Cards and Binos and Reels, Oh My!

The turn of a card or the spin of a reel can add suspense to a game. In this module, you use your knowledge of randomness, probability, and combinations to explore how these games challenge players.

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Introduction

High Tech Games designs and markets video gaming machines. As shown in Figure **1**, the video screen for Cards of Chance displays 16 cards face down. These cards include four from each suit, placed in a random order.

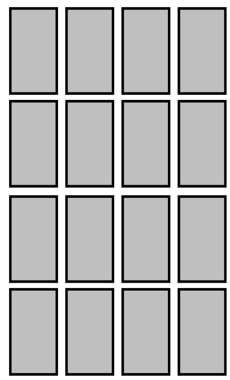


Figure 1: Arrangement of cards for Cards of Chance

Cards of Chance offers the possibility of winning at three different levels involving two, three, and four cards, respectively.

To play Cards of Chance, a player inserts tokens in the machine. When the opening screen appears, the player selects one card and turns it face up. The player then selects any one of the remaining cards and turns it face up. (This constitutes play at the two-card level.)

A player wins credits at each level if all cards turned face up are the same color. If the combination of cards does not meet this condition, the game is over. More credits may be earned at the third level than the second, and more at the fourth level than at the third.

Successful players at the two-card level may collect the credits they have earned or draw a third card, beginning play at the three-card level. Successful players at the three-card level may collect their credits or draw a fourth card. The game ends at the four-card level. Winning players may then collect their earned credits. If a player loses at any level, then all credits earned to that point are lost. Figure **2** shows some possible card combinations for three sample games:

	two-card level	three-card level	four-card level
Game 1 win at all levels	♦ ♥	♦ ♥ ♦	♦ ♥ ♦ ♥
Game 2 win at all levels	♦ ♦	♦ ♦ ♦	♦ ♦ ♦ ♦
Game 3 win at two-card level, loss at three-card level	♠ ♠	♠ ♠ ♦	play stopped with loss at three-card level

Figure 2: Card combinations in three sample games

Exploration

In this exploration, you investigate the probabilities of winning Cards of Chance at the two-card level.

a. Select 16 cards, 4 from each suit, from a deck of ordinary playing cards. To simulate the game, shuffle the 16 cards and draw 2 randomly.

b. Play 10 games of Cards of Chance, stopping each game at the two-card level and shuffling the cards between games. Record your wins and losses.

c. Determine the experimental probability of winning a game of Cards of Chance at the two-card level.

d. Compile the class results. Use the class data to determine the experimental probability of winning at the two-card level. **Note:** Save your data for use in Activity **1**.

e. List all the possible outcomes for Cards of Chance at the two-card level.

f. Determine the theoretical probability of each possible outcome.

g. Recall that for two events A and B, the theoretical probability of either A or B occurring can be found as follows:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

Given this fact, determine the theoretical probability of winning Cards of Chance at the two-card level.

Discussion

a. Describe how you determined the theoretical probabilities in Part **f** of the exploration.

b. What is the theoretical probability of winning at the two-card level? Explain your response.

Mathematics Note

According to the **law of large numbers**, as the number of trials increases, the experimental probability of an event tends to approach its theoretical probability.

For example, when tossing a fair coin, the theoretical probability that the coin lands heads up is 0.5. As the number of tosses increases, the experimental probability that the coin lands heads up will tend to get closer to 0.5.

c. How should increasing the number of times you play the game affect the experimental probability of winning?

Activity 1

Video gaming machines are designed to be durable, not portable. They are usually too big and too heavy for easy transportation. Imagine that you are a sales representative for High Tech Games. To make a sales pitch to potential buyers, you need a quick, creative simulation of the game that can be presented easily.

In this activity, you develop your own simulation of Cards of Chance and determine theoretical probabilities for the three-card level.

Exploration

a. Develop a simulation of Cards of Chance that could be used in a presentation to prospective buyers. The model must demonstrate the rules of the game, provide representative sample results, and be easy to transport.

b. Use your simulation to play 10 games at the three-card level. Use the results to calculate the experimental probability of winning at the three-card level.

c.
1. Compile the class results. Use this data to determine the experimental probability for winning at the three-card level.
2. Compare this probability with the experimental probability for winning at the two-card level obtained in the introduction.

Mathematics Note

Conditional probability is the probability of an event occurring, given that an initial event, or **condition**, has already occurred. The probability of event B occurring, given that event A has already occurred, is denoted $P(\mathrm{B}|\mathrm{A})$.

In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A:

$$P(\mathrm{A\ and\ B}) = P(\mathrm{A}) \bullet P(\mathrm{B}|\mathrm{A})$$

For example, consider drawing two playing cards from a standard deck, one at a time, without replacement, and observing their colors. The cards may be either red (R) or black (B). The tree diagram in Figure **3** shows the probabilities in this situation.

