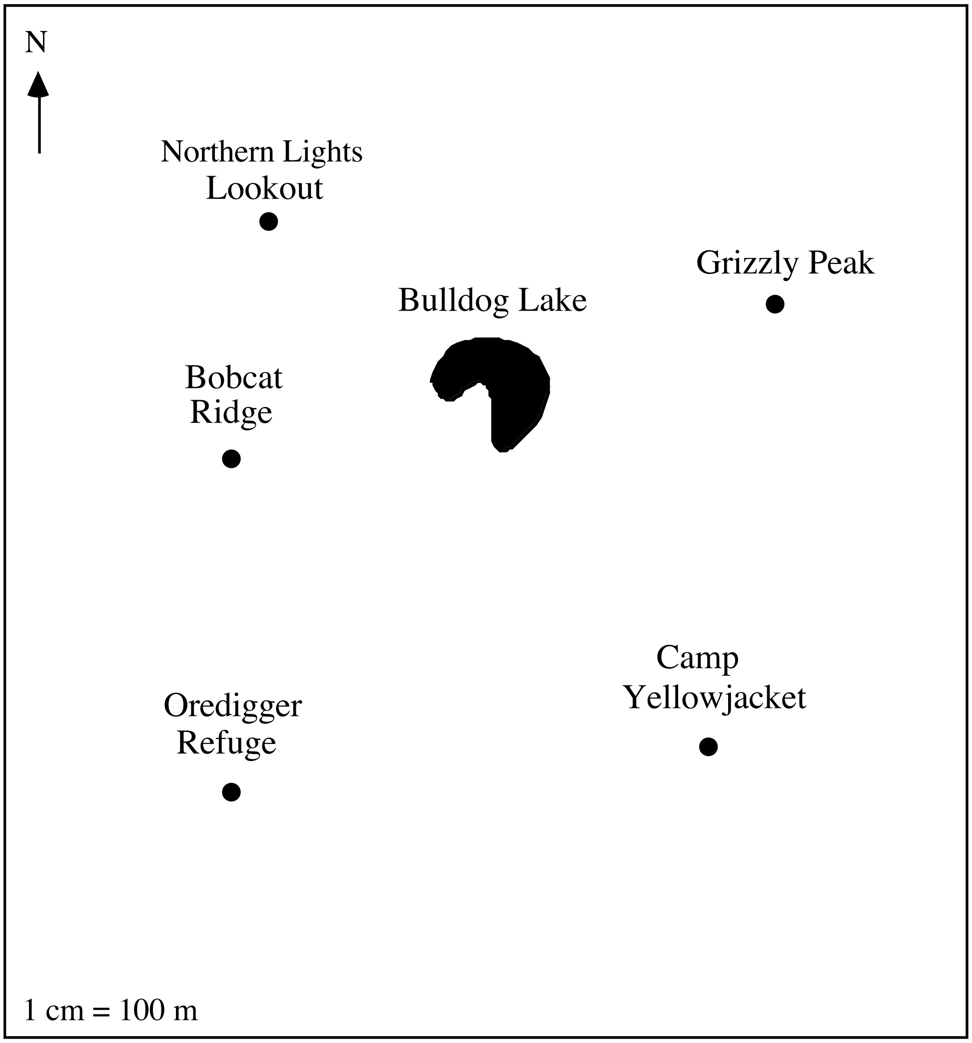

Figure 3: A hiking area

a. Using a copy of Figure **3**, create a polar coordinate system with the pole at Oredigger Refuge and the polar axis extending due east.

b. Use polar coordinates to describe the locations of Grizzly Peak and Camp Yellowjacket.

c. **1.** A polar angle describes an amount of rotation about the pole. This allows the coordinates of a single point to be represented by more than one positive value of θ.

Find two different polar representations for the locations of Grizzly Peak and Camp Yellowjacket, using positive values of θ different from those in Part **b**.

2. Since the polar angle is a directed angle, θ also can have negative values. (A positive value of θ represents an angle measured counterclockwise from the polar axis, a negative value of θ represents an angle measured clockwise from the polar axis.)

Find two different polar representations for the locations of Grizzly Peak and Camp Yellowjacket using negative values of θ.

d. Many hikers use Northern Lights Lookout as a base camp for day trips in the region. To help these hikers plan their trips, create another polar coordinate system with the pole located at Northern Lights Lookout and the polar axis extending due west.

Repeat Parts **b** and **c** using this polar coordinate system.

e. Repeat Part **d** on a polar coordinate system with the pole at Northern Lights Lookout and the polar axis extending due east.

f. Compare the polar coordinates you found for Grizzly Peak and Camp Yellowjacket in Parts **b–e** with those of your classmates.

Discussion 1

a. Describe how to locate the point (r, θ) on a polar coordinate system when θ is positive.

b. How many different ordered pairs of the form (r, θ) can be used to represent a given point on a specific polar coordinate system? Explain your response.

c. **1.** How do the locations of the pole and the polar axis affect the coordinates of a point?

2. Why might it be desirable for all users of a map to agree on the same locations of the pole and polar axis?

Exploration 2

Because of the dramatic changes in elevation between destinations, many hikers purchase a topographic map of the region you examined in Exploration **1**. A topographic map is a two-dimensional representation of a three-dimensional surface. On such maps, **contour lines** are used to depict points of equal elevation. In the map in Figure **4**, for example, each labeled contour line indicates points at the corresponding elevation (in meters) above sea level.

