An Imaginary Journey Through the Real World

Can you find the square root of a negative number? In this module, you discover that you can!

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Introduction

You are familiar with many different sets of numbers: the natural numbers, the whole numbers, the integers, the rational numbers, and the real numbers. Each set was developed as a social—or mathematical—need arose.

Written symbols for the natural numbers 1, 2, 3, … are at least as old as the pyramids. Around 200 A.D., the number 0 was introduced in India to represent an empty column in a counting board that resembled an abacus. The set of numbers consisting of 0 and the natural numbers make up the set of whole numbers.

The need for negative numbers emerged in China in the 6th and 7th centuries, though they were not used in Europe until the 15th century. Negative numbers were useful for representing quantities above or below a given level. The natural numbers, their opposites (negatives), and zero make up the set of integers.

The ancient Greeks introduced the positive rational numbers to represent fractional parts of a quantity. The term *rational* was coined to describe numbers that are ratios of two natural numbers, where the denominator is not 0. During this time, the Greeks believed that rational numbers could be used to describe exactly all measurements in the physical world.

This hypothesis about rational numbers was incorrect. When Greek mathematicians tried to find a rational number to describe the length of the diagonal of a square like the one shown in Figure **1**, they realized that no such rational number existed.

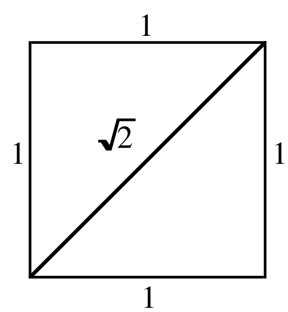

Figure 1: A square and one of its diagonals

As a result, the Greeks extended their number system to include irrational numbers. Eventually, the sets of rational and irrational numbers were combined to form the set of real numbers.

Activity 1

Is the set of real numbers sufficient to describe everything in the physical—or mathematical—world? In this module, you will investigate situations in which another set of numbers is useful.

Discussion

a. **1.** What are the solutions to the equation $x^2 - 2 = 0$?

2. How are these solutions related to the factors of the polynomial $x^2 - 2$?

b. In general, the **difference of squares** $x^2 - a^2$ has two factors: $(x - a)$ and $(x + a)$. In other words, $x^2 - a^2$ can be factored as $(x - a)(x + a)$, or $x^2 - a^2 = (x - a)(x + a)$.

Given this fact, what are the solutions to a polynomial equation of the form $x^2 - a^2 = 0$?

c. Are there any real-number solutions to the equation $x^2 + 1 = 0$? Explain your response.

Mathematics Note

The notation for the **imaginary unit *i***, where $i = \sqrt{-1}$ and $i^2 = -1$, was first introduced by Swiss mathematician Leonhard Euler (1707–1783). The adoption of i by Gauss in his classic *Disquisitiones arithmeticae* in 1801 secured its use in mathematical notation. This notation was generalized to define the square root of any negative number as: $\sqrt{-a} = \sqrt{-1} \bullet \sqrt{a} = i\sqrt{a}$ for any number $a > 0$.

For example, $\sqrt{-3} = \sqrt{-1} \bullet \sqrt{3} = i\sqrt{3}$ and $\sqrt{-9} = \sqrt{9} \bullet \sqrt{-1} = 3i$.

A **complex number** is any number in the form $a + bi$, where both a and b are real numbers. For example, $4.3 + i\sqrt{5}$ and $\pi - 2i$ are complex numbers. So are $7i$ and 11, since they may be represented as $0 + 7i$ and $11 + 0i$, respectively.

A **pure imaginary number** is a complex number $a + bi$ for which $a = 0$ and $b \neq 0$. For example, $5i$, $8i$, and $i\sqrt{5}$ are pure imaginary numbers.

A real number is a complex number $a + bi$ for which $b = 0$. For example, $5 + 0i = 5$ and $-3 + 0i = -3$ are real numbers.

The Venn diagram in Figure **2** shows the relationships among the sets of complex numbers, pure imaginary numbers, and real numbers.

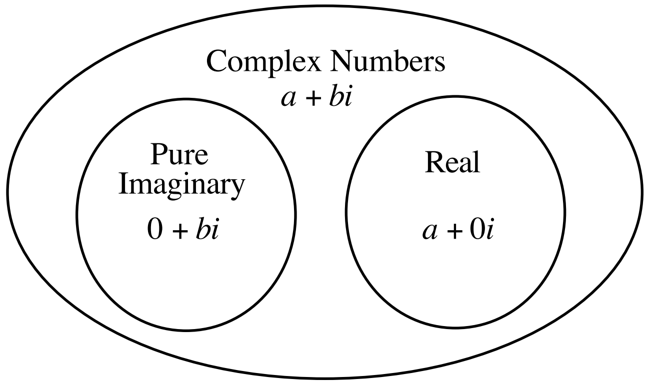

Figure 2: Venn diagram of complex numbers

In the set of complex numbers, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

d. **1.** Considering the information given in the mathematics note, along with the factors of a difference of squares, determine the factors of $x^2 + 4$.

2. What are the solutions to the equation $x^2 + 4 = 0$?

e. Describe how you could factor any expression of the form $x^2 + a^2$ and identify its zeros.

f. **1.** The Pythagorean theorem states that in a right triangle with legs of lengths a and b and hypotenuse of length c, $a^2 + b^2 = c^2$. Another way to describe this relationship is to say that the sum of the areas of two squares is the area of a third square.

Given the lengths of the sides of two of these squares, can you always find the length of the side of the third square? Explain your response.

2. In the set of real numbers, can you always find a value for b given the values of a and c in the equation $a^2 + b^2 = c^2$? Explain your response.

Assignment

1.1 Using your knowledge of the distributive property of multiplication over addition and subtraction, find the sum and the difference of each pair of complex numbers below. (Find the difference by subtracting the second complex number from the first.) Write each result in the form $a + bi$.

a. i and $9i$

b. 4 and $7 + 3i$

c. $21 - 6i$ and $15i$

d. $-13 + 4i$ and $3 - i$

e. $12 + 5i$ and $12 - 5i$

f. $a + bi$ and $c + di$

1.2 Complex numbers also can be multiplied. Use technology to multiply each of the following pairs of complex numbers:

a. i and 3

b. $2i$ and i

c. 7 and $6 - 11i$

d. $5 + i$ and $6 - 3i$

e. $a + bi$ and $c + di$

1.3 Use your results in Problem **1.2e** to show a pencil-and-paper method for multiplying complex numbers.

1.4 **Complex conjugates** are pairs of complex numbers in the form $a + bi$ and $a - bi$.

a. Create a pair of complex conjugates.

b. Find the sum and product of the numbers in Part **a**.

c. Suggest a method for finding the sum and product of complex conjugates.

1.5 In the set of real numbers, the multiplicative identity is 1. In other words, when a is a real number, $a \bullet 1 = 1 \bullet a = a$. Demonstrate that $1 + 0i$ is the multiplicative identity in the set of complex numbers.