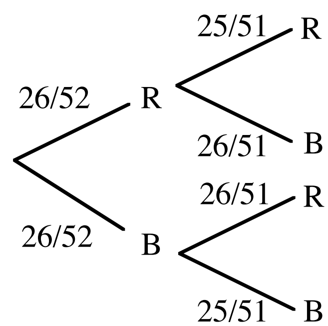


Figure 3: Tree diagram for drawing two cards without replacement

In this case, the conditional probability of obtaining a red card on the second draw, given that the first card is black, is $P(\mathrm{R}|\mathrm{B}) = 26/51$. The conditional probability of obtaining a red card on the second draw, given that the first card is also red, is $P(\mathrm{R}|\mathrm{R}) = 25/51$.

The probability of obtaining two red cards is:

$$P(\mathrm{RR}) = P(\mathrm{R}) \bullet P(\mathrm{R}|\mathrm{R}) = \frac{26}{52} \bullet \frac{25}{51} = \frac{25}{102}$$

d. Create a tree diagram showing each possible outcome and its probability for the three-card level of Cards of Chance. **Note:** Save your work for use in the assignment.

Mathematics Note

An experiment is **random** if individual outcomes are chance events.

A **random variable** *X* is a variable that takes on each of its possible values with a specific probability. Given possible values for *X* of $x_1, x_2, \ldots, x_k$, each has its corresponding probability $p_1, p_2, \ldots, p_k$. The sum of these probabilities is 1.

For example, rolling an ordinary six-sided die is a random experiment because the outcome of the roll is uncertain. If the outcomes of the experiment are assigned to the random variable *X*, then the possible values for *X* are 1, 2, 3, 4, 5, or 6. The probability of each outcome is $1/6$.

A **probability distribution** for a random variable *X* assigns probabilities $p_1, p_2, \ldots, p_k$ to the values $x_1, x_2, \ldots, x_k$ for *X*.

Table **1** shows the probability distribution of the random variable *X* when rolling a six-sided die.

Table 1: Probability distribution of random variable *X*

Value of X (x_i)	1	2	3	4	5	6
Probability (p_i)	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

e. Make a table that shows the probability distribution for all possible outcomes of Cards of Chance at the three-card level.

f. Calculate the theoretical probability of winning at the three-card level.

Discussion

a. Is the theoretical probability of winning Cards of Chance at the two-card level the same as winning at the three-card level? Explain your response.

b. What is the sum of all the probabilities in a probability distribution? What does this sum represent?

c. In Cards of Chance, the first draw and the second draw are not independent events. How does this affect the probabilities of drawing two cards of the same color?

d. How could you change the rules for Cards of Chance so that the game involves independent events?

e. What is the probability of each of the following events in the three-card level of Cards of Chance?

1. drawing a black card, given that the first two cards are black
2. drawing a black card, given that the first two cards are red
3. drawing a black card, given that the first two cards are different colors

Assignment

1.1 **a.** What is the theoretical probability of drawing three red cards at the three-card level of Cards of Chance?

b. Describe how you could use your tree diagram from the exploration to determine the probability in Part **a**.

1.2 **a.** Extend the tree diagram from the exploration to the four-card level of Cards of Chance.

b. Make a table that shows the probability distribution for all possible outcomes at the four-card level.

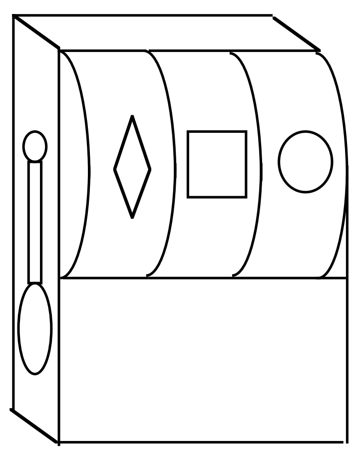
1.3 **a.** Determine the probability of drawing a fourth card of the same color, given that three cards of the same color were drawn at the three-card level.

b. Determine the probability of losing at the four-card level, given that three cards of the same color were drawn at the three-card level.

c. Determine the probability of winning at the four-card level, with no previous conditions given. Justify your response.

d. Determine the probability of losing at the four-card level, with no previous conditions given. Justify your response.

1.4 The Reel Game is another product of High Tech Games. As shown below, each machine has three spinning reels. Each reel has three equally likely symbols—a diamond, a square, and a circle. A player wins by matching the symbols on all three reels when the reels stop.



a. Draw a tree diagram that shows all the possible outcomes of the Reel Game. Label each branch with the appropriate probability.

b. Find the probability of getting a diamond on the second reel.

c. Find the probability of getting a diamond on the third reel.

d. Are getting a diamond on the second reel and a diamond on the third reel independent events? Explain your response.

e. Determine the probability of winning the Reel Game described with three diamonds, given that two diamonds have already appeared on the first two reels.

f. How does $P(3 \text{ diamonds} | 2 \text{ diamonds})$ compare to the probability of getting a diamond on the third reel?

1.5 **a.** Consider a game that involves drawing two cards from a standard deck, one at a time with replacement. Is obtaining an ace on the first draw independent of obtaining an ace on the second draw?

b. For two independent events A and B, $P(\text{B}|\text{A}) = P(\text{B})$. How does this fact support your response to Part **a**?

c. If each card is not replaced after it is drawn, does the game still involve independent events? Explain your response.