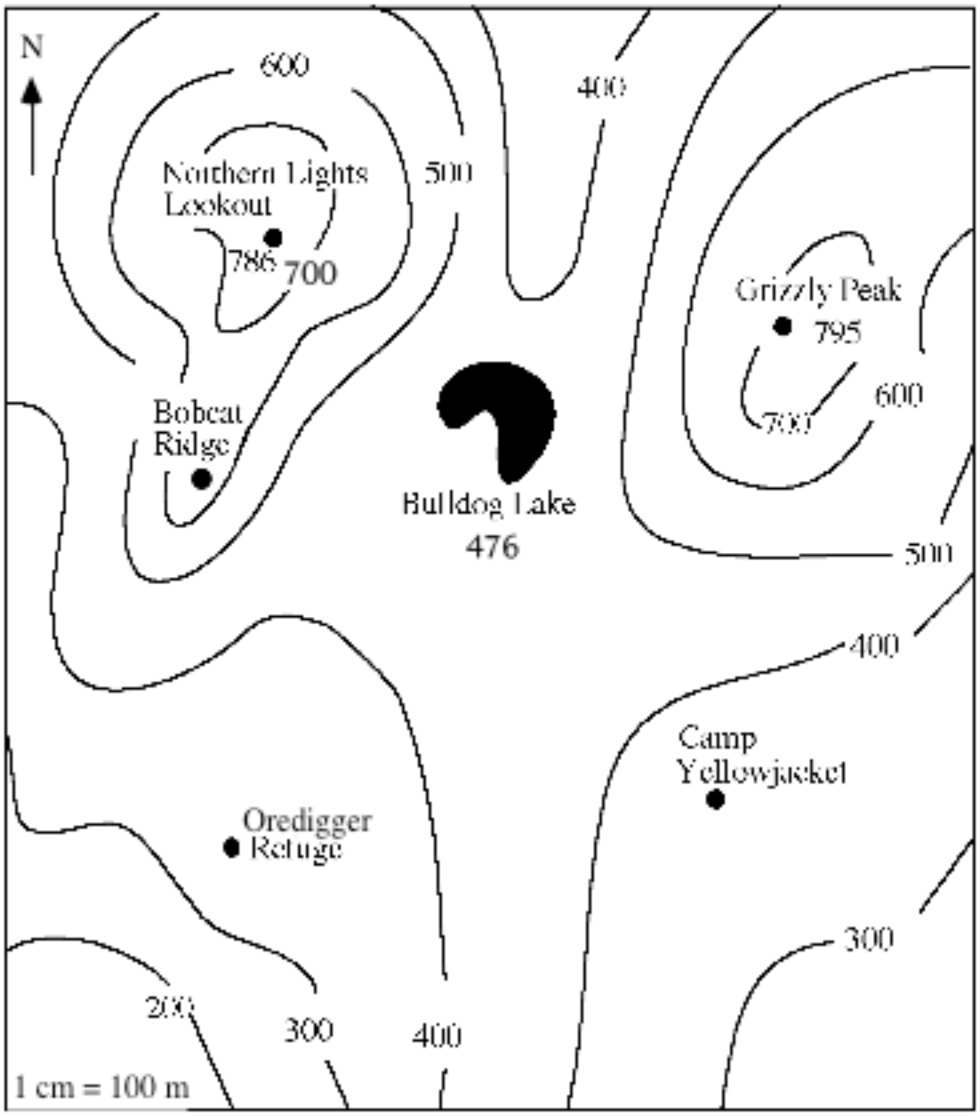

Figure 4: Topographic map of a hiking area

Mathematics Note

Like a rectangular coordinate system, a polar coordinate system can be extended to three dimensions by adding a third dimension z.

In a **cylindrical coordinate system**, a point in space is represented by an ordered triple of the form (r, θ, z). The values of r and θ are measurements in the polar plane. The value of z is the directed distance between the point and the polar plane (the plane containing the polar axis). A positive value for z represents a distance above the polar plane. When a cylindrical coordinate system is used to describe locations on earth, the polar plane is typically located at sea level.

For example, Figure **5** shows the locations of three points on a cylindrical coordinate system. The coordinates of point A are $(5, \pi/3, 1)$, the coordinates of point B are $(4, 5\pi/6, -3)$, and the coordinates of point C are $(2, -\pi/6, 2)$.

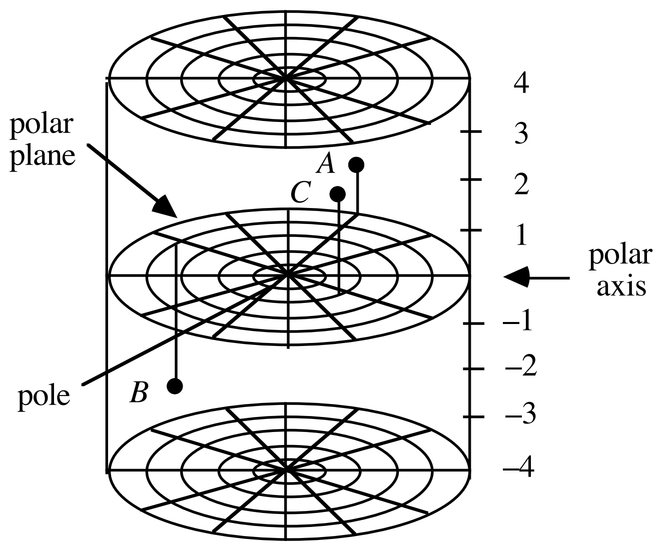

Figure 5: A cylindrical coordinate system

a. On a copy of Figure **4**, create a cylindrical coordinate system with the pole located at the southwest corner of the map, the polar axis extending due east, and the polar plane at sea level.

b. Using your coordinate system from Part **a**, determine cylindrical coordinates for Oredigger Refuge in which:

1. r, θ, and z are all positive
2. r is positive, θ is negative, and z is positive.

Discussion 2

a. How can hikers use the contour lines on a topographic map to help them determine the character of the terrain?

b. Considering the map in Figure **4**, which point do you think would provide the best location for the pole of a cylindrical coordinate system?

c. In a cylindrical coordinate system, describe the geometric figure formed by all points that have the same value for each of the following:

1. r
2. θ
3. z

d. In Part **b** of Exploration **2**, you expressed the location of Oredigger Refuge using both positive and negative values for θ. Are there any points on a cylindrical coordinate system that could not be represented using only positive coordinates? Explain your response.

Assignment

1.1 Re-express the coordinates for each point in Parts **a** and **b** using only positive values.

a. $C\ (7, -\pi/2)$

b. $D\ (5, -19\pi/6)$

1.2 **a.** Graph and label each of the following points on the same cylindrical coordinate system.

1. $A\ (5, \pi/4, 4)$
2. $B\ (5, 7\pi/4, -2)$
3. $C\ (3, \pi/4, 3)$
4. $D\ (2, \pi/2, -2)$
5. $E\ (5, 3\pi/4, -4)$
6. $F\ (1, \pi/4, 1)$
7. $G\ (4, 3\pi/4, 0)$

b. Which of the points in Part **a** lie on the same cylinder centered about the z-axis? Explain your response.

c. Which two points in Part **a** lie in a plane parallel to the polar plane? Explain your response.

d. Which three points in Part **a** determine a plane that contains the pole and is perpendicular to the polar plane? Explain your response.

1.3 Obtain a copy of the map in Figure **4**. Locate the pole of a cylindrical coordinate system at Northern Lights Lookout with the polar axis extending due east. Using this system, find cylindrical coordinates to describe the locations of Grizzly Peak and Camp Yellowjacket. Describe the process you used in each case.