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1.6 Show that the solutions to the equation $x^2 + 27 = 0$ are $x = 3i\sqrt{3}$ and $x = -3i\sqrt{3}$.

1.7 **a.** Describe the solutions to $x^2 + 12 = 0$ where the domain is the set of real numbers.

b. Describe the solutions to $x^2 + 12 = 0$ where the domain is the set of complex numbers.

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Activity 2

Some sets of numbers have the **closure property** under certain operations. For example, consider the set of even natural numbers. If you add any two even natural numbers, you obtain an even natural number. Therefore, this set of numbers is closed under addition.

Does the set of complex numbers have the closure property under addition? Do you think that this set is closed under multiplication? Do you think that each complex number has a multiplicative inverse? In this activity, you answer these questions and more.

Exploration

If complex numbers behave like real numbers, then the reciprocal of $3 + 4i$ can be written as $1/(3 + 4i)$. By definition, the reciprocal of $3 + 4i$ is $a + bi$ if and only if $(a + bi)(3 + 4i) = 1 + 0i$. In the following exploration, you discover how this reciprocal also can be represented in the form $a + bi$.

a. **1.** Expand the left side of the equation below by multiplying the complex numbers.

$$(a + bi)(3 + 4i) = 1 + 0i$$

2. Write the product on the left in the form $m + ni$.

b. In order for the complex number $m + ni$ found in Part **a** to equal $1 + 0i$, the real part (m) must equal 1 and the imaginary part (n) must equal 0.

1. Write each of these relationships as an equation.

2. Solve these two equations to find the values of a and b in the complex number $a + bi$ that is the reciprocal of $3 + 4i$.

c. Verify that the complex number found in Part **b** is the reciprocal of $3 + 4i$ by determining that its product with $3 + 4i$ is 1.

d. The conjugate plays an important role in writing the reciprocal of a complex number in the form $m + ni$.

Use technology to evaluate $1/(3 + 4i)$. Write the result so that m and n are reduced fractions.

e. **1.** Evaluate the following expression:

$$\left(\frac{1}{3+4i}\right)\left(\frac{3-4i}{3-4i}\right)$$

2. Compare the result to the complex number determined in Part **b** and your response to Part **d**.

3. Suggest a method for finding the reciprocal of a complex number $a + bi$ using the conjugate.

f. In the set of real numbers, division by a non-zero number can be interpreted as the product of the dividend and the reciprocal of the non-zero divisor. In other words,

$$a \div b = a \bullet \frac{1}{b}$$

where $b \neq 0$. Division among the complex numbers can be interpreted in the same way. Use this information to perform the following division:

$$\frac{7-5i}{3+4i}$$

Discussion

a. Does the set of natural numbers have the closure property under subtraction? Explain your response.

b. In the real-number system, the commutative property of addition is stated as $a + b = b + a$. How could you show that the commutative property of addition is preserved in the set of complex numbers?

c. In the real-number system, the associative property of addition is stated as $a + (b + c) = (a + b) + c$. How could you show that the associative property of addition is preserved in the set of complex numbers?

d. In the real-number system, the commutative property of multiplication is stated as $ab = ba$. How could you show that the commutative property of multiplication is preserved in the set of complex numbers?

e. In the real-number system, the associative property of multiplication is stated as $a(bc) = (ab)c$. How could you show that the associative property of multiplication is preserved in the set of complex numbers?

Assignment

2.1 Write each of the following expressions in the form $a + bi$.

- **a.** $\sqrt{-49} + \sqrt{-1} + \sqrt{-9}$
- **b.** $\sqrt{-25} - 5\sqrt{-9}$
- **c.** $(3 - 4i) + (-8 + 6i)$
- **d.** $(8 - 7i) - (2 + 6i)$

2.2
- **a.** Determine the values of $i^1, i^2, \ldots, i^{10}$.
- **b.** Describe any patterns you observe in your response to Part **a**.
- **c.** Evaluate i^{90}.
- **d.** Write a rule for evaluating i^n for any positive integer n.

2.3 Write each of the expressions below in the form $a + bi$.

- **a.** $(-9i)(22i)$
- **b.** $(4 - i)(7 + 2i)$
- **c.** $i^3(5 + 7i)(3 - 4i)$

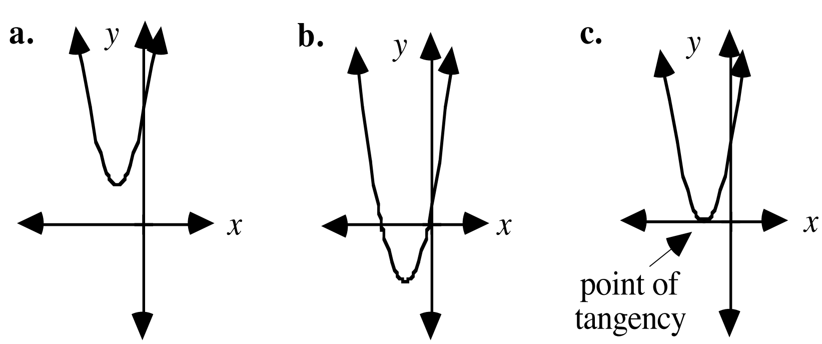
2.4 Using the method developed in the exploration, simplify each expression below to the form $a + bi$.

- **a.** $\dfrac{3}{5 - 6i}$
- **b.** $\dfrac{-8 - i}{-3 - 9i}$

2.5 Determine the roots of each equation below in the set of complex numbers and write the equation in factored form.

- **a.** $y = x^2 - 28$
- **b.** $y = x^2 + 28$

2.6 Describe the roots of each quadratic equation graphed below.



2.7 Write an equation with real coefficients for which each expression below is a solution.

a. $5i$

b. $2i\sqrt{7}$

c. $6-7i$

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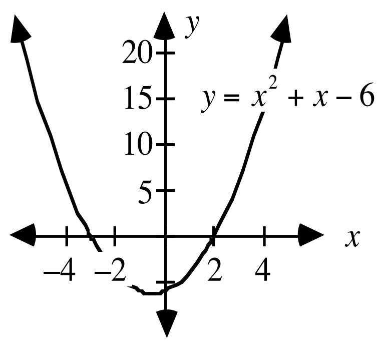
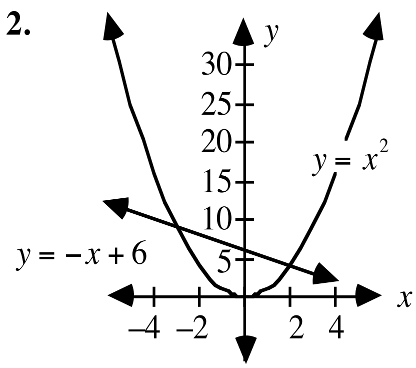
2.8 **a.** Explain why finding the zeros for $y = x^2 + x - 6$ is equivalent to solving the equation $-x + 6 = x^2$.

b. Describe how the zeros of $y = x^2 + x - 6$ are represented in each of the following graphs:

1.

