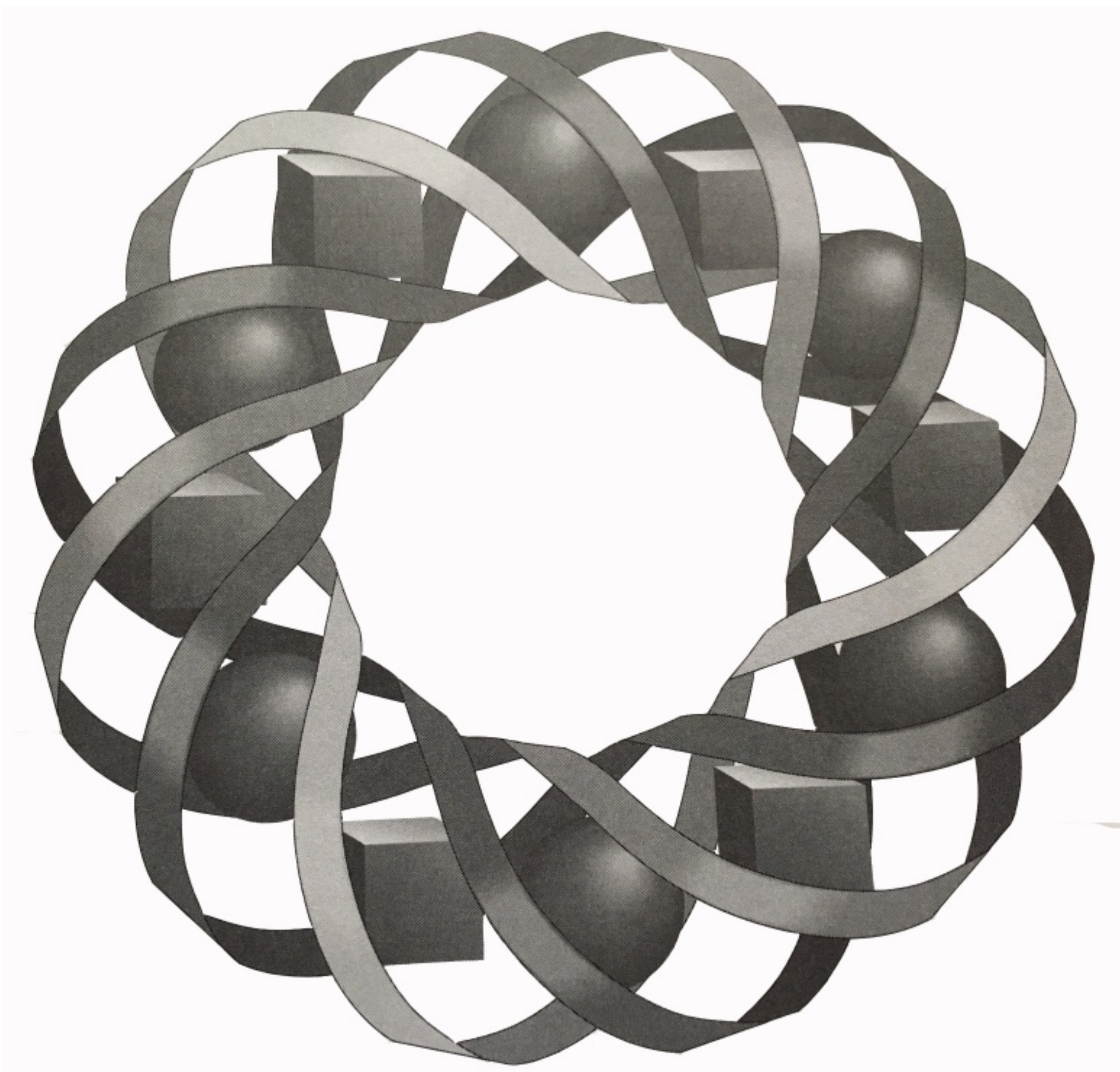
Slow Down! You're Deriving over the Limit



How does the velocity of a falling object change over time? And exactly how fast is it traveling at any instant during its fall? In this module, you discover how to answer these questions.