1.4 Imagine that you are camped at Bobcat Ridge on the map in Figure **4**.

a. Using your camp as the pole and a polar axis extending due east, find polar coordinates for each of the other landmarks on the map: Northern Lights Lookout, Grizzly Peak, Camp Yellowjacket, Oredigger Refuge, and the center of Bulldog Lake.

b. Assume Bobcat Ridge is 625 m above sea level. Find cylindrical coordinates for each of the five landmarks named in Part **a**.

c. There are four other points of interest near your camp: Argonaut Alley, Bear Crossing, Saint's Cave, and Devil's Den. Using a copy of Figure **4** and the following polar coordinates, find and label each of these points: Argonaut Alley $(250, \pi/3)$, Bear Crossing $(400, 5\pi/3)$, Saint's Cave $(150, 7\pi/6)$, and Devil's Den $(125, 3\pi/4)$.

d. Write the approximate cylindrical coordinates for each of the four points of interest named in Part **c**.

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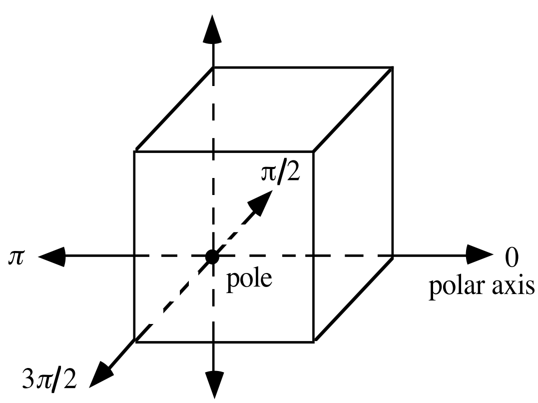
1.5 The diagram below shows a cube with a volume of s^3. The lower left-hand vertex of the cube's rear face is located at the pole of a cylindrical coordinate system.



a. Determine cylindrical coordinates for each vertex of the cube.

b. Determine cylindrical coordinates for the center of the cube.

1.6 Suppose that the pole in Problem **1.5** were moved to the cube's center.

a. Describe how this affects the r-coordinates of the cube's vertices.

b. Describe how this affects the z-coordinates of the cube's vertices.

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Activity 2

There are many methods for creating a flat map of a spherical surface—all of which involve mapping points with coordinates in three dimensions to points with coordinates in two dimensions. As you saw in the introduction to this module, flat maps of three-dimensional surfaces may contain distortions in shape, area, and distance. The distortions produced by a particular type of projection depend on several factors. In this activity, you investigate the distortion caused by a stereographic projection of points on a sphere to a flat map.

Mathematics Note

A **stereographic projection** is a projection of the points on a sphere onto a plane perpendicular to a given diameter of the sphere. The plane is the **plane of projection**. The endpoints of the diameter are the **poles** of the sphere.

The image of a point on the sphere is the point of intersection of a ray and the plane perpendicular to the diameter that contains the poles. The ray contains one of the poles, designated as the **center of projection**, and the point being projected. In any stereographic projection, there are points that have no image in the plane.

In Figure **6**, for example, point C' is a stereographic projection of point C, where point N is the center of projection and the plane of projection is perpendicular to the diameter $\overline{NS}$ at point S. In this projection, point N has no image on the plane, while the image of point S is itself.

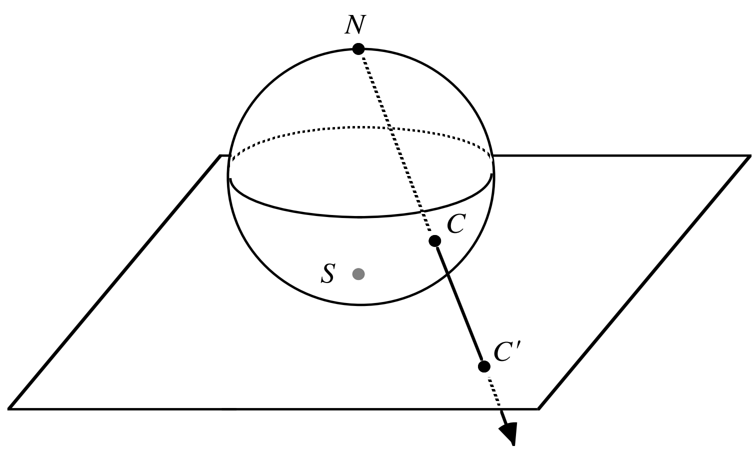


Figure 6: Stereographic projection of point *C*

Exploration 1

Because a sphere is a closed surface and a plane is not, you should expect to observe some differences between a figure on a sphere and its projected image on a plane. In this exploration, you examine how a stereographic projection affects the image of a line on a sphere.

a. Label the two points at opposite ends of a diameter of a sphere *N* and *S*. Let these points represent the north and south poles, respectively. In this exploration, *N* will serve as the center of projection.

b. On a sphere, lines are defined as great circles. The great circles that contain the poles are lines of longitude. Complete the steps below to represent the projection of a line of longitude.

1. Stretch a piece of string from *N* to *S* to represent part of a line of longitude. Mark the string where it touches these two points.
2. Remove the string from the sphere. Mark three other points on the string so that the length determined in Step **1** is divided into four equal parts.
3. Stretch the string from *N* to *S* again. Mark the three points from Step **2** on the sphere. Label these points *A*, *B*, and *C*.

c. As shown in Figure **7**, position the sphere so that a large sheet of paper is tangent to it at *S*. Insert three guides into the sphere and mark the points where the tips of the guides touch the paper. These marks will serve as reference points to maintain the same position of the sphere throughout the exploration.

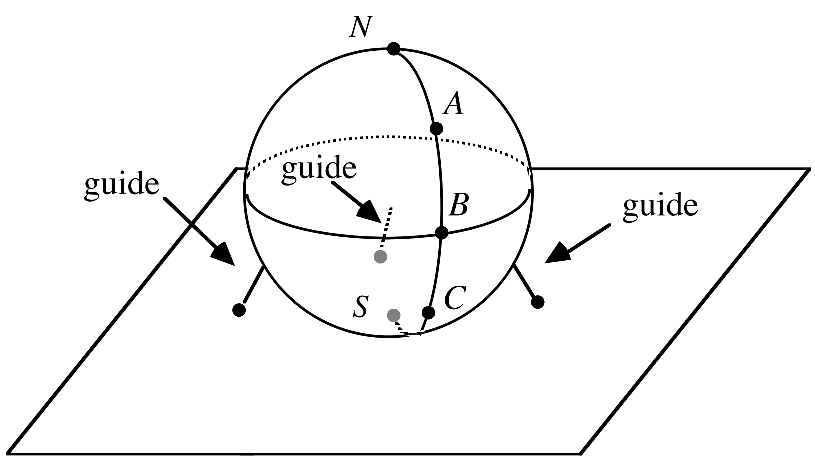


Figure 7: Sphere with points *A*, *B*, and *C* marked

d. To find C', the image of point *C* under a stereographic projection, use a skewer to model a ray. Carefully pass a skewer through the sphere from point *N* through point *C* until the tip of the skewer touches the paper. Mark the point of intersection of the skewer with the paper.

e. Repeat Part **d** for points *A*, *B*, and *S*.

f. Measure $\overline{C'S'}$, $\overline{B'C'}$, and $\overline{A'B'}$.

g. Since the equator is a great circle, it also represents a line on a sphere. Use the steps below to project an image of the equator on the same sheet of paper used above.

