Brilliant Induction

In this module, you use the principle of mathematical induction to establish the validity of mathematical statements.

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Brilliant Induction

Introduction

When dominos are stood on end, each one slightly behind another, tipping over the first domino will cause the second one to fall. As the second domino falls, it will cause the next one to fall, and so on. Figure **1** shows this chain of events.

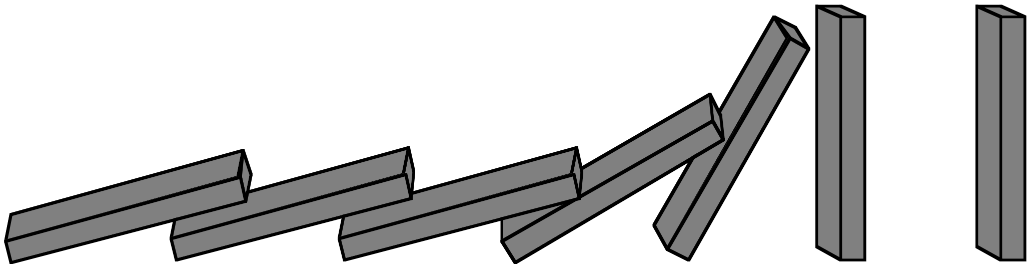

Figure 1: Falling dominos

A process in which each falling domino causes the next one to fall resembles in some ways a method of proof known as **mathematical induction**. This technique has been in use at least since the 16th century, and may have been recognized much earlier, perhaps by the Pythagoreans. In this module, you explore the conditions under which such a method of proof might work, as well as investigate situations where it does not apply.

Discussion

a. When 100 dominos are stood on end, what conditions are necessary for all the dominos to fall when the first one is knocked over?

b. The process that causes the 50th domino to fall is similar to the process that causes the 5th domino to fall. Describe the similarities.

c. How could you prove, without actually knocking the first domino over, that all the dominos will fall if the first one falls?

d. Many situations involve the successful completion of a chain of similar events. In a 400-m relay, for example, the first person must run 100 m, then successfully pass the baton to the second person. The second person also must run 100 m and successfully pass the baton to the next person, and so on, until the race ends. Describe the different ways in which a 400-m relay team might not finish a race.

e. To climb to the top of a ladder, you must start on the first rung, then advance to the second. Once on the second rung, you can advance to the third, and so on. Describe how this process is similar to the one which causes dominoes to fall.

Activity 1

In this activity, you consider the conditions necessary to prove a statement using mathematical induction.

Exploration 1

Figure **2** shows a point on a line. Disregarding the point itself, it can be thought of as separating the line into two regions R_1 and R_2: one on either side of the point.

Figure 2: A point separating a line into two regions

In the following steps, you consider the number of regions into which n distinct points separate a line.

a. Draw a picture showing the number of regions formed when a second distinct point is placed on the line. Label each region as in Figure **2**.

b. Suppose that a third distinct point is placed on the line. Does the number of regions formed depend on the location of the point?

c. Repeat the process described in Parts **a** and **b** for three more points. Record the results in a table with headings like those in Table **1** below.

Table 1: Number of regions into which *n* distinct points separate a line

Number of Distinct Points (n)	Number of Regions Added with Each Additional Point	Total Number of Regions
1		2
2		
3		
4		
5		
6		

Discussion 1

a. Judging from your results in Exploration **1**, what happens to the number of regions formed when an additional point is placed on the line?

b. Does it matter where each additional point is placed on the line, as long as each one is distinct from any previous points?

c. Do you think that your response to Part **b** is true regardless of the number of points already placed on the line?

d. If you knew that your conjecture in Part **a** was true for all points up to and including some *k*th point, how would you argue that the conjecture was true when a $(k + 1)$st point is placed on the line?

Exploration 2

Figure **3** shows three rectangles constructed with toothpicks. Each rectangle has dimensions $1 \times n$, where *n* is 1, 2, or 3 toothpicks. The total number of toothpicks required to build each rectangle with dimensions $1 \times n$ can be described by the following formula: $a_n = 2n + 2$ for $n = 1, 2, 3$.