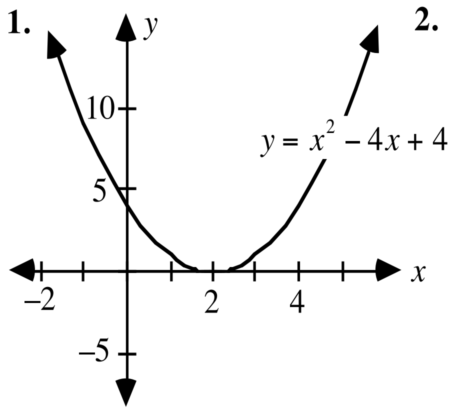


2.

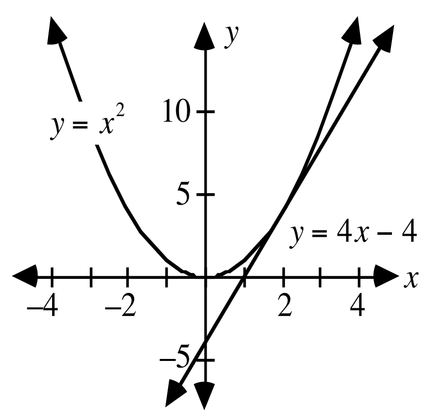


c. Describe how the zeros of $y = x^2 - 4x + 4$ are represented in each of the graphs below:

1.



2.



d. Create a pair of graphs like those shown in Parts **b** and **c** to illustrate the zeros of $y = x^2 + x + 4$.

e. Given an equation of the form $y = ax^2 + bx + c$, there are three possible cases for the roots of the equation:

1. two real roots

2. one real root

3. two non-real roots.

For each of these cases, sketch the graph of $y = ax^2 + bx + c$ on one set of axes. Sketch the corresponding graph of $y = ax^2$ and $y = -bx - c$ on a second set of axes.

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Activity 3

Many quadratic equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$, have no real-number solutions. However, when using the set of complex numbers, solving an equation such as $x^2 + 0x + 1 = 0$ results in two solutions: $x = \sqrt{-1} = i$ and $x = -\sqrt{-1} = -i$.

Exploration 1

In this exploration, you investigate the solutions to second-degree polynomial equations of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

a. **1.** Find integer values for a, b, and c so that $y = ax^2 + bx + c$ has two real-number roots, r_1 and r_2. **Note:** Recall that if $ax^2 + bx + c = 0$ has solutions r_1 and r_2, then $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

2. To check your results, substitute the values you used for a, b, and c into the general equation $y = ax^2 + bx + c$ and graph it.

b. Repeat Part **a** so that $ax^2 + bx + c = 0$ has one real root r. **Note:** If r is the only solution, then $ax^2 + bx + c = a(x - r)(x - r)$. This solution is a double root.

c. Repeat Part **a** so that $ax^2 + bx + c = 0$ has no real roots.

d. Use technology to solve for x in the general quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.

e. Substitute the values of a, b, and c chosen for each case in Parts **a–c** into the solutions found in Part **d**. Confirm that the results agree with the values for the roots in Parts **a–c**.

f. Determine the value of $b^2 - 4ac$ in each equation from Parts **a–c**.

Discussion 1

a. Describe how you could use a graph to demonstrate that a quadratic function has each of the following numbers of roots:

1. two real roots

2. one double root

3. no real roots.

b. How are the two complex-number solutions to the equation in Part **c** of Exploration **1** related?

Mathematics Note

Second-degree polynomial equations of the form $ax^2 + bx + c = 0$ with $a \neq 0$, always have two solutions when solved over the complex numbers:

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

These two solutions make up the **quadratic formula**.

When a, b, and c are real numbers and $b^2 - 4ac < 0$, the solutions are complex and occur in conjugate pairs. For example, for $x^2 + 2x + 5 = 0$, $b^2 - 4ac = -16$. Since $-16 < 0$, $x^2 + 2x + 5 = 0$ has two complex-number solutions:

$$x = \frac{-2}{2} + \frac{\sqrt{2^2 - 4 \bullet 5}}{2} = -1 + 2i \text{ and } x = \frac{-2}{2} - \frac{\sqrt{2^2 - 4 \bullet 5}}{2} = -1 - 2i$$

c. The expression $b^2 - 4ac$, which appears under the radical sign in the solution of the general quadratic equation $ax^2 + bx + c = 0$, is known as the **discriminant**.

As described in the mathematics note above, the discriminant can help you determine whether the roots of a quadratic equation are real or complex. Explain why this is true.

d. A polynomial is **reducible** over the real numbers if it can be expressed as the product of two or more polynomials of degree 1 and with real coefficients. Is every second-degree polynomial reducible over the real numbers? Explain your response.

Exploration 2

In this exploration, you investigate the solutions to polynomial equations of degrees 3 and 4 with real-number coefficients.

a. If the third-degree polynomial equation $ax^3 + bx^2 + cx + d = 0$ has roots r_1, r_2, and r_3, the equation can be expressed in the form:

$$a(x - r_1)(x - r_2)(x - r_3) = 0$$

Using this fact, find a combination of three distinct, real-number roots r_1, r_2, and r_3, such that $a(x - r_1)(x - r_2)(x - r_3)$ results in a third-degree polynomial with real coefficients.

Check your response by graphing the resulting equation.

b. Repeat Part **a** where r_1 is a real root and r_2 and r_3 are complex roots that are not real.

c. Consider fourth-degree polynomial equations of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ where a, b, c, d, and e are real numbers and $a \neq 0$.

Find a combination of four distinct, real-number roots r_1, r_2, r_3, and r_4, such that $a(x - r_1)(x - r_2)(x - r_3)(x - r_4)$ results in a fourth-degree polynomial with real coefficients.

Check your response by graphing the resulting equation.

d. Repeat Part **c** where r_1 and r_2 are distinct, real roots and r_3 and r_4 are complex roots that are not real.

e. Repeat Part **c** where r_1, r_2, r_3, and r_4 are complex roots that are not real.

Discussion 2

a.

1. How may graphs be used to check the number of real solutions to a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c, and d are real numbers and $a \neq 0$?
2. How many real-number solutions are possible for an equation of the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c, and d are real numbers and $a \neq 0$?
3. How many complex-number solutions are possible?

b.

1. How many real-number solutions are possible for an equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ where a, b, c, d and e are real numbers and $a \neq 0$?
2. How many complex-number solutions are possible?

c. Describe the relationship among the complex solutions of the form $a + bi$, where $b \neq 0$, of a polynomial equation when that equation has real-number coefficients.

Mathematics Note

The **fundamental theorem of algebra** states that every polynomial equation of degree $n \geq 1$ with complex coefficients has at least one root which is a complex number.