1. Stretch a piece of string around the sphere to represent the equator.
2. Remove the string from the sphere. Mark five points on the string so that the distance along the equator is divided into six equal parts.
3. Stretch the string around the equator again. Mark the five points and the point where the endpoints of the string meet. Label these points D, E, F, G, H, and I.
4. Use skewers to determine the image of each point under a stereographic projection where N is the center of projection and the plane is tangent to the sphere at S. **Note:** Save your work for use in the assignment.

Discussion 1

a. In Part **e** of Exploration **1**, what is the relationship between the south pole S and its image S'?

b. Consider a flat map of a globe created using a stereographic projection like the one in Exploration **1**.

1. Describe how the images of the lines of longitude and the equator would appear on the flat map.
2. What appears to be the relationship between the size of the equator and the size of its image?
3. Why must the image of a line of longitude be a line in the plane?

c.

1. Recall that in a one-to-one correspondence, each element in the domain is paired with exactly one element in the range, and each element in the range is paired with exactly one element in the domain.

   Does the mapping of the points of a sphere to a plane as described in Exploration **1** represent a one-to-one correspondence? Defend your answer.
2. A stereographic projection can be considered a function. Explain why this is true.

d. **1.** Are lines preserved under a stereographic projection? In other words, are the projected images of lines on a sphere also lines in the plane? Explain your response. (Remember that a line on a sphere is defined as a great circle.)

2. Is collinearity preserved under a stereographic projection? In other words, if the preimage points lie on the same great circle, do the image points lie on the same line in the plane? Explain your response.

e. **1.** On a sphere, lines of longitude are perpendicular to the equator. Is perpendicularity preserved under a stereographic projection? Justify your response.

2. Two lines of longitude can be perpendicular to each other at the poles. Would their images under a stereographic projection also be perpendicular? Explain your response.

f. Consider three points on a sphere—A, B, and C—where A and B are equidistant from C. Under a stereographic projection, would A' and B' be equidistant from C'? In other words, is distance preserved in a stereographic projection? Justify your response.

g. **1.** Considering your results in Exploration **1**, which regions on a sphere appear to be most distorted in a stereographic projection?

2. What does this imply about a map of Greenland created using this type of projection?

3. Which regions on a sphere appear to be least distorted in a stereographic projection?

4. In general, where would you place the point of tangency of the plane to obtain the least distortion of a preimage?

h. Consider a stereographic projection in which the globe's south pole is the center of projection and the plane is located tangent to the globe at the north pole. How would the amount of distortion in the image of Iceland compare with the amount of distortion in the image of Florida?

i. Describe the map created by a stereographic projection in which the globe's north pole is the center of projection and the plane contains the equator.

j. On a sphere, there are an infinite number of lines (great circles) that do not contain the poles. Describe the images of these lines under a stereographic projection like the one in Exploration **1**.

Exploration 2

In Exploration **1**, you discovered that the distance between points on a sphere generally is not preserved when those points are projected onto a plane. In this exploration, you create a mathematical model of a stereographic projection and use it to investigate these distortions.

a. Figure **8** shows a cross section of a sphere, where $\overline{NS}$ is a diameter of the sphere, and $\overrightarrow{NP}$ intersects the sphere in the plane of the cross section.

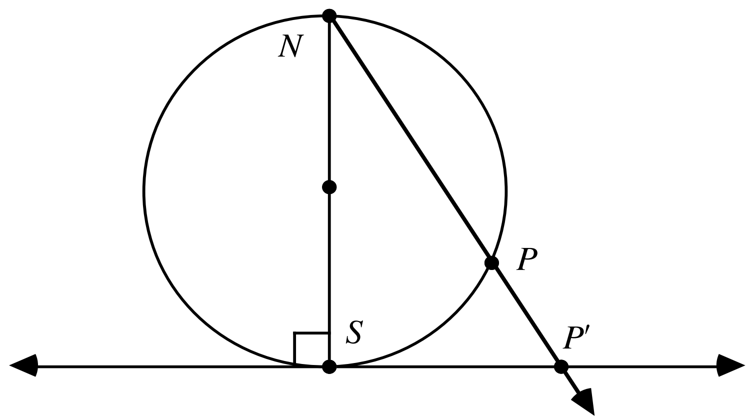


Figure 8: Cross section of a stereographic projection

Reproduce this diagram using a geometry utility. Anchor $\overrightarrow{NP}$ at N, allowing point P to move around the circle. Make sure that the position of point P' changes as P moves around the circle.

b. Describe what happens to P' as you move P around the circle.

c. Construct the three points that correspond with A, B, and C in Part **b** of Exploration **1**. Make sure that $\overset{\frown}{NA} \cong \overset{\frown}{AB} \cong \overset{\frown}{BC} \cong \overset{\frown}{CS}$.

d.
1. Move point P so that it is concurrent with point C.
2. Record the length of $\overline{P'S}$.
3. Repeat Steps **1** and **2** for points A, B, S, and N.

e. Construct $\overline{PZ}$ perpendicular to $\overline{NS}$ as shown in Figure **9**. Make sure $\overline{PZ}$ changes length as point P moves around the circle.

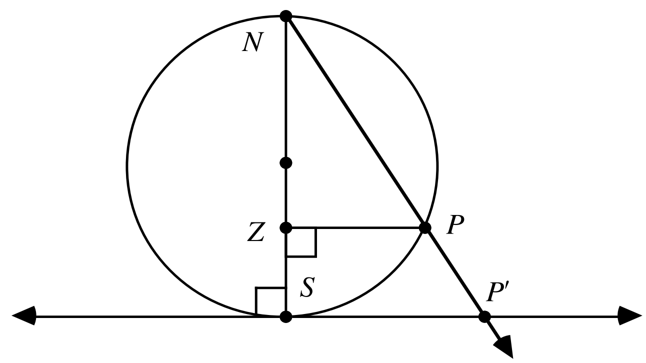


Figure 9: Cross section with $\overline{PZ}$

f. Express the length of $\overline{P'S}$ in terms of the lengths of $\overline{NZ}$, $\overline{NS}$, and $\overline{PZ}$.