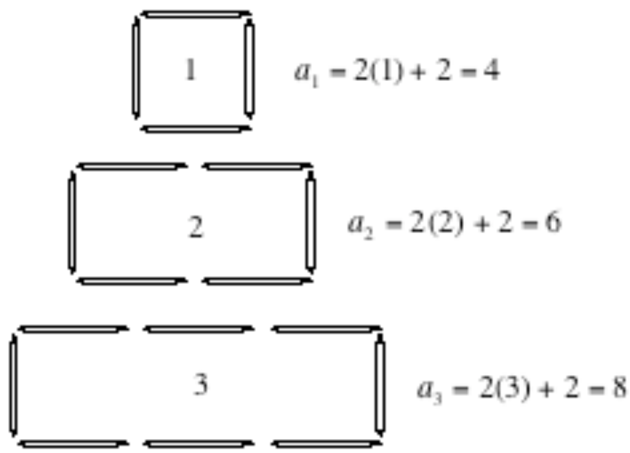

Figure 3: Rectangles constructed with toothpicks

Suppose that the pattern shown in Figure **3** is continued for all natural numbers. Does the formula still work for $n = \{1, 2, 3, \ldots\}$?

a. To argue that the formula $a_n = 2n + 2$ is correct for all natural numbers *n*, you must start by examining the first rectangle of dimensions $1 \times n$. When $n = 1$, the rectangle requires four toothpicks. Therefore, $a_1 = 4$.

   1. What happens to the first rectangle in order to create the second rectangle?

   2. How is a_2 related to a_1?

b. Explain how the process you described in Part **a** can be used to create a third rectangle given the second rectangle, and determine a_3 given a_2.

c. Suppose that the process you described in Parts **a** and **b** continues to work for all natural numbers up to *k*. Do you think that it can then be used to determine a_{k+1} given a_k? Explain your response.

d. The number of toothpicks required to build each rectangle in Figure **3** describes a sequence: $a_1,\ a_2,\ a_3$, where $a_1 = 4$, $a_2 = 6$, and $a_3 = 8$. The corresponding series S_3 represents the total number of toothpicks required to build three rectangles.

Determine $S_1,\ S_2,\ S_3,\ S_4$, and S_5.

e. Using the techniques you learned in the Level 6 module, "The Sequence Makes a Difference," verify that one possible formula for S_n is $S_n = n(n+3)$.

f. Note that $S_1 = a_1$ and $a_n = 2n+2$. Use this fact to verify that the formula suggested in Part **e** is true for S_1.

g. Since $S_2 = S_1 + a_2$ and $a_n = 2n+2$, the algebraic process below demonstrates that the formula $S_n = n(n+3)$ is true for $n = 2$, given that it is true for $n = 1$.

$$\begin{aligned} S_2 &= S_1 + a_2 \\ &= 1(1+3) + 2(2) + 2 \\ &= 4 + 6 \\ &= 10 \\ &= 2(2+3) \end{aligned}$$

1. Use the same process to verify that the formula $S_n = n(n+3)$ is true for $n = 3$, given that it is true for $n = 2$.
2. Verify that the formula is true for S_4, given that it is true for S_3.

h.

1. Assuming that the formula is true for S_{100}, the total number of toothpicks required to build the first 100 rectangles, verify that it also is true for S_{101}.
2. Assuming that the formula is true for S_{752}, verify that it also is true for S_{753}.

Discussion 2

a. Is it possible to prove that the formulas in Exploration **2** are true by checking them for every possible value of n?

b. Describe how you could verify that the formula $S_n = n(n+3)$ is true for S_{k+1} given that it is true for S_k. Hint: The process is the same as the one used in verifying that the formula is true for S_2 given that it true for S_1.

Assignment

1.1 At a very young age, the mathematician Karl Gauss (1777–1855) devised a method for adding consecutive natural numbers $1+2+3+\cdots+n$. This method has many applications.

a. By examining a pattern or using the methods developed in "The Sequence Makes a Difference," suggest a formula for the sum of the first n natural numbers.

b. **1.** Assume that the formula is true for $n = 100$. Use this assumption to verify that the formula also is true for $n = 101$.