One consequence of the fundamental theorem of algebra is that *n*th-degree polynomial equations have exactly *n* roots in the set of complex numbers. This total may include some multiple roots. For example, the roots of the fifth-degree polynomial $x^5 - 4x^4 - 15x^3 + 50x^2 + 64x - 96 = 0$ are –3, –2, 1, and 4. One of these (4) is a double root. Therefore, the polynomial has a total of five roots in the set of complex numbers.

d. Describe the number of real solutions possible for polynomial equations of the form $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, where the coefficients are real numbers and $a \neq 0$.

e. In general, how many complex roots of the form $a + bi$, where $b \neq 0$, can an *n*th-degree polynomial equation with real coefficients have? How are these complex roots related?

f. For what type of polynomial equations must there always be at least one real root? Explain your response.

Assignment

3.1 Determine the solutions to each of the following equations over the complex numbers. Use these solutions to express each equation as the product of first-degree polynomials.

a. $9x^2 + 12x + 4 = 0$

b. $9x^2 + 35x - 4 = 0$

c. $x^2 + 4x + 9 = 0$

d. $3x^3 - 12x^2 + 12x - 48 = 0$

e. $2x^4 - 6x^3 + 12x^2 + 4x - 120 = 0$

3.2 When considering solutions to polynomial equations with real coefficients, the fact that the product of a complex number and its conjugate is a real number has special significance.

a. Find a polynomial equation in the form $ax^2 + bx + c = 0$ with real coefficients that has solutions $r_1 = 2 + i$ and $r_2 = 2 - i$.

b. Find the solutions to $x^2 - 6x + 13 = 0$. Describe the relationship between the two solutions.

c. Determine four complex-number solutions for an equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ that result in coefficients that are real numbers. Give the values of these coefficients.

3.3 If $2 + 3i$, 2, and –5 are solutions to the polynomial equation $x^4 - x^3 + cx^2 + 79x - 130 = 0$, determine the value of c. Describe how you made this determination.

3.4 Write a paragraph describing the different numbers of real solutions that are possible for sixth-degree polynomial equations of the form $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$, where the coefficients are real numbers and $a \neq 0$.

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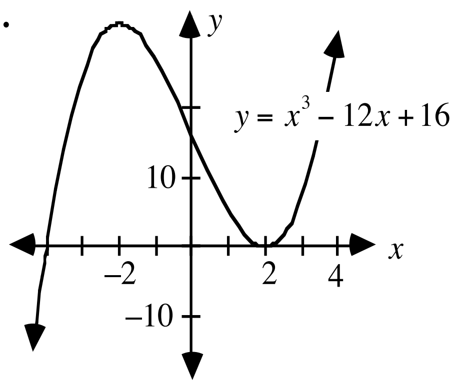
3.5 **a.** Explain why finding the zeros for $y = x^3 - 12x + 16$ is equivalent to solving the equation $x^3 = 12x - 16$.

b. Describe how the zeros of $y = x^3 - 12x + 16$ can be interpreted using each of the following graphs:

1.



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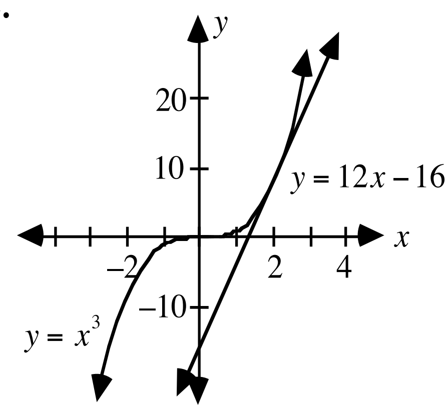

c. Changing the constant in the equations graphed in Part **b** will change the zeros of the equation. Suppose that the constant is changed from 16 to 10, resulting in the equation $y = x^3 - 12x + 10$.

1. Predict the number of times that the graph of $y = x^3$ intersects the graph of $y = 12x - 10$.
2. Predict the number of real zeros for $y = x^3 - 12x + 10$.
3. Confirm your predictions by finding the zeros for $y = x^3 - 12x + 10$.

d. In Part **b**, the graph of $y = x^3 - 12x + 16$ is tangent to the x-axis at one point, while the graph of $y = 12x - 16$ is tangent to $y = x^3$ at one point.

Use the graphs to predict how the zeros of $y = x^3 - 12x + 16$ would change if the constant 16 is increased.

e. Given an equation of the form $y = ax^3 + bx^2 + cx + d$, there are three possible cases for the roots of the equation. For each of these cases, sketch the graph of $y = ax^3 + bx^2 + cx + d$ on one set of axes. Sketch the corresponding graph of $y = ax^3 + bx^2$ and $y = -cx - d$ on a second set of axes. Then describe the types of roots represented by the graphs.

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Activity 4

The use of the word *imaginary* reflects some of the original uneasiness that mathematicians had with numbers involving $\sqrt{-a}$, where a is a positive real number. However, the phrase "imaginary numbers" seems inappropriate in today's world, where such numbers are routinely used in analyzing electrical circuits, in cartography, and in quantum mechanics.

Mathematics Note

Swiss clerk Jean Robert Argand (1768–1822) and Danish mathematician Caspar Wessel (1745–1818) were the first two people to graph complex numbers on a plane. They represented a complex number $a + bi$ as an ordered pair (a,b) where a is the real part and b is the imaginary part.

Each complex number can be graphed as a point in the complex plane. Any point on the horizontal axis is a real number and any point on the vertical axis is a pure imaginary number.

For example, Figure **3** shows the graphs of the ordered pairs (2,3), (0,3), (3,0) and (4,–3), which represent the complex numbers $2 + 3i$, $0 + 3i$, $3 + 0i$ and $4 - 3i$, respectively.

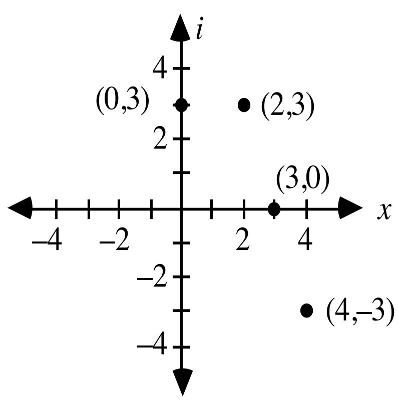


Figure 3: The complex plane

Exploration

Argand's geometric interpretation of complex numbers provides many advantages when exploring the complex-number system.

a. Graph a complex number of the form $a + 0i$ on the complex plane.

b.
1. Multiply the number by i and graph the result as a point.
2. Multiply the result from Step **1** by i and graph the resulting point. Continue this process of multiplying by i to obtain two more points.

c. Make a conjecture about the effects of multiplying by i with respect to the movement of a point on the complex plane.

d.
1. Predict the result of multiplying $a + 0i$ by $-i$.
2. Test your prediction by repeating Parts **a** and **b** using the factor $-i$.

e.
1. Multiply $a + bi$ by i.
2. Does your conjecture from Part **c** appear to apply to all complex numbers? If not, revise it so that it does.

Discussion

a. Describe the transformation that occurs when a complex number is multiplied by each of the following:
1. i
2. $-i$

b. How do the transformations in Part **a** affect the ordered pair that represents $a + 0i$?

c. Describe what occurs when $0 + 0i$ is multiplied by $-i$.