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Slow Down! You're Deriving over the Limit

Introduction

In the Level 6 module "Mathematics in Motion," you used parametric equations to model the position, with respect to time, of freely falling objects. Recall that freely falling objects are acted on only by the force of gravity. Ignoring air resistance, for example, a ball dropped from some initial height is a freely falling object, as is a ball thrown with some initial velocity. In this module, you continue your exploration of this type of motion.

Discussion

a. Consider a ball dropped from an initial height of 100 m. What would a graph of the ball's height versus time look like? Explain your response.

b.
1. Recall that velocity describes an object's change in position with respect to time. After 1 sec, the ball described in Part **a** is 95.1 m from the ground. Describe the ball's average velocity during this interval.
2. In this case, what does the sign of the velocity indicate?

c. Will the ball travel the same distance during each second of its fall? Explain your response.

Activity 1

In this activity, you collect data on the motion of a freely falling object and determine an equation that models its height with respect to time.

Exploration

a. Drop an object from an initial height of 2 m. As it falls, use a range finder and science interface device to collect data on the object's height with respect to time.

b. Create a scatterplot of the object's height versus time.

c. Determine an appropriate equation to model the data. **Note:** Save this equation, the scatterplot from Part **b**, and your data for use in Activity **2**.

d. Graph your function from Part **c** on the scatterplot from Part **b**.

e. Use your equation to predict the time required for the object to reach a height of 0 m.

f. The acceleration due to gravity is 9.8 m/sec^2 in a direction toward Earth's center. Estimate the ball's velocity at the time it hit the ground.

Discussion

a. Describe your scatterplot of the data for the falling object.

b. What is represented by the slope of a line containing any two points on the scatterplot?

c.
1. Describe the equation you obtained in the exploration.
2. How well does your equation appear to model the data?

d. Recall that the height of a freely falling object after t sec can be described by the following equation:

$$h(t) = -\frac{1}{2}gt^2 + v_0t + h_0$$

where g is the acceleration due to gravity, v_0 is the object's initial velocity in the vertical direction, and h_0 is the object's initial height.

1. Given this general equation, describe a function that would model the height with respect to time of a ball dropped from the same initial height as in the exploration.
2. Compare this equation to the one you obtained in the exploration.

e. Figure **1** shows a graph of height versus time for a ball thrown straight into the air.

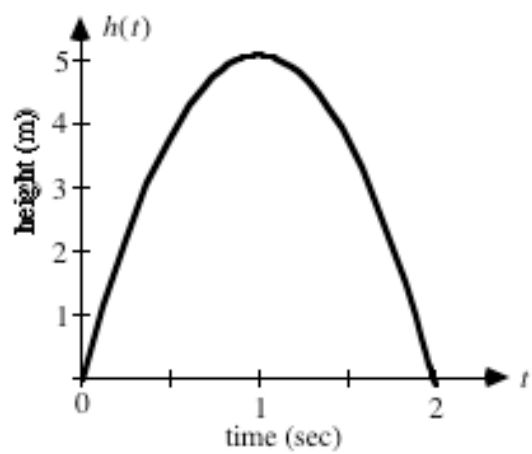

Figure 1: Graph of distance versus time

1. During what time interval is the ball's velocity positive?
2. During what time interval is the ball's velocity negative?
3. What is the ball's velocity when it reaches its highest point?

Assignment

1.1 Consider an object propelled upward at a velocity of 49 m/sec from the top of a 98-m tower.

a. Determine a function $h(t)$ that models the object's height with respect to time and graph it.

b. Identify the interval for which the object's velocity is positive.

c. Identify the interval for which the object's velocity is negative.

d. **1.** What is the greatest height reached by the object?

2. How long will it take the object to reach this height?

3. What is the velocity of the object when it reaches this height?

e. How many seconds will it take the ball to return to its initial height of 98 m?

1.2 A ball dropped from the top of a tower strikes the ground after 3 sec.

a. How tall is the tower?

b. Determine an equation that describes the ball's height with respect to time.

1.3 **a.** If an object is propelled straight up from the ground with an initial velocity of 34.3 m/sec, how long will it remain in the air?

b. What will the object's velocity be just before it strikes the ground?

c. If a similar object remained in the air for 6 sec, what was its initial velocity?