1.6 Given rules similar to those of the Reel Game in Problem **1.4**, determine the probability of winning each of the following:

a. a game with two spinning reels, each having two symbols, a diamond and a triangle

b. a game with three spinning reels, each having three symbols, a square, a diamond, and a circle.

1.7 Consider a game that involves rolling two standard dice. If the sum of the faces is greater than 10, the player earns 10 points. If the sum equals 7, the player earns 5 points. Any other roll of the dice is worth 0 points. Show the probability distribution for the random variable S, where S represents the number of points won.

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1.8 A history test contains 5 multiple-choice questions. Each question has 5 possible responses, only 1 of which is correct. To pass this test, students must answer at least 4 of the questions correctly.

If a student selects responses at random, what is the probability of each of the following events?

a. All 5 questions are answered correctly.

b. None of the questions are answered correctly.

c. Exactly 4 of the questions are answered correctly.

d. The student passes the test.

e. The student does not pass the test.

1.9 Consider a game in which 26 cards, each labeled with a different letter of the alphabet, are placed in a container and mixed thoroughly. If cards are drawn without replacement, what is the probability of each of the following?

a. When three cards are drawn, all three cards are vowels.

b. When five cards are drawn, all five cards are consonants.

c. Given that two vowels have been drawn, the next card is a vowel.

d. Given that three vowels and two consonants have been drawn, the next card is a vowel.

e. When seven cards are drawn, they include the letters of the word *fortune*.

1.10 Louis has four $1 bills, three $5 bills, and two $10 bills in his pocket. If he randomly draws two bills from his pocket, one at a time without replacement, what is the probability that the total is $15?

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Activity 2

To expand its share of the market, High Tech Games is developing another kind of gaming machine, the binostat. As shown in Figure **4**, a binostat game involves dropping a ball through a triangular grid. At each level of the grid, the ball deflects to the right or left of a peg with equal probability, until it enters a numbered slot. For example, the ball in Figure **4** passed through the grid and landed in slot 2.

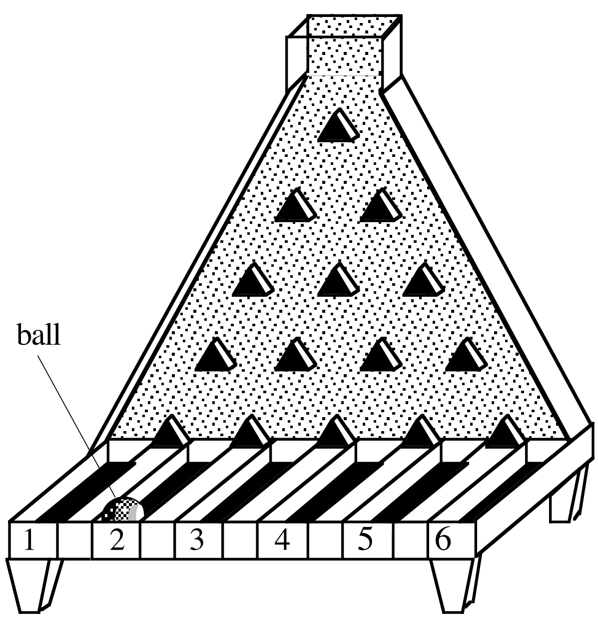

Figure 4: A binostat game

To play a binostat game, a player inserts tokens in the machine and selects a slot number. The player then drops the ball into the top of the machine and watches as it falls through the grid. If the ball lands in the selected slot, the player wins the game and earns credits. If the ball lands in any other slot, the game is over and the player loses the tokens or credits played.

Exploration

Figure **5** shows a simple binostat with only one level. In this binostat, a ball falling through the grid deflects either to the right or to the left. If it deflects to the left, it ends up in slot 1. If it deflects to the right, it ends up in slot 2.

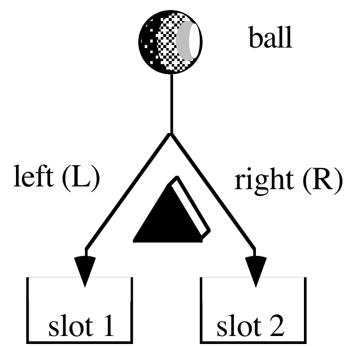

Figure 5: A one-level binostat

a. In the one-level binostat game shown in Figure **5**, a ball falling through the grid has only two possible paths: right (R) or left (L).

For a ball to reach slot 1 in the two-level binostat game shown in Figure **6**, the ball must deflect left and then left again (LL). To reach slot 2, a ball can deflect left and then right (LR), or it can deflect right and then left (RL). To reach slot 3, the ball must deflect right and then right again (RR).