Assignment

4.1
a. Multiply the complex number $3 + 2i$ by the each of the following numbers. Write the products as ordered pairs.
1. i
2. i^2
3. i^3
4. i^4

b. Plot the products from Part **a** in the complex plane. What is the geometric relationship among these points?

4.2 Describe how complex conjugates are related in terms of their graphs in the complex plane.

4.3 **a.** Let $u = 4 + 9i$ and $v = 5 + 4i$. Find $u + v$ and $u - v$.

b. Graph the four complex numbers from Part **a**, u, v, $u + v$, and $u - v$ as ordered pairs on the complex plane.

c. Define the addition and subtraction of complex numbers $a + bi$ and $c + di$ using ordered pairs.

4.4 **a.** Consider the polynomial equation $x^3 - x^2 + x - 1 = 0$. Determine which of the following are solutions to this equation: $1, -1, i$, or $-i$.

b. Rewrite $x^3 - x^2 + x - 1 = 0$ as a product of factors in the form $(x - k)$ where k is a root of the equation.

c. Multiply the factors to verify your response to Part **b**.

* * * * *

4.5 Julia sets are sets of complex numbers that often make interesting patterns when graphed on the complex plane. Julia sets are generated by the recursive formula: $a_n + b_n i = (a_{n-1} + b_{n-1}i)^2 + a_1 + b_1 i$, where n is a natural number.

a. The second term of the Julia set where $a_1 + b_1 i = 2 + 3i$ is:

$$\begin{aligned} a_2 + b_2 i &= (2 + 3i)^2 + 2 + 3i \\ &= 4 + 6i + 6i + 9i^2 + 2 + 3i \\ &= -3 + 15i \end{aligned}$$

Find the third term of this Julia set.

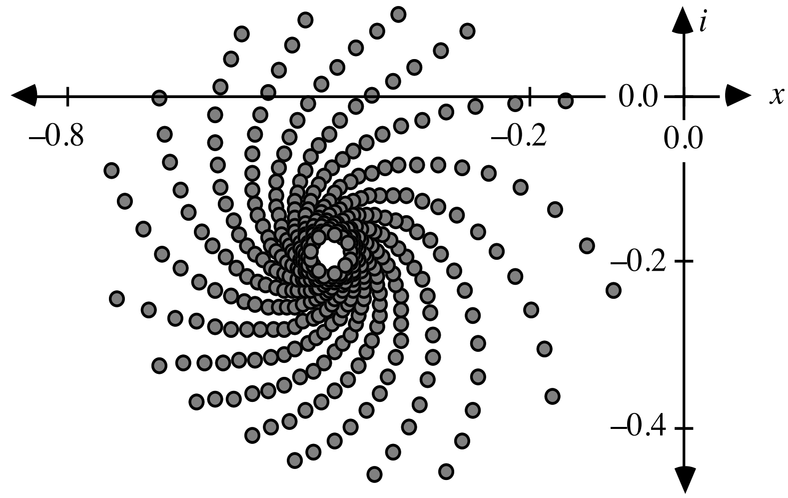
b. In order to create a scatterplot of a Julia set using technology, each term is written as an ordered pair. The first two terms of the Julia set from Part **a** can be written as (2,3) and (–3,15). Write the third term of this Julia set as an ordered pair.

c. **1.** Expand the recursive formula for Julia sets, writing the result in the form indicated below:

$$\begin{aligned} a_n + b_n i &= (a_{n-1} + b_{n-1}i)^2 + a_1 + b_1 i \\ &= \text{real part} + \text{imaginary part} \end{aligned}$$

2. Write the nth term as an ordered pair in the form (a_n, b_n).

d. For certain values of a_1 and b_1, the scatterplots of Julia sets make interesting patterns. For example, the following graph shows a scatterplot of the first 400 terms of the Julia set where $a_1 = -0.63$ and $b_1 = -0.37$:



The table below shows the first six terms (rounded to the nearest 0.0001) of the Julia set with $a_1 = -0.63$ and $b_1 = -0.37$.

Term Number (*n*)	**Real Part (a_n)**	**Imaginary Part (b_n)**
1	–0.63	–0.37
2	–0.3700	0.0962
3	–0.5024	–0.4412
4	–0.5723	0.0733
5	–0.3079	–0.4539
6	–0.7412	–0.0906
⋮	⋮	⋮

1. Use a spreadsheet to extend the table to 200 terms.
2. Create a scatterplot of the ordered pairs (a_n, b_n) on the complex plane.

e. Small changes in the first term of a Julia set result in a dramatically different set.

1. Create a scatterplot of the first 200 terms of the Julia set where $a_1 = -0.63$ and $b_1 = -0.38$.
2. Create new scatterplots by making small modifications in a_1 and b_1.

f. The table below shows the first terms in some other Julia sets. Use these first terms to explore patterns in the resulting scatterplots.

a_1	b_1
0.2477	0.56
–0.61	–0.405
0.29	0.45
–1.195	0.45
–1.2	0.15

4.6 Consider the set of matrices of the form below, where a and b are real numbers.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

a. Show that this set of matrices has the closure property under addition. In other words, show that if any two matrices in this set are added together, the sum is a matrix from the set.

b. Show that the set of matrices is closed under multiplication.

c. Find an additive identity and a multiplicative identity for this set of matrices.

d. Find the multiplicative inverse of the following matrix, if it exists:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

e. The arithmetic of the set of matrices defined in Parts **a–d** behaves almost exactly the same as arithmetic with real numbers. However, this arithmetic has one property that arithmetic with real numbers does not. Square the following matrix and describe how the result compares to the multiplicative identity for the set of matrices.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

f. Considering your results in Part **e**, the arithmetic of this set of matrices behaves like the arithmetic of the set of complex numbers. Now suppose that every complex number of the form $a + bi$ can be identified with the matrix

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

What is the matrix representation of the complex number $0 + i$?

g. Recall that multiplication by a matrix of the form below produces a rotation of θ about the origin.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What matrix produces a 90° rotation about the origin?

h. Compare the matrices found in Parts **f** and **g**.

* * * * * * * * * *

Activity 5

In Activity **4**, you used ordered pairs of the form (a,b) to represent complex numbers. However, when performing multiplication of complex numbers, it can be more convenient to use their **trigonometric form**. Using the trigonometric form also can simplify finding powers of complex numbers.

Mathematics Note

Figure **4** shows the complex number $a + bi$ represented as the ordered pair (a,b).

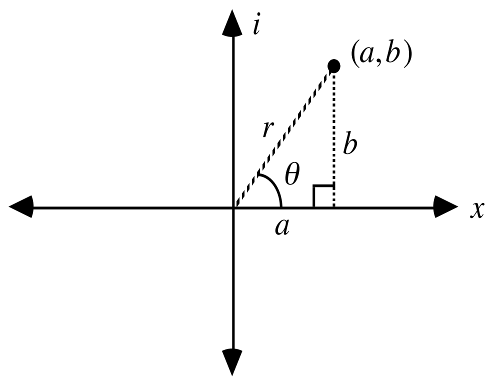


Figure 4: Graph of *a + bi*

A complex number $a + bi$ can be written in **trigonometric form** as follows:

$$a + bi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

The value of r is the **absolute value** or **modulus** of the complex number and is determined by $r = \sqrt{a^2 + b^2}$. Note that r is always a non-negative number.