Discussion 2

a. What do the results found in Part **d** of Exploration **2** tell you about distances on a map made using a stereographic projection?

b. Describe what happens to the length of $\overline{P'S}$ as the length of $\overline{SZ}$ gets close to the length of $\overline{NS}$.

c. Triangles NPZ and $NP'S$ in Figure **9** are similar triangles. Explain how you know this is true.

d. In Exploration **1**, you observed that the image of the equator under a stereographic projection with the plane tangent to the sphere at the south pole appeared to be twice the diameter of the preimage. Use similar triangles to explain why this is true.

e. The image of a point *P* on a sphere under a stereographic projection like the one in Exploration **2** can be described using polar coordinates.

Describe how the distance from point *S* to point P' can be used to help find the polar coordinates of P', where the pole of the graph is at *S* and the polar axis is opposite of $\overrightarrow{SP'}$.

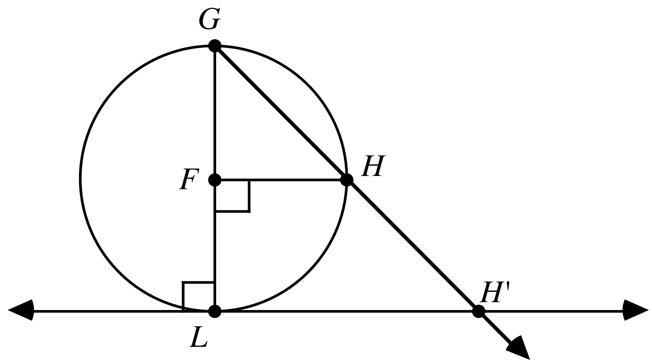
Assignment

2.1 In Exploration **1**, you found that distance and collinearity are not preserved in a stereographic projection. Given this fact, do you think that a figure's perimeter would be preserved? Describe how your response affects the use of flat maps to compare boundaries of countries and continents.

2.2 Virtually every map includes a scale to help users find the distance from one point to another. Describe the dangers in using this scale to determine precise distances.

2.3 Lines of latitude on a globe are not considered lines on a sphere. Using a stereographic projection like the ones in the explorations, what do the images of lines of latitude look like?

2.4 An angle whose vertex is a point on a circle and whose sides contain chords of a circle is an **inscribed angle**. In the diagram below, for example, $\angle LGH$ is an inscribed angle.

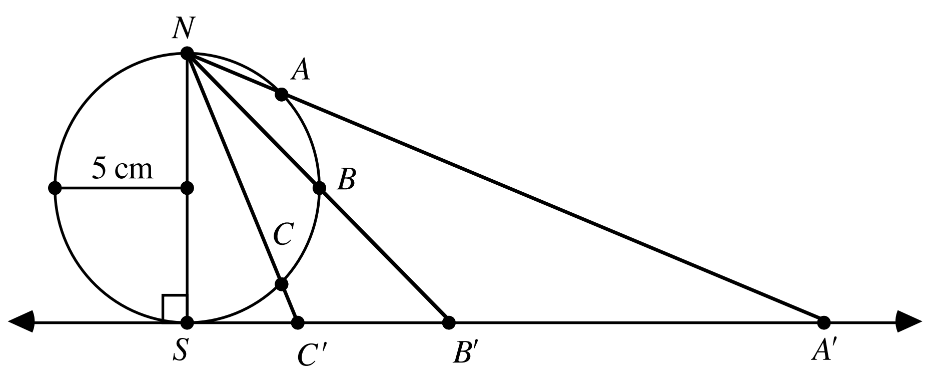


a. Given that circle F in the diagram above has a radius of 3 cm, determine the length of $\overline{LH'}$.

b. Determine the relationship between the measure of $\angle LGH$ and its intercepted arc HL.

c. The relationship you found in Part **b** is true for any inscribed angle and its intercepted arc. Use the diagram above to help prove why this is so.

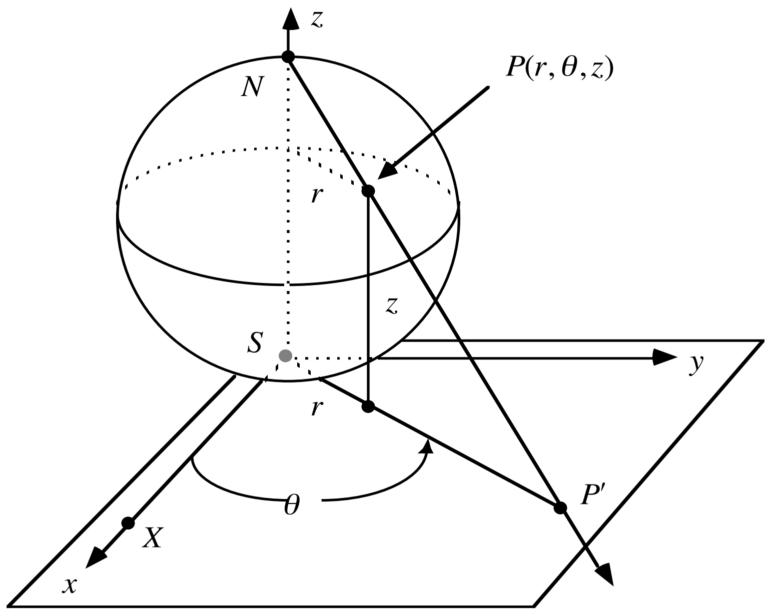
2.5 The following diagram shows a cross section of a sphere like the one in Exploration **1**, where A, B, and C divide the length of $\overset{\frown}{NS}$ into four equal parts. Use this diagram to find the lengths of $\overline{SA'}$, $\overline{SB'}$, and $\overline{SC'}$. Hint: Use a technique similar to that described in Exploration **2**.



2.6 **a.** On your sheet of paper from Exploration **1**, mark a polar axis with its endpoint at point S'.

b. Using this polar axis, find the polar coordinates of points A', B', C', D', E', and F'.

2.7 In the following diagram, S is the pole and $\overrightarrow{SX}$ is the polar axis of a polar coordinate system. Suppose point P represents a city on the globe with coordinates (r, θ, z) and P' is the stereographic projection of P. The diameter d of the sphere equals the length of $\overline{NS}$.



a. Describe how to find the polar coordinates of P' on the plane.

b. Determine the polar coordinates of P' in terms of d, r, z, and θ.

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2.8 Consider a stereographic projection in which a globe's north pole is the center of projection and the plane contains the equator. In this case, is the image of a line of longitude also a line on the plane? Explain your response.

2.9 Points A, B, and C lie on a sphere of radius 10 cm. The pole of a cylindrical coordinate system is located at the south pole of the sphere. The approximate cylindrical coordinates of the three points are: A (8.66,1.13,15.00), B (10.00,1.13,10.00), and C (8.66,1.13,5.00).

a. Describe the arrangement and position of A, B, and C on the sphere.

b. Determine the distance along the line of longitude from A to B and from B to C.

c. Find the polar coordinates of the images of A, B, and C under a stereographic projection in which the center of projection is at the north pole and the plane is tangent to the sphere at the south pole. Assume that the same pole and polar axis are used for both the cylindrical and the polar coordinate systems.

d. Describe the arrangement and position of the images A', B', and C' on the plane.

e. Find the distance from A' to B' and from B' to C'. Is distance preserved under this mapping?