2. Assuming that the formula is true for $n = 101$, show that it also is true for $n = 102$.

3. Assuming that the formula is true for $n = 102$, show that it also is true for $n = 103$.

c. Describe how your work in Part **b** is similar to the following statement: "If the 100th domino falls, so will the 101st, the 102nd, and the 103rd."

1.2 **a.** A student has suggested the following formula for the sum of the first n natural numbers:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}+1$$

Show that if this conjecture is true for some natural number k, then it also is true for the next natural number $k + 1$.

b. Show that the conjecture is in fact false when $n = 1$.

c. Your responses to Parts **a** and **b** are comparable to showing that if the first domino in a row of dominoes is knocked down, then all the others will fall, when in fact, the first one cannot be knocked down.

Use a truth table to illustrate how a false hypothesis leads to a true conditional statement.

d. Is there any way to prove that the conjecture in Part **a** is true? Explain your response.

1.3 Consider the following inequality: $2^{(n+1)} < 3^n$, where n is a natural number. This relationship is not true for $n = 1$ but is true for $n = 2$.

a. Graph the sequences $t_n = 2^{(n+1)}$ and $t_n = 3^n$ on the same coordinate system for $n = \{2, 3, 4, 5\}$.

b. Does the inequality appear to be true for $n \geq 2$?

c. Assuming that the inequality is true for $n = 577$, explain how the following steps verify that it also is true for $n = 578$.

$$\begin{aligned} 2^{(578+1)} &= 2^{(577+1)} \bullet 2^1 \\ &< 3^{577} \bullet 2^1 \\ &< 3^{577} \bullet 3^1 = 3^{578} \end{aligned}$$

d. Make a conjecture about the set of natural numbers for which the inequality is true.

1.4 **a.** Determine an explicit formula for finding the sum of the first n positive even integers: $2 + 4 + 6 + \cdots + 2n$.

b. Assume that the formula is true for $n = 50$. Write the equation that is implied by this assumption.

c. Show that if the assumption from Part **b** is true, then the formula also is true for $n = 51$.

1.5 Consider a meeting room containing the members of a civic group. Each person shakes hands with every other person in the room. When 2 people are in the room, 1 handshake occurs. When 3 people are in the room, 2 handshakes occur.

Use the process described in Problem **1.1** to suggest a formula for the number of handshakes that occur when n people are in the room.

1.6 Consider the following conjecture: "The quantity $3^n + 1$ is divisible by 2 for all natural numbers n."

a. Assuming that the conjecture is true for $n = 10,003$, show that it also is true for 10,004 using the steps described below.

1. The expression $3^{10,004} + 1$ can be rewritten as follows:

$$\begin{aligned} 3^{10,004} + 1 &= 3^{10,003+1} + 1 \\ &= 3^{10,003} \bullet 3^1 + 1 \end{aligned}$$

Using the fact that $3^1 = 2 + 1$, rewrite the above expression.

2. Argue that the result is divisible by 2, given that the conjecture is true for $n = 10,003$.

b. Does your work in Part **a** alone guarantee that the conjecture is always true? Explain your response.

1.7 Each of the following mathematical statements is false. To prove that each is false, identify a counterexample for each one.

a. $1 \bullet 2 \bullet 3 \bullet \cdots \bullet n = n^n - 2^{n-1}$ for all natural numbers n

b. $5^n \geq n^5$ for all natural numbers n

c. 8 is a factor of $12^n - 8^n$ for all natural numbers n

d. $2n+1$ is prime for all natural numbers n

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1.8 A sequence can be defined by the recursive formula below:

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \; n > 1 \end{cases}$$

a. Determine the first five terms of the sequence.

b. Write an explicit formula for the sequence.

c. Assume that the explicit formula is true for $n = 35$. Use this assumption to show that the formula also is true for $n = 36$.