The angle θ is an **argument** of the complex number and is measured from the positive portion of the real axis. Angles generated by counterclockwise rotations are assigned positive measures; those generated by clockwise rotations are assigned negative values. The ray passing through the point (a,b) representing the number $a + bi$ is the **terminal ray** of the argument.

For example, consider the complex number $\sqrt{3} + i$ represented by the point $(\sqrt{3},1)$. Using right-triangle trigonometry, an argument is $\theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$. The absolute value is $r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Therefore, the trigonometric form of $\sqrt{3} + i$ is:

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Because θ can be any angle measured from the positive portion of the real axis whose terminal ray contains $(\sqrt{3},1)$, the trigonometric form of $\sqrt{3}+i$ is not unique. In this case, θ can be any angle of the form $\theta = \pi/6 + 2n\pi$, where n is an integer. Two additional representations of $\sqrt{3}+i$ in trigonometric form are $2(\cos(13\pi/6) + i\sin(13\pi/6))$, where $\theta = \pi/6 + 2\pi$, and $2(\cos(-11\pi/6) + i\sin(-11\pi/6))$, where $\theta = \pi/6 - 2\pi$.

Exploration 1

a. The following complex numbers are written in the form $a + bi$. Evaluate $\tan^{-1}(b/a)$ for each number. Use the resulting angle measure to determine an argument of the given number.

1. $1 + \sqrt{3}i$

2. $-1 + \sqrt{3}i$

3. $-1 - \sqrt{3}i$

4. $1 - \sqrt{3}i$

b. Determine the measure of two additional positive arguments and two additional negative arguments for each of the complex numbers in Part **a**.

Discussion 1

a. Given the complex number $a + bi$, in which quadrants can the point (a,b) lie if $\theta = \tan^{-1}(b/a)$ is an argument of $a + bi$?

b. If $\theta = \tan^{-1}(b/a)$ is not an argument of $a + bi$, how can an argument be determined using $\tan^{-1}(b/a)$?

c. Describe the methods you used to determine the additional positive and negative arguments in Part **b** of Exploration **1**.

Mathematics Note

If the graph of the complex number $a + bi$ is in the first or fourth quadrants, $\theta = \tan^{-1}(b/a)$ is an argument of the number and every argument is represented by the expression $\theta = \tan^{-1}(b/a) + 2n\pi$, where n is any integer.

For complex numbers represented by points in the second and third quadrants, arguments have the form $\theta = (\tan^{-1}(b/a) + \pi) + 2n\pi$, where n is an integer.

For example, the graph of the complex number $-2 + 2i$ is in the second quadrant. In this case, its arguments can be found as follows, where n is an integer:

$$\theta = (\tan^{-1}(2/-2) + \pi) + 2n\pi = 3\pi/4 + 2n\pi$$

Exploration 2

In Activity **4**, you learned that points on the complex plane are rotated when multiplied by i or $-i$. In this activity, you multiply complex numbers to discover other patterns.

a. Multiply each of the following pairs of complex numbers. Plot each pair of complex numbers and their product as points on a complex plane.

1. $v = 2 + i$ and $s = 2 + 3i$
2. $t = 0 + 2i$ and $u = -1 + i$
3. $m = -1 + 2i$ and $w = -2 - i$
4. $g = 3 + 2i$ and $h = 2 + 4i$

b. For each complex number in Part **a**, determine its absolute value, as well as an argument (to the nearest 0.01 radians) between -2π and 2π. Leave the absolute value in radical form, even if the square root is an integer. Record these values in a table with headings like those in Table **1** below.

Table 1: Complex numbers and their products

Number	$a + bi$	Absolute Value	Argument
v	$2 + i$		
s	$2 + 3i$		
$v \bullet s$			
t	$0 + 2i$		
u	$-1 + i$		
$t \bullet u$			
m			
w			
$m \bullet w$			
g			
h			
$g \bullet h$			

c. Select at least three conjugate pairs of complex numbers. Repeat Parts **a** and **b** using these pairs.

d.
1. Use a symbolic manipulator to verify the following rule for multiplying complex numbers in trigonometric form:
$$[a(\cos x + i\sin x)] \bullet [b(\cos y + i\sin y)] = ab[\cos(x + y) + i\sin(x + y)]$$
2. Using the terms *absolute value* and *argument*, describe the rule for multiplying complex numbers in trigonometric form.
3. Determine if this rule is illustrated in Table **1**.

e. 1. Write $1 - i$ and $-2 + i$ in trigonometric form, rounding both r and θ to the nearest 0.01.

2. Use the rule from Part **d** to multiply the trigonometric forms of $1 - i$ and $-2 + i$.

3. Write the product in the form $a + bi$.

4. Use the distributive property to multiply $1 - i$ and $-2 + i$.

5. Compare the products in Steps **2** and **4**.

Discussion 2

a. Describe the relationship between conjugates when they are expressed in trigonometric form.

b. What is the argument of the product when a complex number and its conjugate are multiplied?

c. Is multiplication of complex numbers in trigonometric form commutative? Justify your response.

d. Compare the process of multiplying complex numbers in the form $a + bi$ with the process of multiplying the same numbers in trigonometric form.

Assignment

5.1 The complex numbers in Parts **a–c** are given in trigonometric form. Multiply each number by its respective conjugate and write the products in trigonometric form.

a. $3(\cos(\pi/12) + i\sin(\pi/12))$

b. $11(\cos(8.83) + i\sin(8.83))$

c. $\sqrt{7}(\cos(-\pi/15) + i\sin(-\pi/15))$

5.2 Use the rule developed in Exploration **2** to multiply the following pairs of complex numbers. Write each product in trigonometric form.

a. $3(\cos(\pi/4) + i\sin(\pi/4))$ and $2(\cos(8.85) + i\sin(8.85))$

b. $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$

5.3 The ordered pairs (–1,2) and (3,–2) represent two complex numbers on the complex plane.

a. Find their product using two different methods.

b. Compare the results and explain any differences you observe.

5.4 Multiplication by the complex number $2(\cos(\pi/6)+i\sin(\pi/6))$ can be thought of as a dilation by a scale factor of 2 and a rotation of $\pi/6$ with center at (0,0). What complex number produces the same dilation but the opposite rotation? Describe the relationship between these two numbers.

5.5 **a.** Evaluate $(3-4i)^2$ by converting the expression to trigonometric form before multiplying.

b. Find the trigonometric form of $(3-4i)^3$ by multiplying the trigonometric form of $(3-4i)^2$ by the trigonometric form of $(3-4i)$.

c. Write $(3-4i)^5$ in trigonometric form.

d. Express $(3-4i)^n$, where n is an integer, in trigonometric form.