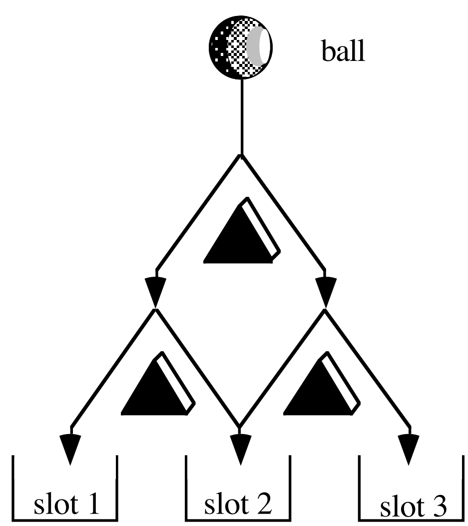

Figure 6: A two-level binostat

In other words, there are a total of four ways—LL, LR, RL, and RR—to reach the three slots. Two of these paths lead to slot 2.

List all the paths in a binostat game with each of the following:

1. three levels

2. four levels.

b. At each junction in the binostat, the probability of going left is equal to the probability of going right. Determine the probability of the ball landing in each slot in a one-level binostat. Express each probability as a fraction in which the denominator equals the number of possible paths.

c. Repeat Part **b** for two-level, three-level, and four-level binostat games.

d. Obtain a template of the binostat paths shown in Figure **7**.

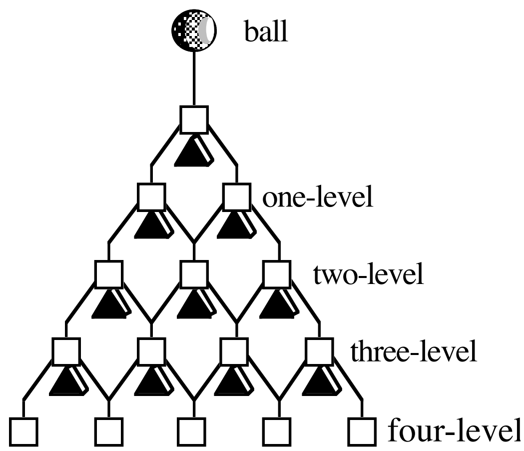

Figure 7: Number of paths for binostat games

1. Fill in each square with the number of paths that can be taken to get to that particular slot.

2. Describe any patterns you observe in this figure.

e. In a one-level binostat, there are a total of 2 possible paths. In a two-level binostat, there are a total of 4 possible paths.

1. Use your results from Part **d** to express the total number of paths as a power of 2 for one-level, two-level, three-level, and four-level binostats.

2. Describe any relationship you observe between the total number of paths and the corresponding level of the binostat.

f. Use the patterns you observed in Parts **d** and **e** to extend the template through the 10th level.

Mathematics Note

The diagram you developed in the exploration is part of a pattern of numbers known as **Pascal's triangle**. Figure **8** shows the first five rows of Pascal's triangle.

				1					Row 0
			1		1				Row 1
		1		2		1			Row 2
	1		3		3		1		Row 3
1		4		6		4		1	Row 4

Figure 8: A portion of Pascal's triangle

Discussion

a. Describe how you could generate any row of Pascal's triangle, given the previous row.

b. Explain how you could use your diagram to find the number of paths a ball can take to land in a particular slot of a binostat game.

c. Explain how you could use your diagram to find the total number of paths in a binostat game of any level.

d. In Part **e** of the exploration, the total number of paths in a given row was expressed as a power of 2. Explain why 2 was chosen as the base.

e. Explain how you could use your diagram to find the probability of a ball landing in slot x of an n-level binostat.

f. Why is a 1-level binostat like tossing a fair coin?

Assignment

2.1 Consider a three-level binostat game.

a. How many slots does this game have?

b. How many different paths are possible for the ball?

c. **1.** How many paths can the ball take to land in slot 2?

2. What is the probability of the ball landing in slot 2?

2.2 **a.** What is the sum of the theoretical probabilities of a ball landing in the slots of:

1. a one-level binostat?
2. a two-level binostat?
3. a three-level binostat?

b. Explain your responses to Part **a**.

2.3 **a.** How could you use exponential notation to express the theoretical probability of a ball landing in slot 1 in a one-, two-, or three-level binostat game?

b. What is the theoretical probability of a ball landing in slot 1 of a 10-level binostat?

2.4 **a.** Complete the probability distribution table below for each slot in a three-level binostat game.

Slot	1	2	3	4
Probability				

b. Can you use Pascal's triangle to create a probability distribution for each slot in an n-level binostat game? If so, how?

2.5 Create a probability distribution table for a six-level binostat.

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2.6 An industrial plant has 8 robots for use on its assembly line. Six of the robots must be functioning for the plant to be operational.

a. If "I" represents an inoperable robot and "O" represents an operable robot, then one possible situation can be written as [OIOOIOOI]. Find the total number of situations that can occur.

b. Which row of Pascal's triangle corresponds with your response to Part **a**?

c. Use Pascal's triangle to find the number of outcomes in which exactly 6 out of 8 robots are operational.

d. Use Pascal's triangle to find the number of outcomes in which at least 6 out of 8 robots are operational.