1.4 Consider a ball thrown straight up from the ground with a velocity of 49 m/sec.

a. Write an equation that models the ball's height with respect to time.

b. Determine how long the ball will remain in the air.

c. Determine the height of the ball at the end of each second of its flight.

d. Determine the ball's average velocity during each 2-sec interval of its flight. Record these values in a table like the one shown below.

Interval (sec)	**Average Velocity**
$[0, 2)$	
$[2, 4)$	
$\vdots$	

e. What do the values in the table from Part **d** indicate about the ball's flight?

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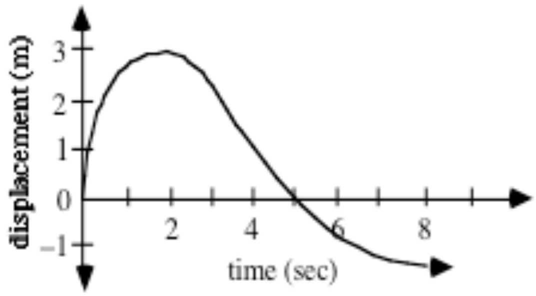
1.5 Consider a wind-up toy moving along a linear track. The graph below shows its displacement over time. In this case, a positive displacement indicates movement to the right of the starting position. Assume that the toy's initial velocity is 0 m/sec.



a. During what time interval is the toy's velocity positive? During what interval is its velocity negative? Explain your response.

b. What is the toy's average velocity during the first 2 sec?

c. When did the toy change direction? Describe how this change is indicated on the graph.

d. When did the toy return to its starting position? Describe how this is indicated on the graph.

e. During which 1-sec interval does the magnitude of the toy's velocity appear to be the greatest? How is this indicated on the graph?

f. Describe the toy's motion from the time it started moving until the time it stopped.

1.6 The Empire State Building in New York City is approximately 381 m high. Consider the motion of an object dropped from the top of this building.

a. Write an equation that models the height of the object over time.

b. Determine the time required for the object to reach the ground.

c. Estimate the object's velocity just before it reaches the ground.

d. Because of the danger to pedestrians, it is illegal to drop objects from tall buildings. Write a statement explaining how such laws protect the public.

1.7 The height of a ball in meters after t seconds can be modeled by the following equation: $h(t) = 24.5 + 19.6t - 4.9t^2$.

a. From what height was the ball thrown?

b. What initial velocity was given to the ball?

c. How long was the ball in the air?

d. What was the maximum height reached by the ball?

e. What was the ball's velocity just before it struck the ground?

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Activity 2

In Activity **1**, you investigated the change in position with respect to time for a falling object, as well as its average velocity over particular intervals of time. In this activity, you examine one method for approximating an object's velocity at any given instant. This is known as **instantaneous velocity**.

Exploration

a. Examine your scatterplot of the falling-ball data from Activity **1**. Select a subset of the data that appears to accurately describe the motion of the ball for approximately 0.5 sec. Identify a data point *A* so that there are an equal number of data points before and after *A*.

b. One way to estimate the ball's instantaneous velocity at point *A* is to examine the average velocities for intervals that include *A*.

1. Using your chosen subset of data, identify the data point *P* that is farthest to the left of *A*.
2. Draw the line that contains *A* and *P*. Your graph should now resemble the one shown in Figure **2** below.