1.9 Consider the inequality $(n-3)^2 \leq 3n$, where n is a natural number.

a. Assume that this inequality is true for $n = 7$. Use this assumption to show that it also is true for $n = 8$.

b. Graph the sequences $t_n = (n-3)^2$ and $t_n = 3n$ on the same coordinate system.

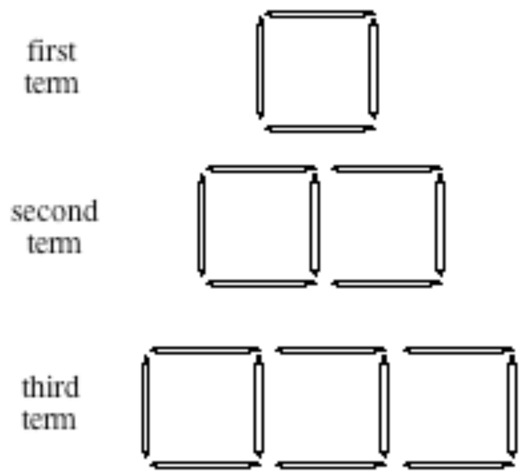
c. Make a conjecture about the set of natural numbers for which the inequality is true.

1.10 Consider the conjecture: "The quantity $2^n - 2$ is divisible by n whenever n is a positive odd integer."

a. Find a value of n that supports this conjecture.

b. Does the evidence you provided in Part **a** constitute a proof? Explain your response.

1.11 The following diagram shows a sequence of three figures constructed with toothpicks.



a. Describe the process required to create successive terms in this sequence.

b. Develop a formula for the number of toothpicks needed to construct a figure with n congruent squares.

c. Assuming that the formula is true for $n = 50$, show that it also is true for $n = 51$.

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Activity 2

As you observed in Activity **1**, many conjectures can be verified for a finite number of cases. However, this does not necessarily prove that a conjecture is true for all cases. In this activity, you use what you have learned to investigate a proof by mathematical induction.

Discussion 1

a. Consider an endless row of dominos standing on end, each one slightly behind another. Describe how this arrangement ensures that if the first domino in the row is tipped over, then:

1. the millionth domino will fall

2. the rest of the dominoes will continue to fall as well.

b. Describe the results in Part **a** if the fifth domino in the row is tipped over instead of the first.

c. Consider a non-empty subset T of the natural numbers with the following property: for any natural number that is in T, the next consecutive natural number also is in T.

Explain how the "domino effect" described in Part **a** guarantees that T contains all natural numbers greater than the least natural number in T.

Mathematics Note

The principle of **mathematical induction** can be described as follows:

Suppose that for any natural number n, $P(n)$ is a mathematical statement involving n. If,

- $P(1)$ is true, and
- whenever k is a natural number such that $P(k)$ is true, $P(k+1)$ is also true

then $P(n)$ is true for all natural numbers n**.**

For example, consider the following conjecture: "The square of each natural number n is the sum of the first n odd numbers." Figure **4** shows a geometric representation of this conjecture for $n = \{1,\ 2,\ 3,\ 4\}$.

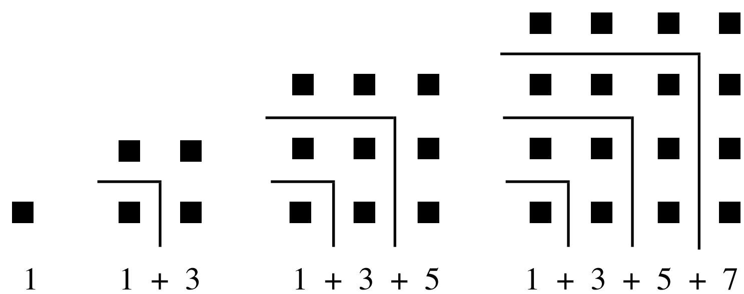


Figure 4: Geometric depiction of square numbers

The numbers of dots in the terms of this sequence are 1, 4, 9, and 16, respectively, or $1^2,\ 2^2,\ 3^2$, and 4^2. Since the sum of the first n odd numbers can be represented as the series $S_n = 1 + 3 + 5 + \cdots + (2n-1)$, the conjecture can be expressed as follows: $1 + 3 + 5 + \cdots + (2n-1) = n^2$**.**

This conjecture can be proven true for all natural numbers n using mathematical induction, as described below.