2.7 Consider an experiment in which 10 supermarket customers are selected at random and asked to taste 2 different brands of tomato soup. Each person must state a preference for one brand or the other. The company sponsoring the taste test would like to claim that 80% of consumers prefer their brand over their competitor's brand.

a. Find the number of possible outcomes for the experiment. Which row of Pascal's triangle corresponds with this total?

b. Use Pascal's triangle to find the number of ways that exactly 8 of the 10 customers could choose a specific brand.

c. Use Pascal's triangle to find the number of ways that at least 8 of 10 customers could choose a specific brand.

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Activity 3

Although Pascal's triangle can be used to calculate the probabilities of winning and the numbers of possible paths in binostat games, High Tech Games wants a mathematical model to determine the probabilities for its machines.

Mathematics Note

A **binomial experiment** has the following characteristics:

- It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
- The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
- Each trial has only two possible outcomes: a success or a failure. (The prefix *bi-* means "two.")
- The probability of a success remains constant from trial to trial.
- The total number of successes is observed.

For example, consider an experiment that consists of tossing a fair coin 3 times and observing the number of heads that occur. In this case, there is a fixed number of trials, 3. For each trial, there are only two possible outcomes: either heads or tails. The probability that heads occurs remains constant for each toss, and the result of one toss does not influence the result of any other. Therefore, this represents a binomial experiment.

Discussion 1

a. Explain whether or not each of the following represents a binomial experiment.

1. A fair coin is flipped six times. The total number of heads is recorded.
2. Two cards are drawn from a standard deck of playing cards, one at a time with replacement. The total number of diamonds is recorded.
3. Two cards are drawn from a standard deck of playing cards, one at a time without replacement. The total number of diamonds is recorded.

b. Given $P(\text{success})$ for any trial in a binomial experiment, describe how to find $P(\text{failure})$.

Exploration

Consider a ball falling through the four-level binostat in Figure **9**. Before reaching a slot, the ball makes a total of four deflections. At each peg, the ball has an equal probability of deflecting to the right or to the left.

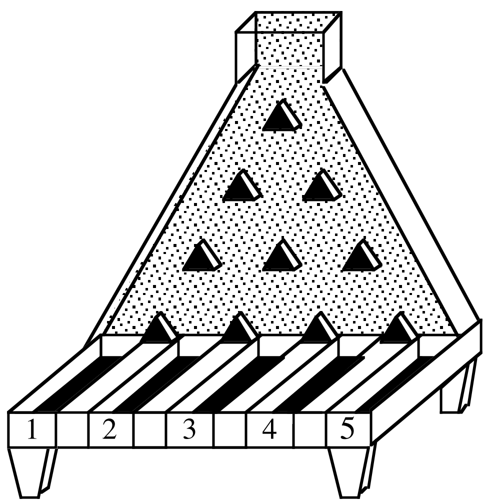

Figure 9: Four-level binostat game

Using Pascal's triangle, you can determine that there is one possible path to slot 1, four possible paths to slot 2, six possible paths to slot 3, four possible paths to slot 4, and one possible path to slot 5.

a. To land in slot 2, the ball can make only one deflection to the right out of a total of four deflections. The four possibilities are RLLL, LRLL, LLRL, and LLLR.

Recall that an arrangement of *r* items out of *n* items, where order is not important, is a **combination**. Use the notation $C(n,r)$, where *n* is the total number of deflections and *r* is the number of deflections to the right, to describe the number of paths the ball can take to land in slot 2.

b. List the possible paths the ball can take to land in each of the other slots of a four-level binostat. What patterns exist between the number of deflections to the left or right for each path to a given slot?

c. Use combination notation to describe the number of paths possible for each of the following slots in a four-level binostat.

1. slot 1

2. slot 3

3. slot 4

4. slot 5

d. As noted previously, there are four possible paths to slot 2. Each path has a total of four deflections, with only one deflection to the right: RLLL, LRLL, LLRL, and LLLR.

Since the probability of a deflection to the left or to the right is $1/2$, the probability that the ball will take path RLLL can be found as follows:

$$\begin{aligned} P(\text{RLLL}) &= P(\text{R}) \bullet P(\text{L}) \bullet P(\text{L}) \bullet P(\text{L}) \\ &= [P(\text{R})]^1 \bullet [P(\text{L})]^3 \\ &= \left(\frac{1}{2}\right)^1 \bullet \left(\frac{1}{2}\right)^3 = \frac{1}{16} \end{aligned}$$

Similarly, the probability for each of the other three paths to slot 2 is also $[P(\text{R})]^1 \bullet [P(\text{L})]^3$, or $1/16$. Because there are 4 possible paths, each with a probability of $[P(\text{R})]^1 \bullet [P(\text{L})]^3$, the probability of the ball landing in slot 2 can be expressed as follows:

$$\begin{aligned} P(\text{slot 2}) &= 4 \bullet [P(\text{R})]^1 \bullet [P(\text{L})]^3 \\ &= 4 \bullet \left(\frac{1}{2}\right)^1 \bullet \left(\frac{1}{2}\right)^3 \\ &= \frac{4}{16} = \frac{1}{4} \end{aligned}$$

Using the process described for slot 2, determine the probabilities for slots 1, 3, 4, and 5.

e. Use combinations and exponents to write a formula for the probability of a ball ending in slots 1, 2, 3, 4, and 5.