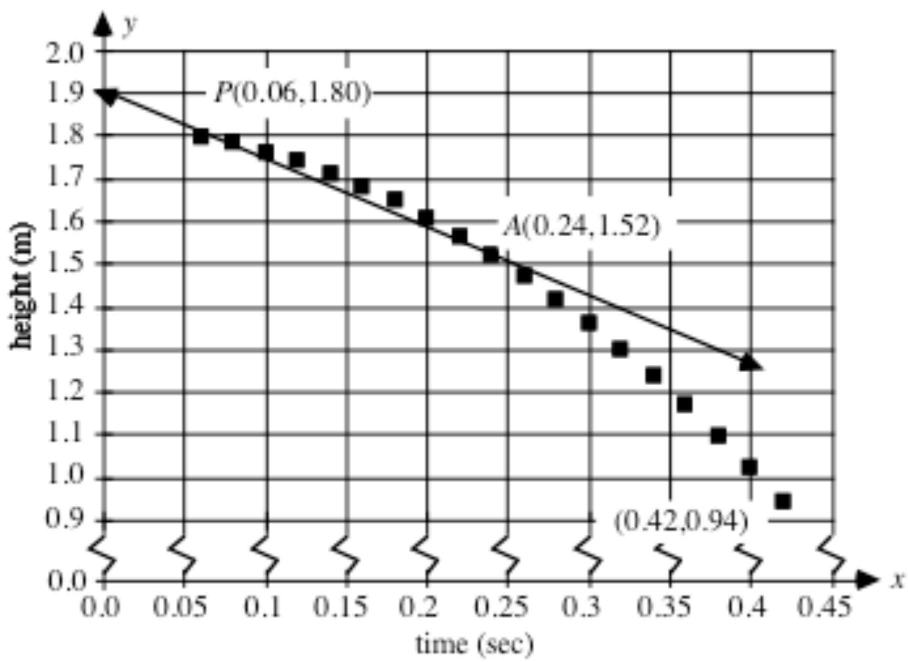


Figure 2: Graph of sample falling-ball data

3. Calculate the difference *d* in the *x*-coordinates of this pair of points. For *A* and *P* in Figure **2**, for example, this value is 0.24 – 0.06 = 0.18 sec.
4. Determine the slope of the line from Step **2** and describe what this value represents in terms of the falling ball.

c. Draw a line through A and a data point Q that is d sec to the right on the scatterplot. In Figure **2**, for example, this is the point (0.42,0.94). Determine the slope of this line and describe what its value represents in terms of the falling ball.

Record your findings from Parts **b** and **c** in a table with headings like those in Table **1**. In this case, $(x-d)$ is the x-coordinate of a point P to the left of A, while $(x+d)$ is the x-coordinate of a point Q to the right of A. (The cells in the first row of Table **1** show the appropriate sample values from Figure **2**.)

Table 1: Approximating velocity at point $A(x,y)$

Value of d	**$(x-d)$**	**y-coord. of P**	**Slope of $\overleftrightarrow{PA}$**	**$(x+d)$**	**y-coord. of Q**	**Slope of $\overleftrightarrow{AQ}$**
0.18	0.06	1.80	–1.56	0.42	0.94	–3.22
0.16						
0.14						
⋮						
0.04						
0.02						

d. What is the relationship among the slope of $\overleftrightarrow{PA}$, the slope of $\overleftrightarrow{AQ}$ and the instantaneous velocity of the ball at A? Explain your response.

e. Repeat Parts **b** and **c**, using a pair of points P and Q that are closer to A. Record your results in Table **1**.

f. Continue the process described in Parts **b** and **c** for each successive pair of points, until you have used the pair closest to A.

g. Use your results in Parts **b–f** to estimate the ball's instantaneous velocity at point A.

Discussion

a. The slope of the line that passes through the first and last data points in Figure **2** is:

$$m = \frac{1.80 - 0.94}{0.06 - 0.42} \approx -2.39 \text{ m/sec}$$

Describe what this value represents in terms of the falling ball.

b. Describe how you approximated the instantaneous velocity of the falling ball at point A.

c. In Activity **1**, you used a regression equation to model the data for the falling ball. Figure **3** shows the graph of a function, $f(x) = -4.65x^2 - 0.14x + 1.82$, that models the scatterplot in Figure **2**.