- Show $P(1)$ is true:

$$S_1 = 2(1) - 1 = 1 = 1^2$$

- Showing that $P(1)$ implies that $P(2)$ is true may suggest a method for proving that $P(k)$ implies that $P(k+1)$ is true. In this case, $P(1)$ can be used to prove that $P(2)$ is true as follows:

$$\begin{aligned} S_2 &= S_1 + (2 \bullet 2 - 1) \\ &= 1 + (2 \bullet 2 - 1) \\ &= 2 \bullet 2 + 1 - 1 \\ &= 2 \bullet 2 \\ &= 2^2 \end{aligned}$$

- Let k be a natural number such that whenever $P(k)$ is true,

$$1 + 3 + 5 + \cdots + (2k - 1) = S_k = k^2$$

Use this assumption to prove that $P(k+1)$ is true.

$$\begin{aligned} S_{k+1} &= S_k + \big(2(k+1) - 1\big) \\ &= k^2 + \big(2(k+1) - 1\big) \\ &= k^2 - 1 + 2k + 2 \\ &= (k-1)(k+1) + 2(k+1) \\ &= (k - 1 + 2)(k+1) \\ &= (k+1)^2 \end{aligned}$$

Since it has been shown that $P(1)$ is true, and that if $P(k)$ is true, then $P(k+1)$ also is true, $P(n)$ is true for all natural numbers n.

d. **1.** Consider a non-empty subset T of the integers with the same property described in Part **c** of this discussion: for any integer that is in T, the next consecutive integer also is in T.

Could the principle of mathematical induction be used to show that a set contains all integers greater than the least integer in the set? Explain your response.

2. Could it be used to show that a set contains all real numbers greater than the least in the set? Explain your response.

e. Consider the following conjecture: "The inequality $2^{n+1} < 3^n$ is true for all natural numbers n." Could the principle of mathematical induction be used to prove this conjecture? Justify your response.

f. Describe how you could prove the following conjecture using a process similar to mathematical induction: "The inequality $2^{n+1} < 3^n$ is true for all natural numbers greater than 1."

Exploration

In Exploration **2** of Activity **1**, you examined the series $S_n = 4 + 6 + 8 + \cdots + (2n + 2)$ and suggested a possible formula for it. In the following exploration, you use mathematical induction to prove that $S_n = n(n + 3)$ for all natural numbers n.

a. In Activity **1**, you showed that the following conjecture is true for $n = 1$ and $n = 2$, as well as for some other natural numbers.

$$S_n = 4 + 6 + 8 + \cdots + (2n + 2) = n(n + 3)$$

Assume that this conjecture is true for any natural number k. Write the equation that is implied by this assumption.

b. Use the equation you wrote in Part **a** to show that if k is a natural number and $P(k)$ is true, then $P(k + 1)$ also is true.

Hint: Begin by adding the next term of the sequence, $\left(2(k + 1) + 2\right)$, to both sides of the equation. Then manipulate the right-hand side of the equation until it is equal to $(k + 1)\left((k + 1) + 3\right)$.

Discussion 2

a. How does manipulating the right-hand side of the equation in Part **b** of the exploration until it is equivalent to $(k + 1)\left((k + 1) + 3\right)$ verify that $P(k + 1)$ is true?

b. In Activity **1**, you verified that $P(1)$ is true. You also showed that if $P(1)$ is true, then $P(2)$ is true. Do these verifications, along with the steps in the exploration, constitute a proof that the conjecture is true for all natural numbers?

c. Consider the false conjecture: "The inequality $(n + 1)! > 2^{n+3}$ is true for all natural numbers n." How could this conjecture be disproved?

d. Describe the steps needed for a proof by mathematical induction.

e. How do the requirements for proof by mathematical induction guarantee that a conjecture is true for all natural numbers?