Discussion 2

a. How can you use Pascal's triangle to determine the number of possible paths that include r deflections to the right in an n-level binostat?

b. How can you use the notation $C(n,r)$ to represent any element of Pascal's triangle?

Mathematics Note

A **binomial distribution** is the probability distribution associated with repeated trials of a binomial experiment.

The probability of obtaining r successes in n trials can be determined using the following formula, where p is the probability of success in any one trial:

$$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

For example, consider a binomial experiment that involves rolling a fair die. In this experiment, rolling a six is designated a success while any other roll is designated a failure. In 7 trials, the probability of getting 3 sixes is:

$$P(3 \text{ successes}) = C(7,3) \bullet \left(\frac{1}{6}\right)^3 \bullet \left(\frac{5}{6}\right)^4 = 35 \bullet \frac{625}{279{,}936} \approx 8\%$$

c. Using the process described in Part **e** of the exploration, how can you use combinations to express the probability of a ball landing in a slot that requires each of the following?

1. r deflections to the right in an n-level binostat
2. r deflections to the left in an n-level binostat

d. Can you use the formula for a binomial distribution to calculate the probability of getting at least 3 twos with 5 rolls of a fair die? If so, how?

Assignment

3.1 Which of the following are binomial experiments? Justify your responses.

a. A die is tossed three times. After each toss, the number that appears on the top face is recorded.

b. Two playing cards are drawn from a deck of 16 cards, one at a time, without replacement. After each card is drawn, its color is recorded.

c. A coin is flipped until it lands heads up.

3.2 Use the formula for a binomial distribution to calculate the probabilities of a ball landing in each of the slots of a five-level binostat game. Check your answers using Pascal's triangle.

3.3 Consider a reel game with five reels. Each reel has the same five symbols—a diamond, a club, a heart, a spade, and an automobile. Each of the five symbols is equally likely to appear on each reel.

a. If you wanted to get an automobile on each reel, how would you define a "success" in this game?

b. What is the probability of a "success" on any one reel?

c. What is the probability of a "failure" on any one reel?

3.4 Consider the reel game described in Problem **3.3**.

a. What is the probability of getting a diamond on every reel?

b. What is the probability of getting a heart on exactly three of the reels?

c. What is the probability of getting a heart on at least three reels?

d. The middle reel on one machine always sticks, displaying a diamond. Calculate the probability of getting three diamonds on this machine.

3.5 Binomial expressions such as $(x + y)^3$ can be expanded using the distributive property. For example, the expanded forms of several binomial expressions are shown below:

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

a. Examine the expanded form of each of the binomial expressions above. Describe the sum of the exponents on each term in the expanded form with respect to the exponent of the original expression.

b. For each expression, describe the relationship between the coefficients of each term and Pascal's triangle.

c. Rewrite the expanded form of $(x + y)^4$ using the notation for combinations, $C(n,r)$.

d. Use your results in Parts **a–c** to suggest an expanded form for the general binomial $(x + y)^n$.

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3.6 Experimental data shows that when a thumbtack is tossed in the air, it will land point up 75% of the time, and point down 25% of the time.

a. Does tossing a thumbtack in the air and observing the outcome represent a binomial experiment?

b. What is the probability that when a tack is tossed 10 times, it lands point down exactly 6 times?

3.7 A married couple plans to have four children. Assuming that the probability that each child is a girl is 0.5, what is the probability that their four children will include:

a. 3 boys and 1 girl?

b. 4 girls?

c. 2 boys and 2 girls?

3.8 Consider a test which contains 10 multiple-choice questions. Each question offers 4 possible choices for the answer, but only one of them is correct. To pass this test, students must obtain at least 6 correct answers. If a student guesses the answer to each question at random, what is the probability of passing the test?

3.9 A certain airplane has two engines. The probability that any one engine will fail during a transcontinental flight is 0.001. Assuming that the event of one engine failing is independent of the other engine failing, determine the probability of each of the following.

a. a transcontinental flight will be completed without engine failure

b. both engines will fail

c. at least one engine will fail.

3.10 On January 28, 1986, the space shuttle *Challenger* exploded shortly after launch. The cause of this tragedy was traced to the failure of 1 of the 6 sealed joints on the booster rockets. Assuming that each joint has a 0.977 success rate, and that the failure of any one joint is independent of the failure of any of the others, calculate the probability that at least 1 of the 6 joints fails.

3.11 According to the **binomial theorem**, the expansion of $(x+y)^n$, where n is a whole number, is the sum of $(n+1)$ terms, as follows:

$$(x+y)^n = C(n,n)x^n + C(n,n-1)x^{n-1}y + C(n,n-2)x^{n-2}y^2 + \cdots + C(n,0)y^n$$

Use the binomial theorem to expand each of the binomials below.

a. $(x-y)^6$

b. $(x^2+3y)^3$

c. $(2xy-5)^4$

* * * * * * * * * *

Activity 4

The video gaming machine industry is highly competitive. Besides designing games that keep players interested and entertained, High Tech Games must also ensure that its products generate a profit.