2.10 Points D, E, and F lie on a sphere of radius 10 cm. When the pole of a cylindrical coordinate system is located at the south pole of the sphere, their approximate coordinates are: D (9.06,0.69,14.22), E (9.06,2.95,14.22), and F (9.06,5.25,14.22).

a. Describe the arrangement and position of D, E, and F on the sphere.

b. Find the distance along the line of latitude from D to E and from E to F.

c. Find the polar coordinates of the images of D, E, and F under a stereographic projection in which the center of projection is at the north pole and the plane is tangent to the sphere at the south pole. Assume that the same pole and polar axis are used for both the cylindrical and the polar coordinate systems.

d. Describe the arrangement and position of the images D', E', and F' on the plane.

e. Find the distance from D' to E' and from E' to F'. Is distance preserved under this mapping?

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Activity 3

Over the history of map making, satellite imagery and supercomputers have replaced captain's logs and quill pens as the tools of choice. However, many centuries-old techniques are still useful and relevant to modern cartography.

For example, one common type of flat map is based on the work of the 16th-century Flemish cartographer Gerardus Mercator. A Mercator projection maps the surface of a sphere to a cylinder tangent to the sphere along a great circle (often the equator).

Mathematics Note

A **cylindrical projection** is a projection of the points of the sphere onto a tangent right circular cylinder. The image of a point on the sphere is the intersection of a ray and the cylinder. The ray contains the center of the sphere, designated as the **center of projection**, and the point being projected. In any cylindrical projection of a sphere, there are points that have no images on the cylinder.

For example, Figure **10** illustrates the cylindrical projection of point C on a sphere. Note that $\overrightarrow{OC}$ intersects the cylinder in at most one point.

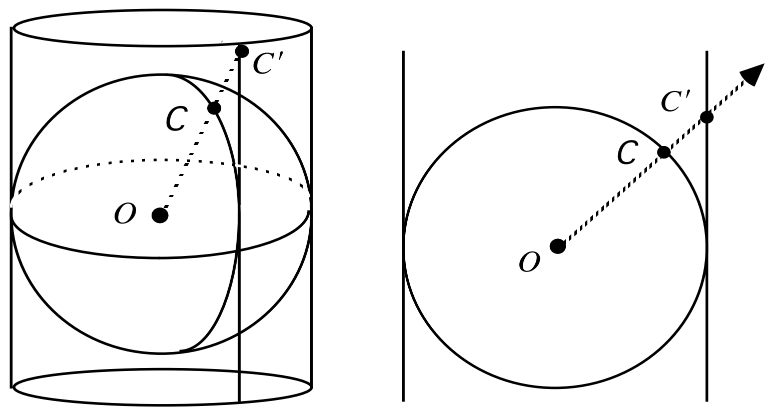


Figure 10: A cylindrical projection and its cross-sectional view

Exploration 1

In this exploration, you investigate cylindrical projections of lines of longitude and of the equator. To simplify locating the sphere's center, you work with a hemisphere, as shown in Figure **11**.

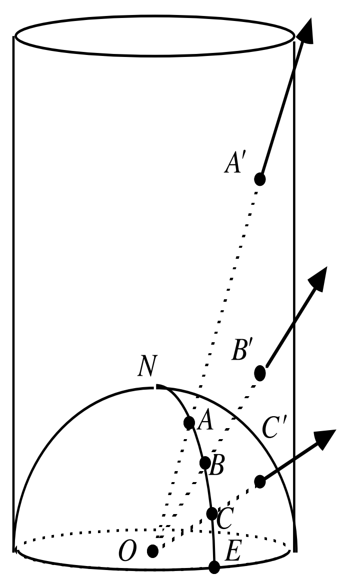


Figure 11: Cylindrical projections of points on a hemisphere

a. Cut a sphere in half along a great circle. **Note:** Save the other half of the sphere for use in the assignment.

b. Label the center of the flat face of the hemisphere *O*. The outer edge of the hemisphere represents the equator.

c. Label the north pole *N*. Label a point on the equator *E*.

d. Label three equally spaced points—*A*, *B*, and *C*—along the line of longitude from point *N* to point *E*, as shown in Figure **11**.

e. Wrap a sheet of paper around the hemisphere to make a right circular cylinder that fits over the hemisphere tangent to the equator. Place a mark on the bottom edge of the cylinder at point *E* and label it E'. Keep points *E* and E' aligned throughout the exploration.

f. Pass a skewer from *O* through point *C* on the hemisphere to model the corresponding ray. Mark and label the intersection of the skewer with the cylinder on the cylinder's outside surface.

g. Repeat Part **f** for points *A* and *B*.

h. Repeat the mapping process for at least two other lines of longitude and for the equator.

i. Open the cylinder and lay it flat with the outside surface facing up. This is the map produced by a cylindrical projection.

Discussion 1

a. Describe the map you produced using a cylindrical projection.

b.
1. What would happen if you used the process described in the exploration to project a point close to *N* onto the cylinder?
2. Where is the image of point *N* using this projection? Explain your response.

c. Suppose that all the points on a sphere that have an image under a cylindrical projection were mapped onto one cylinder.
1. Describe the surface that would result when the cylinder was "unrolled" to produce a flat map.
2. How would the image of a line of longitude appear on the map?

d. On a sphere, any pair of distinct lines (great circles) intersect in exactly two points. In other words, there are no parallel lines on a sphere.
1. Under a stereographic projection, can the images of great circles form parallel lines on the flat map?
2. Under a cylindrical projection, can the images of great circles form parallel lines on the flat map?

e. Is perpendicularity preserved under a cylindrical projection? Justify your response.

f. Describe a map of the lines of longitude and lines of latitude in the northern hemisphere created under a cylindrical projection.

g. Consider a cylindrical coordinate system in which the polar plane contains the equator of a sphere.

1. What points on the sphere would have negative z-coordinates?
2. Using a cylindrical projection, where would points with negative z-coordinates be projected on the flat map?

Exploration 2

In this exploration, you use a geometry utility to continue your investigation of cylindrical projections. Figure **12** shows a cross section of a sphere and a tangent right circular cylinder. Lines l and m represent parallel lines in the surface of the cylinder and $\overline{NS}$ is a diameter of the sphere.