Assignment

2.1 Consider the following conjecture: $2+4+6+\cdots+2n=n(n+1)$, for all natural numbers n. Complete the following steps to prove, by mathematical induction, that this conjecture is true.

a. Show that $P(1)$ is true.

b. Show that your response to Part **a** implies that $P(2)$ also is true.

c. Suppose the conjecture is true for a natural number k. Write the equation that is implied if $P(k)$ is true.

d. Write the equation which is implied if $P(k+1)$ is true.

e. Prove that if $P(k)$ is true, then $P(k+1)$ also is true. Hint: Manipulate the right-hand side of the equation from Part **d**.

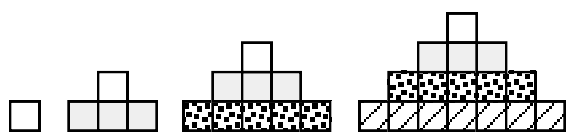
2.2 In Problem **1.6**, you examined the conjecture, "The quantity 3^n+1 is divisible by 2 for all natural numbers n." This can be restated as follows: "For all natural numbers n, $3^n+1=2p$, where p is some integer."

a. Show that $P(1)$ is true.

b. Show that your response to Part **a** implies that $P(2)$ also is true.

c. Continue using the principle of mathematical induction to prove that the conjecture is true for all natural numbers.

2.3 The diagram below shows the first four terms of a sequence generated by combining unit squares into triangular patterns.



a. Make a conjecture about an explicit formula for S_n, the number of unit squares in the nth term of the sequence.

b. Use mathematical induction to prove that your conjecture is true for all natural numbers n.

2.4 Consider the conjecture below for all natural numbers n:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Explain what is wrong with the following proof of this conjecture.

- As shown below, $P(1)$ is true:

$$S_1 = 1 = \frac{1(1+1)}{2}$$

- Given that $P(1)$ is true, $P(2)$ also is true:

$$\begin{aligned} S_2 &= S_1 + 2 \\ &= \frac{1(1+1)}{2} + 2 \\ &= 3 \\ &= \frac{2(2+1)}{2} \end{aligned}$$

- Assuming that $P(k+1)$ is true, it can be shown that $P(k)$ is true:

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)\big((n+1)+1\big)}{2}$$

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1) + 2(n+1)}{2}$$

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- Therefore, by the principle of mathematical induction, the conjecture is true for all natural numbers.

2.5. Use mathematical induction to prove that the following conjecture is true for all natural numbers n.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

2.6 Use mathematical induction to prove that the following conjecture is true for all natural numbers n: “3 is a factor of $n^3 + 5n + 6$.”

2.7 To prove a conjecture using mathematical induction, you must first prove that the statement $P(1)$ is true. However, some conjectures may be true only for a subset of the natural numbers (for example, $n \geq 2$).

In such cases, it may be possible to prove the conjecture for a particular subset of natural numbers using a form of induction in which the first statement is not $P(1)$. After showing that the conjecture is true for some initial natural number, the conjecture is proven true for the next natural number. From there, you can generalize and prove that if $P(k)$ is true, then $P(k+1)$ also is true.

a. Consider the following conjecture: $n! > 2^n$. This conjecture is not true for $P(1)$, since $1! \ngtr 2!$.

1. Find the first value of n for which the conjecture is true by graphing the sequences $t_n = n!$ and $t_n = 2^n$ on the same coordinate system for $n \geq 1$.
2. Show that the conjecture is true for the value of n you identified in Step **1**. This is the first step of the induction process.
3. Does the conjecture appear to be true for all values of n greater than the number you identified in Step **1**?

b. The second step of the induction process is to show that $P(5)$ is true, given that $P(4)$, or $4! > 2^4$, is true. This can be done as follows:

$$(4+1)! = 5! = 5 \bullet 4! > 2 \bullet 4! > 2^1 \bullet 2^4 = 2^5$$

So, $5! > 2^5$ is true.

Use the same method to show that if $P(k)$ is true, then $P(k+1)$ also is true. This is the final step of the induction process.