Whenever someone uses a gaming machine, there are costs involved. For the player, the cost is the price required to play. For the machine's owners, the costs include paying off the credits earned by successful players.

Mathematics Note

The **expected value** or **mean** of a random variable *X*, denoted $E(X)$, is the sum of the products of each possible value of *X* and its corresponding probability.

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_kp_k$$

In mathematics, a sum is often denoted using the Greek letter *sigma*, Σ. Using this notation, the expected value of *X* can be written as follows:

$$E(X) = \sum_{i=1}^{k} x_ip_i$$

This indicates that the values of x_ip_i are added as *i* increases from 1 to *k*.

For example, consider a game that consists of rolling a fair die. A roll of one is worth 1 point, a roll of two is worth 2 points, and so on. The expected value for this game can be calculated as shown below:

$$E(X) = \sum_{i=1}^{6} x_ip_i = 1 \bullet \frac{1}{6} + 2 \bullet \frac{1}{6} + 3 \bullet \frac{1}{6} + 4 \bullet \frac{1}{6} + 5 \bullet \frac{1}{6} + 6 \bullet \frac{1}{6} = \frac{21}{6} = 3.5$$

In this case, an exact value of 3.5 cannot be obtained on any single roll of the die. However, according to the law of large numbers, if the die is rolled many times, the mean of all the rolls is likely to be close to 3.5.

Exploration

In the following exploration, you investigate a five-level binostat that costs 100 tokens to play.

a. To players, the expected value for a game is its expected payoff. This can be determined by multiplying the payoff for each outcome by its theoretical probability, then finding the sum of these products.

The payoffs (in tokens) for each slot in the five-level binostat are shown in Table **2** below. From the exploration in Activity **2**, the probabilities of the ball landing in slot 1, 2, 3, 4, 5, or 6 are 1/32, 5/32, 5/16, 5/16, 5/32, or 1/32, respectively. Determine the expected payoff for this game.

Table 2: Payoffs for five-level binostat game

Slot	**1**	**2**	**3**	**4**	**5**	**6**
Payoff	200	20	10	10	20	200

b. In a **fair game**, the expected payoff equals the cost of playing the game. The game described in Part **a** is not a fair game. Use a spreadsheet to determine a set of payoffs that make this five-level binostat game a fair one.

c. In High Tech's home market, state law requires a 70% minimum return to players. Determine a set of payoffs that satisfies the state requirements, while providing a profit for the owners of the gaming machines.

Discussion

a. For the binostat game represented in Table **2**, the payoffs for landing in slots 1 or 6 are much higher than those of the other slots. What is the relationship between the probabilities for each of the six outcomes and their corresponding payoffs?

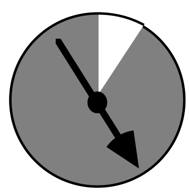
b. Is there more than one way to assign payoffs to make the game fair? Explain your response.

c. What changes did you make in your payoff scheme to turn your fair game into a legal and profitable one?

d. Describe a payoff scheme that might attract players, yet still make the game profitable for owners.

Assignment

4.1 The spinner below is used in a carnival game. If the arrow lands in the unshaded sector, the player receives 50 points. If it lands in the shaded sector, the player receives 5 points. The central angle of the unshaded sector measures 30°. Determine the expected value (in points) for this game.



4.2 In most games, outcomes with lower probabilities have higher payoffs than outcomes with higher probabilities. This can make the game attractive to players, yet still profitable.

a. Suppose that the payoff for the first slot of a six-level binostat is 457 tokens, as shown in the table below. Determine the payoffs for each remaining slot using an inverse relationship between the probability and the payoff. For example, the probability of landing in slot 2 is 6 times the probability of landing in slot 1. To determine the payoff for slot 2, multiply the payoff for slot 1 by $1/6$, or $457 \bullet 1/6 \approx 76$.

Slot	1	2	3	4	5	6	7
Probability	1/64	6/64	15/64	20/64	15/64	6/64	1/64
Payoff	457	76					

b. What should be the cost to play to make this a fair game?

4.3 Consider a six-level binostat that costs 50 tokens to play. If state law requires a minimum 70% return for players, determine a set of payoffs that would make this game both legal and profitable. As in Problem **4.2**, the payoff for each slot should be inversely related to its probability.

4.4 Suppose that Cards of Chance costs 50 tokens to play. If state law requires a minimum 70% return for players, determine a set of payoffs for the entire game—from the two-card level to the four-card level—that would make it both legal and profitable.

4.5 Consider a three-level binostat game that costs 50 tokens to play. If state law requires a minimum 70% return for players, determine a set of payoffs that would make this game both legal and profitable. The payoff for each slot should be inversely related to its probability.

4.6 As part of a market analysis for High Tech Games, you have been asked to compare Cards of Chance, reel games, and binostat games in terms of complexity, player interest, and potential profitability. Write a summary of your comparisons.