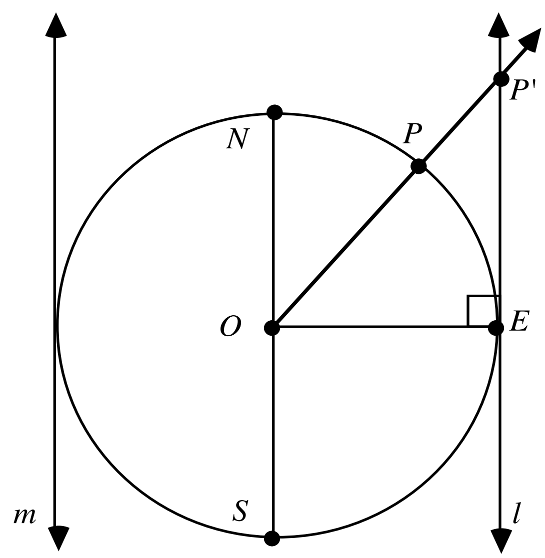


Figure 12: Cross section of cylindrical projection

a. Use a geometry utility to reproduce the diagram in Figure **12**. Anchor $\overrightarrow{OP}$ at O, allowing point P to move around the circle. Make sure that the position of P' changes as P moves around the circle.

b. Describe what happens to P' as you move P from E to N.

c. Construct points A, B, and C so that $\overset{\frown}{NA} \cong \overset{\frown}{AB} \cong \overset{\frown}{BC} \cong \overset{\frown}{CE}$.

d.
1. Move point P so it is concurrent with point C.
2. Record the length of $\overline{EP'}$.
3. Repeat Steps **1** and **2** for points A, B, E, and N.

e. Construct $\overline{PZ}$ as shown in Figure **13** below. Make sure that $\overline{PZ}$ changes length as P moves around the circle.

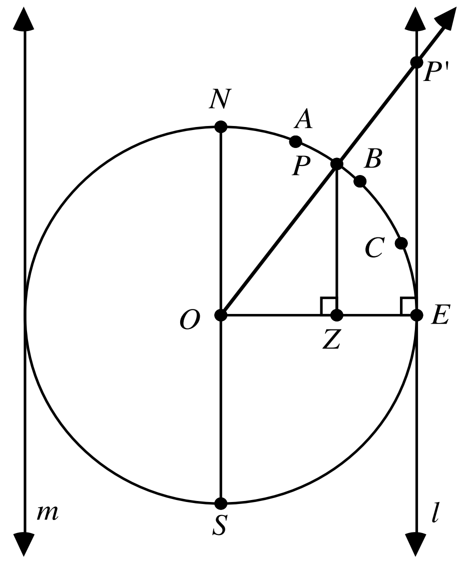


Figure 13: Cross section with $\overline{PZ}$

f. Express the length of $\overline{EP'}$ in terms of the lengths of $\overline{OZ}$, $\overline{PZ}$, and $\overline{OE}$, the radius of the sphere.

Discussion 2

a. Describe what happens to the length of $\overline{EP'}$ as P gets closer and closer to N.

b. In Figure **13**, triangles POZ and $P'OE$ are similar triangles. Explain how you know this is true.

c. A point P on a sphere can be described by cylindrical coordinates of the form (r, θ, z), with the pole located at the sphere's center. Its image on a flat map under a cylindrical projection can be described by rectangular coordinates of the form (x, y).

Suppose that the image of the intersection of the polar axis and the equator has the coordinates (0,0)

1. What is the image of the equator on the flat map?
2. Describe how the value of θ in the cylindrical coordinates of P is related to the x-coordinate of P'.
3. Describe how the distance EP' in Figure **13** can be used to find the y-coordinate of P'.

Assignment

3.1 Are distance, area, or perimeter preserved under a cylindrical projection? Explain your response.

3.2 **a.** Using a cylindrical projection with the cylinder tangent to the equator, would the image of Venezuela be more or less distorted than the image of Greenland? Explain your response.

b. Which would produce a greater distortion of Greenland—a stereographic projection through the south pole or a cylindrical projection? Explain your response.

3.3 Lines of longitude and latitude are important aids for navigation. When navigators use flat maps, what type of projection would you expect them to prefer—stereographic or cylindrical? Justify your response.

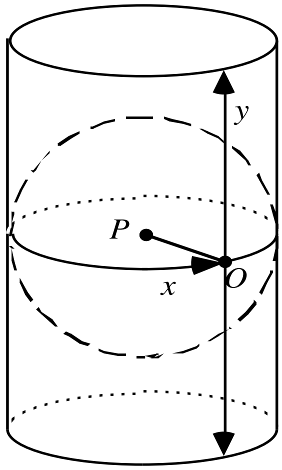
3.4 Consider a figure on a sphere whose image, under a cylindrical projection, is a rectangle. One of the sides of this figure lies along the equator. Describe a possible shape for the preimage.

3.5 Consider a sphere with a paper cylinder wrapped around it, tangent to the equator. Any point on the sphere can be represented by cylindrical coordinates of the form (r, θ, z), with the pole located at the sphere's center.

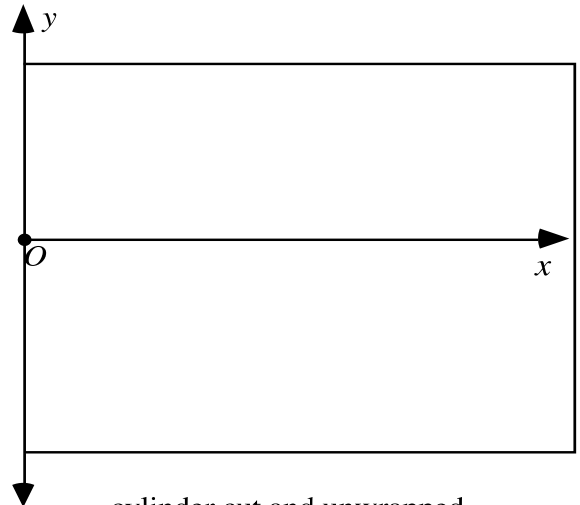
When points on the sphere are projected onto the cylinder, and the cylinder is cut and unwrapped, a flat map is produced. The position of each point on the map can be described by rectangular coordinates of the form (x, y).

As shown in the following diagram on the left, a line drawn on the outside of the cylinder along the equator can represent the positive x-axis. The origin O can be located at the point where the polar axis ($\overrightarrow{PO}$) of the cylindrical coordinate system intersects the equator.

When the paper cylinder is cut through the origin perpendicular to the x-axis, then unwrapped and laid flat with its outside surface facing up, it resembles the diagram on the right.



cylinder wrapped
around sphere



cylinder cut and unwrapped

a. If the sphere's radius is 10 cm, what is the length of the portion of the x-axis on the unwrapped cylinder?

b. Point A on the sphere has cylindrical coordinates $(10.00, \pi/6, 0.00)$. What is the x-coordinate of A', the image of A under a cylindrical projection?

c. Point B on the sphere has coordinates $(2.59, 5.40, -9.66)$. What is the x-coordinate of B', the image of B under a cylindrical projection?

d. Let P represent any point on a sphere with radius w. If P has cylindrical coordinates (r, θ, z), find the x-coordinate of P', the image of P under a cylindrical projection.

e. Describe how to determine the value of the y-coordinate for the image point in a cylindrical projection.