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2.8 Use mathematical induction to prove that the following conjecture is true for all natural numbers n:

$$7 + 11 + 15 + \cdots + (4n+3) = 2n^2 + 5n$$

2.9 Consider the conjecture: $2^{(n+1)} < 3^n$. Complete the following steps to prove that the conjecture is true for all natural numbers greater than 1.

a. Show that the inequality is true for $n = 2$.

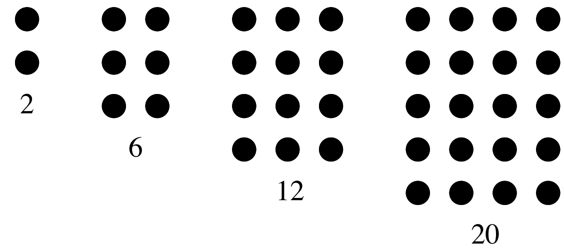
b. Use the fact that the inequality is true for $n = 2$ to show that it is also true for $n = 3$.

c. Write the inequality implied by the assumption that $P(k)$ is true.

d. Write the inequality that is implied if $P(k+1)$ is true.

e. Use the inequality from Part **c** to show that if $P(k)$ is true, then $P(k+1)$ also is true.

2.10 The diagram below shows a geometric model of a sequence. In each rectangular array of dots, the length is always 1 greater than the width.



a. Determine an explicit formula for a_n, the number of dots in each array.

b. Use mathematical induction to prove that your formula is true for all natural numbers n.

2.11 **a.** A diagonal of a polygon connects two non-adjacent vertices. As the number of sides of a polygon increases, the number of diagonals also increases. To explore the patterns created by this situation, complete the following table.

Term No. (n)	**No. of Sides in Polygon**	**No. of Additional Diagonals**	**Total No. of Diagonals (a_n)**
1	3		0
2	4		
3	5		
4	6		
5	7		

b. Determine a recursive formula for a_n, the total number of diagonals.

c. Either prove or disprove the conjecture that an explicit formula for the total number of diagonals is as follows:

$$a_n = \frac{(n+2)(n-1)}{2}$$

2.12 Consider the following argument and determine what is wrong with the proof.

Prove that $4n+3$ is divisible by 4 for all natural numbers n.

Assume the conjecture is true for some natural number k. This means $4k+3 = 4p$ for some integer p. So,

$$4(k+1)+3 = 4k+4+3 = 4k+3+4 = 4p+4 = 4(p+1)$$

Since $4(p+1)$ is divisible by 4, $4(k+1)+3$ is also divisible by 4. Therefore, $4n+3$ is divisible by 4 for all natural numbers n.

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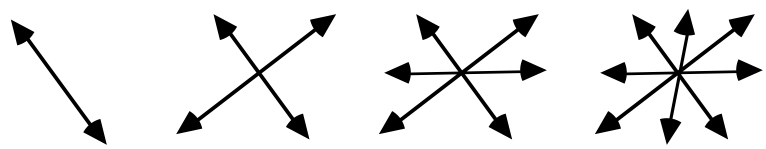
Summary Assessment

1. **Concurrent lines** are two or more lines that intersect at a common point. Two angles are **supplementary** if the sum of their measures is 180°.

   a. Given *n* concurrent lines, how many pairs of supplementary angles are formed, if none of the angles are right angles? To identify a pattern, examine the diagram below and complete the following table.



No. of Lines (n)	**Additional Pairs of Supplementary Angles**	**Total No. of Pairs of Supplementary Angles (a_n)**
1		0
2	4	4
3		
4		

   b. Describe the recursive pattern in the number of pairs of supplementary angles formed.

   c. Use the pattern described in Part **b** to find the number of pairs of supplementary angles for five concurrent lines.

   d. Write a recursive formula for a_n.

2. Prove the conjecture that the explicit formula for the number of pairs of supplementary angles for *n* concurrent lines is $a_n = 2n(n-1)$.

Module Summary

- The principle of **mathematical induction** can be described as follows:

  Suppose that for any natural number n, $P(n)$ is a mathematical statement involving n. If,

  - $P(1)$ is true, and
  - whenever k is a natural number such that $P(k)$ is true, $P(k+1)$ is also true

  then $P(n)$ is true for all natural numbers n**.**

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