* * * * *

4.7 A three-reel game costs 50 tokens to play. Each reel has three distinct symbols, and players must match at least two symbols to earn tokens. If state law requires a minimum of 70% return for players, determine a set of payoffs that would make this game both legal and profitable. The payoff for each slot should be inversely related to its probability.

4.8 An insurance company offers an accident/illness policy with the following benefits.

- If a policyholder becomes seriously ill during the year, the company will pay $1000.
- If a policyholder has an accident, the company will pay $2000.
- If a policyholder has an accident and becomes seriously ill, the company will pay $7500.

The annual premium for this policy is $200.

a. According to the company's statistics, the probability of becoming seriously ill in any one year is 0.06, while the probability of having an accident is 0.04. Assuming that these are independent events, determine the probability of each of the following:

1. a policyholder does not become ill or have an accident
2. a policyholder becomes ill, but does not have an accident
3. a policyholder does not become ill, but does have an accident
4. a policyholder becomes ill and has an accident.

b. What is the company's expected annual profit per policyholder?

* * * * * * * * * *

Summary Assessment

High Tech Games is developing a new video gaming machine called Flip-o-Mania. To begin the game, players insert tokens in the machine. The game then electronically "flips" a coin. The player continues flipping coins electronically. For play to continue, each successive coin flipped must match the first coin. When a coin appears that does not match the others, the game is over. The player wins credits based on the number of coins flipped successfully.

Although High Tech has not yet determined how many coins players should have the option of flipping, the company plans to market a test version of Flip-o-Mania soon. As director of marketing, you must make a presentation at next week's board meeting about the new machine.

1. Design a simulation for Flip-o-Mania that you could use to demonstrate how the game works.

2. **a**. An initial study suggests that players should be offered the chance to flip at least 4 coins, but no more than 10. If each flip is considered a level in the game, determine at what level Flip-o-Mania should conclude.

b. Determine the probabilities of winning at each level.

3. If state law requires a 70% minimum return to players, determine an appropriate set of payoffs for Flip-o-Mania, given that the cost to play is 10 tokens.

4. Write a report that defends your recommendations and findings in Problems **1–3**. Your report should include a discussion of probability distribution tables, binomial experiments, binomial probabilities, and expected value.

Module Summary

- According to the **law of large numbers**, as the number of trials increases, the experimental probability of an event tends to approach its theoretical probability.
- **Conditional probability** is the probability of an event occurring, given that an initial event, or **condition**, has already occurred. The probability of event B occurring, given that event A has already occurred, is denoted $P(\mathrm{B}|\mathrm{A})$.
- In an experiment involving conditional probabilities, the probability of both A and B occurring is found by multiplying the probability of A by the conditional probability of B given A:

$$P(\text{A and B}) = P(\mathrm{A}) \bullet P(\mathrm{B}|\mathrm{A})$$

- An experiment is **random** if individual outcomes are chance events.
- A **random variable** X is a variable that takes on each of its possible values with a specific probability. Given possible values for X of $x_1, x_2, \ldots, x_k$, each has its corresponding probability $p_1, p_2, \ldots, p_k$. The sum of these probabilities is 1.
- A **probability distribution** for a random variable X assigns probabilities $p_1, p_2, \ldots, p_k$ to the values $x_1, x_2, \ldots, x_k$ for X.
- The triangular pattern of numbers shown below is called **Pascal's triangle**:

$$\begin{array}{ccccccccccl}
 & & & & 1 & & & & & & \text{Row 0} \\
 & & & 1 & & 1 & & & & & \text{Row 1} \\
 & & 1 & & 2 & & 1 & & & & \text{Row 2} \\
 & 1 & & 3 & & 3 & & 1 & & & \text{Row 3} \\
1 & & 4 & & 6 & & 4 & & 1 & & \text{Row 4} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

- A **binomial experiment** has the following characteristics:
  1. It consists of a fixed number of repetitions of the same action. Each repetition is a **trial**.
  2. The trials are independent of each other. In other words, the result of one trial does not influence the result of any other trial in the experiment.
  3. Each trial has only two possible outcomes: a success or a failure. (The prefix *bi-* means "two.")
  4. The probability of a success remains constant from trial to trial.
  5. The total number of successes is observed.
- A **binomial distribution** is the probability distribution associated with repeated trials of a binomial experiment. The probability of obtaining r successes in n trials can be determined using the following formula, where p is the probability of success in any one trial:

  $$P(r \text{ successes in } n \text{ trials}) = C(n,r) \bullet p^r \bullet (1-p)^{n-r}$$

- The **expected value** or **mean** of a random variable X, denoted $E(X)$, is the sum of the products of each possible value of X and its corresponding probability.

  $$E(X) = x_1p_1 + x_2p_2 + \cdots + x_kp_k$$

  In mathematics, a sum is often denoted using the Greek letter *sigma*, Σ. Using this notation, the expected value of X can be written as follows:

  $$E(X) = \sum_{i=1}^{k} x_ip_i$$

  This indicates that the values of x_ip_i are added as i increases from 1 to k.
- In a **fair game**, the expected payoff equals the cost of playing the game.

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