5.6 Given that the trigonometric form of $a+bi$ is $r(\cos\theta+i\sin\theta)$, write the trigonometric forms of $(a+bi)^2$, $(a+bi)^3$, $(a+bi)^4$, and $(a+bi)^n$.

* * * * *

5.7 In previous modules, you have used several different methods to draw regular polygons. In this assignment, you investigate another way to construct regular polygons.

a. Consider the complex number $3+4i$. What is the modulus r of this number?

b. Plot the complex number $3+4i$ on a grid.

c. What is the radius of the circle with center at the origin that contains the point $3+4i$?

d. What is the measure θ of a central angle of a regular pentagon?

e. To construct a regular pentagon with the point representing $3+4i$ as one of the vertices, one could rotate the point (and its successive images) by θ, with center at the origin.

To do this, multiply $3+4i$ by a complex number in the form $\cos\theta+i\sin\theta$. Plot the coordinates of the product. Continue this process to find the five vertices of the regular pentagon.

5.8 Use the process described in Problem **5.7** to construct a regular hexagon whose sides measure 3 units.

5.9 When designing circuits for use with alternating current, electrical engineers use the complex-number form of Ohm's law:

$$I = \frac{V}{Z}$$

where I is the effective current (a measure of the number of electrons moving in the wires), V is the effective voltage (a measure of the force moving the electrons), and Z is the impedance (a measure of the resistance to the flow of electrons caused by magnetic fields). In this relationship, I, V, and Z are complex numbers.

In Parts **a–c**, write each response in the same form as the original numbers.

a. Determine the effective voltage in a circuit if the effective current is $4(\cos(\pi/18) + i\sin(\pi/18))$ and the impedance is $29(\cos(\pi/9) + i\sin(\pi/9))$.

b. Find the effective current in a circuit if the effective voltage is $120(\cos 0 + i\sin 0)$ and the impedance is $44(\cos(11\pi/36) + i\sin(11\pi/36))$.

c. Find the impedance of a circuit when the effective voltage is $77 + 77i$ and the effective current is $2.9 - 0.35i$.

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Activity 6

Using the trigonometric form of a complex number, you can explore some other interesting properties of complex numbers.

Mathematics Note

De Moivre's theorem states that the non-zero powers of any complex number $a + bi$ can be found in the following manner:

$$(a + bi)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Abraham De Moivre (1667–1754) developed this theorem for positive integer values of n. Later work found it to be true for all real-number values of n. For example,

$$(2 + 0i)^3 = [2(\cos 0^\circ + i\sin 0^\circ)]^3 = 2^3(\cos(3 \bullet 0) + i\sin(3 \bullet 0))$$

Exploration

In previous activities, you examined the square roots of negative numbers. In this activity, you investigate some additional roots of complex numbers.

a. **1.** Graph $2(\cos 0 + i\sin 0)$ on a complex coordinate plane.

2. On the same plane, graph the image of $2(\cos 0 + i\sin 0)$ under a counterclockwise rotation of $\pi/2$ with center at (0,0).

3. Repeat Step **2** with each new image until points begin to repeat. Using this process, determine the coordinates of each unique point generated.

b. **1.** Describe the geometric relationships among the points in Part **a**.

2. Determine the trigonometric form of the complex number represented by each point.

c. Use De Moivre's theorem to raise each complex number in Part **b** to the fourth power. Convert each result to a number in the form $a + bi$.

d. **1.** If the graph of the complex number $2(\cos 0 + i\sin 0)$ represents one vertex of a regular pentagon centered at the origin, determine the angle of counterclockwise rotation about the origin required to locate the next consecutive vertex of the pentagon.

2. Determine the complex numbers that correspond to the five vertices obtained by repeating this rotation on each image. Write these numbers in trigonometric form. **Note:** Save this data for use in the assignment.

e. Use De Moivre's theorem to raise each complex number from Part **c** to the fifth power. Convert each result to a number in the form $a + bi$.

f. Select any complex number $a + bi$ where $a \neq 0$ and $b \neq 0$. Determine the cube roots of this number and write them in trigonometric form.

g. Graph the cube roots found in Part **f** on the complex plane. Describe any geometric relationship among these points.

Mathematics Note

From the fundamental theorem of algebra, the equation $x^n - z = 0$ has *n* roots in the set of complex numbers.

For example, $x^3 - 8 = 0$ has roots 2, $-1 + i\sqrt{3}$, and $-1 - i\sqrt{3}$. The solutions to this equation are the cube roots of 8. Thus, there are 3 cube roots of 8 in the set of complex numbers. In a similar manner, there are 4 fourth roots of 8 and 5 fifth roots of 8. In general, there are exactly *n* distinct *n*th roots of any complex number.

Discussion

a. Describe the significance of your results in Part **c** of the exploration in terms of the fourth roots of a number.

b. Describe the significance of your results in Part **e** of the exploration in terms of the fifth root of a number.

c. Using the examples from the exploration, describe the relationship between the modulus of a complex number in trigonometric form and the modulus of its roots in trigonometric form.

d. Using the examples from the exploration, describe the relationship between the argument of a complex number and the argument of its roots.

e. **1.** When $\theta = 0$, what is $r(\cos\theta + i\sin\theta)$?

2. When $\theta = \pi/2$, what is $r(\cos\theta + i\sin\theta)$?

f. What effect did the initial value of θ have on the polygons formed in the exploration?

Mathematics Note

If a complex number $z = a + bi$ is written as $z = r(\cos\theta + i\sin\theta)$, then the nth roots of z can be found using the following formula:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right)\right)$$

where $k = 0,\ 1,\ 2,\ \ldots,\ n-1$. There will be exactly n of these roots.

For example, $z = 8 + 0i$ can be written as $z = 8(\cos 0 + i\sin 0)$. The three cube roots of z are:

$$r_1 = 2\left(\cos\left(\frac{0}{3} + \frac{0 \bullet 2\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{0 \bullet 2\pi}{3}\right)\right) = 2\left(\text{cis}\,\frac{0\pi}{3}\right) = 2 + 0i$$

$$r_2 = 2\left(\cos\left(\frac{0}{3} + \frac{1 \bullet 2\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{1 \bullet 2\pi}{3}\right)\right) = 2\left(\text{cis}\,\frac{2\pi}{3}\right) \approx -1 + 1.73i$$

$$r_3 = 2\left(\cos\left(\frac{0}{3} + \frac{2 \bullet 2\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{2 \bullet 2\pi}{3}\right)\right) = 2\left(\text{cis}\,\frac{4\pi}{3}\right) \approx -1 - 1.73i$$

Assignment

6.1 **a.** Evaluate $\left(2\sqrt{3}+2i\right)^3$ using the distributive property.

b. Convert $2\sqrt{3}+2i$ to trigonometric form and cube it using De Moivre's theorem. Round the argument to the nearest 0.001 radians.

6.2 The 4 fourth roots of 81 are evenly spaced on a circle in the complex plane centered at the origin.

a. What is the radius of the circle?

b. By how many radians are consecutive roots separated?

c. Graph all 4 fourth roots. Include the circle containing the vertices in your graph.

d. Write each fourth root in the form $a+bi$.