3.6 **a.** Point A lies on a sphere of radius 20 cm. Using a cylindrical coordinate system with the pole at the sphere's center, the coordinates of A are $(7.32, \pi/2, 10)$.

Using a cylindrical projection as described in Problem **3.5**, what are the rectangular coordinates of A'?

b. Point B lies on a sphere of radius 20 cm. Using a cylindrical coordinate system with the pole at the sphere's center, the coordinates of B are $(1.74, 4.96, -19.92)$.

1. In what region is B located on the sphere?
2. Using a cylindrical projection as described in Problem **3.5**, what are the rectangular coordinates of B'?

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3.7 Points A, B, and C lie on a sphere of radius 25 cm. Using a cylindrical coordinate system with the pole at the sphere's center, their coordinates are: A (13.62,5.43,20.97), B (25.00,5.43,0.00), and C (13.62,5.43,–20.97).

a. Describe the arrangement and position of A, B, and C on the sphere.

b. Find the distances along the line of longitude from A to B and from B to C.

c. Using a cylindrical projection as described in Problem **3.5**, find the rectangular coordinates of A', B', and C'.

d. Describe the arrangement and position of A', B', and C' on the flat map.

e. Find the distance from A' to B' and from B' to C'. Is distance preserved under this mapping?

3.8 Points D, E, and F lie on a sphere of radius 25 cm. Using a cylindrical coordinate system with the pole at the sphere's center, their coordinates are: D (19.15,0.25,–16.07), E (19.15,2.01,–16.07), and F (19.15,3.77,–16.07).

a. Describe the arrangement and position of D, E, and F on the sphere.

b. Find the arc lengths along the line of latitude from D to E and from E to F.

c. Using a cylindrical projection as described in Problem **3.5**, find the rectangular coordinates of D', E', and F'.

d. Describe the arrangement and position of D', E', and F' on the flat map.

e. Find the distance from D' to E' and from E' to F'. Is distance preserved under this mapping?

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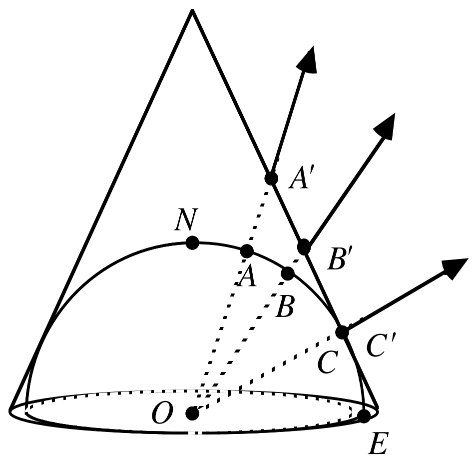
Research Project

Mapmakers use many different types of projections. Write a report that describes three projections not presented in this module. Your report should include the following information:

- how each projection is used to make a map
- how lines of latitude and longitude appear on the resulting map
- the geometric properties that are preserved in each projection
- the types of situations in which each projection is most useful
- the historical background of each projection.

Summary Assessment

Maps made using a conic projection are similar to those made with a cylindrical projection. In a conic projection, however, points are projected onto a tangent right circular cone, instead of a tangent right cylinder, as shown in the diagram below.



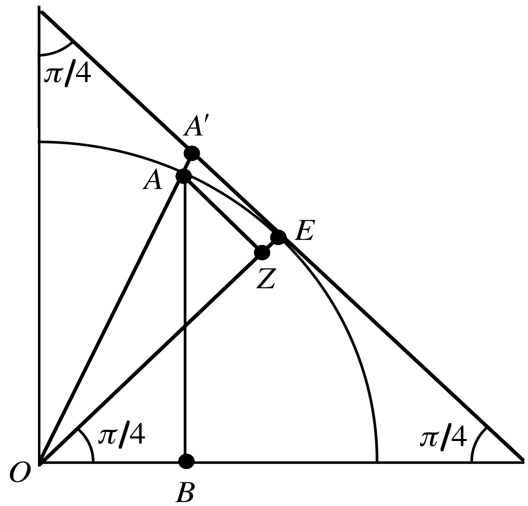
1. **a.** What figure is formed by the intersection of the sphere and the cone?

b. How would the images of lines of latitude and longitude appear in a conical mapping?

c. What geometric properties appear to be preserved under a conic projection? Justify your response.

2. The diagram below shows part of a cross section of a sphere and a tangent cone. The cone's vertex angle is the angle formed by the intersection of the cone and a plane perpendicular to the cone's base and passing through the apex.

Suppose the vertex angle of the cone measures $\pi/2 \approx 1.57$ radians and A is a point on the sphere with cylindrical coordinates $(13.62, 5.43, 20.97)$. The radius of the sphere is 25 cm.



If A' is the image of A under a conical projection, determine the ratio of the length of $\overline{A'E}$ to the length of $\overset{\frown}{AE}$.

3. Does the measure of the vertex angle of the cone affect the amount of distortion in a conic mapping? If so, how? Hint: Use a geometry utility to investigate this situation.

Module Summary

- A **polar coordinate system** describes the location of a point P in a plane using an ordered pair consisting of a **radius** r and a **polar angle** θ.

  The plane containing a polar coordinate system is the **polar plane**. The polar angle is an angle measured from a fixed ray, called the **polar axis**. The endpoint of the polar axis is the **pole**. The distance from the pole to point P, measured in the polar plane, is r.

- In a **cylindrical coordinate system**, a point in space is represented by an ordered triple of the form (r, θ, z). The values of r and θ are measurements in the polar plane. The value of z is the directed distance between the point and the polar plane (the plane containing the polar axis). A positive value for z represents a distance above the polar plane.

- A **stereographic projection** is a projection of the points on a sphere onto a plane perpendicular to a given diameter of the sphere. The plane is the **plane of projection**. The endpoints of the diameter are the **poles** of the sphere.

  The image of a point on the sphere is the point of intersection of a ray and the plane perpendicular to the diameter that contains the poles. The ray contains one of the poles, designated as the **center of projection**, and the point being projected. In any stereographic projection, there are points that have no image in the plane.

- A **cylindrical projection** is a projection of the points of the sphere onto a tangent right circular cylinder. The image of a point on the sphere is the intersection of a ray and the cylinder. The ray contains the center of the sphere, designated as the **center of projection**, and the point being projected. In any cylindrical projection of a sphere, there are points that have no images on the cylinder.

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