6.3 **a.** Pick a complex number of the form $a+bi$ where $a \neq 0$ and $b \neq 0$.

b. Find the three cube roots of this complex number.

c. Write each cube root in trigonometric form.

d. Sketch a graph of the cube roots on a complex coordinate plane.

6.4 Solve the equation $z^8 = -2+3i$ for z and write the roots as ordered pairs (a,b).

6.5 One of the cube roots of 64 is 4, which can be written as $4\cos 0 + 4i\sin 0$.

a. The point (4,0) in the complex plane represents the complex number $4\cos 0 + 4i\sin 0$. Determine the counterclockwise rotation of the point (4,0) about the origin required to locate the next consecutive cube root of 64.

b. Graph the three cube roots of 64 on the complex plane.

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6.6 **a.** Find the three cube roots of 1 in the form $r(\cos\theta + \text{i}\sin\theta)$ and write each cube root in the form $a+bi$.

b. Verify that the roots obtained are in fact cube roots of 1.

6.7 **a.** If the graph of $3\left(\cos(3\pi/4) + i\sin(3\pi/4)\right)$ defines one vertex of a regular octagon centered at the origin of the complex plane, what numbers correspond to the other seven vertices?

b. What do the complex numbers corresponding to each vertex of this octagon represent?

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Research Project

Besides introducing the imaginary unit i, Swiss mathematician Leonhard Euler made many other contributions to mathematics. Read more about Euler's life and work. Complete the following tasks in your report.

a. Describe Euler's formula.

b. Explain how Euler's formula can be used to represent complex numbers in exponential form.

c. Show how substituting π into Euler's formula results in a natural logarithm for -1.

d. Demonstrate that natural logarithms of negative numbers exist in the set of complex numbers.

Summary Assessment

1. Write a paragraph describing the number of real roots possible for the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$, where the coefficients are real numbers and $a_n \neq 0$.

2.
a. Use a symbolic manipulator to find the roots of the polynomial equation $x^4 - 2x^3 + x^2 + 4x - 6 = 0$.

b. Use the results from Part **a** to express the polynomial $x^4 - 2x^3 + x^2 + 4x - 6$ as a product of one or more polynomials such that the constant term in each factor is:

1. a complex number
2. a real number
3. a rational number.

3. According to the binomial theorem:

$$(a+b)^n = C(n,n) \bullet a^n b^0 + C(n,n-1) \bullet a^{n-1} b^1 + C(n,n-2) \bullet a^{n-2} b^2 + \cdots + C(n,1) \bullet a^1 b^{n-1} + C(n,0) \bullet a^0 b^n$$

where $C(n,r)$ is the combination of n things, taken r at a time.

a. Expand $(1+i)^8$ using the binomial theorem.

b. Simplify the expression from Part **a**.

c. Write $(1+i)$ in trigonometric form.

d. Use the trigonometric form to evaluate $(1+i)^8$. Write the result in standard form.

e. Compare the binomial theorem and De Moivre's theorem as methods for raising complex numbers to a power.

4. The coordinates of the vertices of ΔABC in a complex coordinate plane are $A(2,1)$, $B(4,1)$, and $C(3,2)$. The image of ΔABC has vertices with coordinates $A'(-5,5)$, $B'(-7,9)$, and $C'(-8,6)$.

The transformation from ΔABC to $\Delta A'B'C'$ is produced by multiplying by the complex number z, then adding z. For example, A is transformed to A' by $(2,1)\bullet z + z = (-5,5)$.

a. Plot ΔABC and its image in the complex plane. Describe the geometric relationship between these two triangles.

b. What is the ratio of $A'B'/AB$? What does this ratio reveal about the number z that produced the transformation?

c. Find the trigonometric forms of the $B-A$ and $B'-A'$. What do these reveal about the number z that produced the transformation?

d. Find z. Pick a point on ΔABC and show that it is transformed appropriately.

Module Summary

- The **imaginary unit** is ***i*** where $i = \sqrt{-1}$ and $i^2 = -1$.
- A **complex number** is defined as any number in the form $a + bi$, where both a and b are real numbers.
- A **pure imaginary number** is a complex number $a + bi$ for which $a = 0$ and $b \neq 0$.
- A real number is a complex number $a + bi$ for which $b = 0$.
- In the set of complex numbers, $a + bi = c + di$ if and only if $a = c$ and $b = d$.
- **Complex conjugates** are pairs of complex numbers of the form $a + bi$ and $a - bi$. The sum of complex conjugates is a real number. The product of complex conjugates also is a real number.
- The **reciprocal** of a complex number $a + bi$ is $1/(a + bi)$. To express this reciprocal in complex form $m + ni$, it can be multiplied by

  $$\frac{a - bi}{a - bi}$$

  where $a - bi$ is the conjugate of $a + bi$.
- A complex number $a + bi$ can be represented by the ordered pair (a, b) where a is the real part and b is the imaginary part. Using the horizontal axis as the real axis and the vertical axis as the imaginary axis, this ordered pair can be graphed as a point on the complex plane.
- Second-degree polynomial equations of the form $ax^2 + bx + c = 0$ with $a \neq 0$, always have two solutions when solved over the complex numbers:

  $$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

  These two solutions make up the **quadratic formula**. When a, b, and c are real numbers and $b^2 - 4ac < 0$, the solutions are complex and occur in conjugate pairs.
- The **fundamental theorem of algebra** states that every polynomial equation of degree $n \geq 1$ with complex coefficients has at least one root, which is a complex number (real or imaginary).
- One consequence of the fundamental theorem of algebra is that nth-degree polynomial equations have exactly n roots in the set of complex numbers.

- A complex number $a + bi$ can be written in **trigonometric form** as:

$$r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$$

  The value of *r* is the **absolute value** or **modulus** of the complex number and is determined by $r = \sqrt{a^2 + b^2}$. Note that *r* is always a non-negative number. The angle θ is an **argument** of the complex number and is measured from the positive portion of the real axis to the point (a, b) in the complex plane.

  If the graph of the complex number $a + bi$ is in the first or fourth quadrants, $\theta = \tan^{-1}(b/a)$ is an argument of the number and every argument is represented by the expression $\theta = \tan^{-1}(b/a) + 2n\pi$ where *n* is any integer.

  For complex numbers $a + bi$ represented by points in the second and third quadrants, arguments have the form $\theta = \left(\tan^{-1}(b/a) + \pi\right) + 2n\pi$ where *n* is any integer.

- **De Moivre's theorem** states that the powers of any complex number $a + bi$ can be found in the following manner:

$$(a + bi)^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

- Another consequence of the fundamental theorem of algebra is that the equation $x^n - z = 0$ has *n* roots in the set of complex numbers.

- If a complex number $z = a + bi$ is written as $z = r(\cos\theta + i\sin\theta)$, then the *n*th roots of *z* can be found using the following formula:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k \bullet 2\pi}{n}\right)\right)$$

  where $k = 0,\ 1,\ 2,\ \ldots,\ n - 1$. There will be exactly *n* of these roots.

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