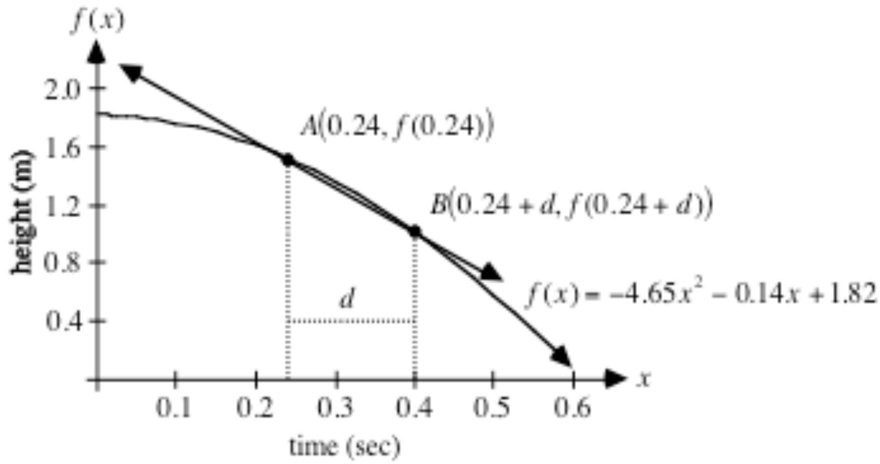


Figure 3: Regression equation for sample data

1. A **secant** is a line that intersects a curve but is not tangent to it. Describe how a secant such as $\overleftrightarrow{AB}$ in Figure **3** can be used to approximate the velocity of the falling ball at 0.24 sec.
2. In Figure **3**, the coordinates of *B* are given in terms of *d* and the coordinates of *A*. How does the value of *d* affect the estimate of instantaneous velocity?
3. As *d* approaches 0, the slope of $\overleftrightarrow{AB}$ approaches the slope of the line tangent to the curve at point *A*. What does the slope of the line tangent to the curve at *A* represent?

d. Table **2** lists the slopes of some secant lines through *A* (0.24,1.52) on the curve in Figure **3**. The values in the columns labeled "Slope" were calculated using the following formulas, where $x = 0.24$:

$$\frac{f(x-d)-1.52}{(x-d)-0.24} \text{ and } \frac{f(x+d)-1.52}{(x+d)-0.24}$$

Describe how you could use this table to approximate the ball's instantaneous velocity at 0.24 sec.

Table 2: Slopes of secant lines through *A*

d	**($x - d$)**	**$f(x - d)$**	**Slope**	**($x + d$)**	**$f(x + d)$**	**Slope**
0.18	0.06	1.798	–1.54	0.42	0.942	–3.21
0.16	0.08	1.781	–1.63	0.40	1.023	–3.11
0.14	0.10	1.762	–1.73	0.38	1.097	–3.02
0.12	0.12	1.739	–1.83	0.36	1.168	–2.93
0.10	0.14	1.711	–1.91	0.34	1.236	–2.84
0.08	0.16	1.682	–2.03	0.32	1.300	–2.75
0.06	0.18	1.646	–2.10	0.30	1.360	–2.67
0.04	0.20	1.608	–2.20	0.28	1.417	–2.58
0.02	0.22	1.566	–2.30	0.26	1.471	–2.45

e. What information would allow you to obtain a better approximation of the velocity of the ball at 0.24 sec? Explain your response.

Assignment

2.1 A group of students obtained the following data during a ball-drop experiment.

Time (sec)	**Height (m)**
0.22	1.564
0.23	1.542
0.24	1.519
0.25	1.494
0.26	1.469
0.27	1.443
0.28	1.416

a. Determine the quadratic regression equation that models the data above and create a graph of this equation.

b. Choose an instant in time x. Use the quadratic regression equation to predict the ball's height y at that time. Plot this point (x, y) on the graph from Part **a**.

c. Use a spreadsheet with headings like those in Table **2** to approximate the ball's instantaneous velocity at this point.

d. Describe what the instantaneous velocity represents in terms of the line tangent to the curve at this point.

2.2 **a.** Select three more instants in time on the graph of the regression equation from Problem **2.1**. Approximate the ball's instantaneous velocities at these times and record these values, along with the one from Problem **2.1c**, in a table like the one below.

Time (sec)	**Velocity (m/sec)**

b. Use your results in Part **a** to create a scatterplot of velocity versus time.

c. Describe any trends you observe in the scatterplot and determine an appropriate model for the data.

d. What does your model in Part **c** indicate about the ball's change in velocity with respect to time?

2.3 Consider a projectile shot upward with an initial velocity of 98 m/sec from a height of 196 m above the ground.

a. Determine and graph a function $h(t)$ that models the height of the projectile with respect to time.

b. Find the projectile's average velocity during each of the following intervals:

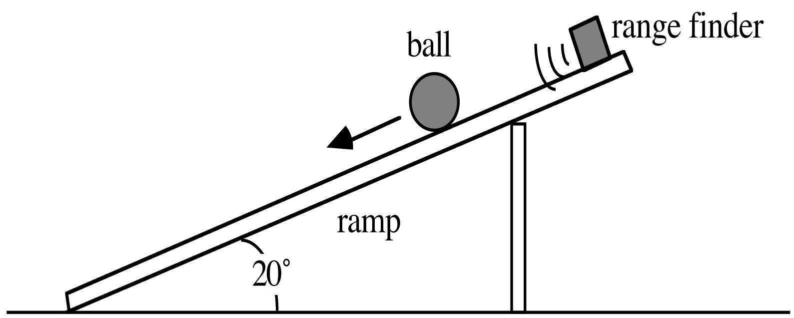
1. from $t = 0$ sec to $t = 2$ sec
2. from $t = 18$ sec to $t = 20$ sec
3. from $t = 2$ sec to $t = 18$ sec

c. The total distance traveled by the projectile from $t = 2$ sec to $t = 18$ sec is approximately 628 m. Half of this distance is in the direction away from Earth's surface, while the other half is directed towards Earth's surface. Explain why the average velocity for this interval cannot be calculated as shown below:

$$\frac{628 \text{ m}}{16 \text{ sec}} = 39.25 \text{ m/sec}$$

d. Approximate the projectile's instantaneous velocity at $t = 16$ sec.

2.4 The diagram below shows an experiment designed to measure the acceleration of a ball rolling down a ramp.



The following table shows some of the data collected in this experiment.

Time (sec)	Distance (m)	Time (sec)	Distance (m)
0.0	0.443	1.0	2.607
0.1	0.571	1.1	2.931
0.2	0.718	1.2	3.276
0.3	0.885	1.3	3.640
0.4	1.072	1.4	4.024
0.5	1.279	1.5	4.427
0.6	1.505	1.6	4.850
0.7	1.751	1.7	5.293
0.8	2.016	1.8	5.755
0.9	2.302	1.9	6.238

a. Predict the shape of a scatterplot of this data.

b. Create a scatterplot of the data and compare its shape with your prediction.

c. Determine an appropriate regression model for the data and graph it on the scatterplot from Part **b**.

d. Use your model to determine the distance the ball has traveled after 1 sec, 2 sec, 3 sec, and 4 sec. Approximate the ball's instantaneous velocity at each of these points.

e. Use your results in Part **d** to create a scatterplot of velocity versus time.

f. Determine an appropriate model for the scatterplot in Part **e**.

g. What does your model in Part **f** indicate about the ball's change in velocity with respect to time?

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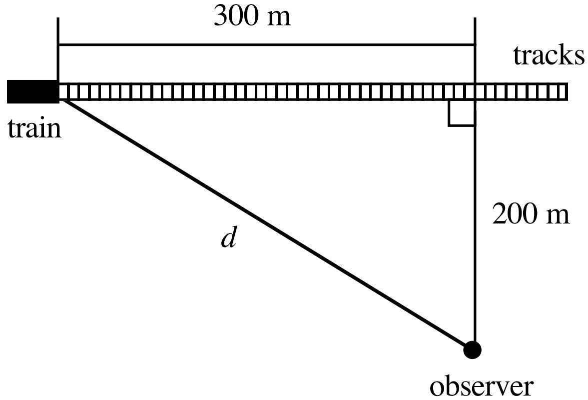
2.5 As a train sounding its whistle approaches and passes an observer, the pitch of the sound changes. (This is known as the Doppler effect.)

a. The following diagram shows the locations of the observer and the train at the moment when the train was first heard ($t = 0$ sec). What is the distance d between the observer and the train at this time?



b. The train is traveling at a constant velocity of 30 m/sec. When will it be closest to the observer?

c. Write an equation that describes the distance from the train to the observer with respect to time.

d. Do you believe that the rate at which the distance between the observer and the train changes is constant? Explain your response.

e. Graph the equation from Part **c** and explain whether or not the graph confirms your response to Part **d**.

f. Approximate the instantaneous rate of change in the distance from the observer to the train at $t = 5$ sec and at $t = 10$ sec.

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Activity 3

In the previous activity, you used the average velocity over smaller and smaller intervals to approximate instantaneous velocity. In this activity, you investigate a method of precisely determining a rate of change at any given instant.

Exploration

Figure **4** shows a portion of a graph of a semicircle in the first quadrant, a point on the semicircle $(x, f(x))$, and two secant lines *u* and *t*. As described in Activity **2**, the slope of the line tangent to the curve at $(x, f(x))$ can be approximated using the slope of *u* or *t*. Better approximations of the slope are found for values of *h* near 0.

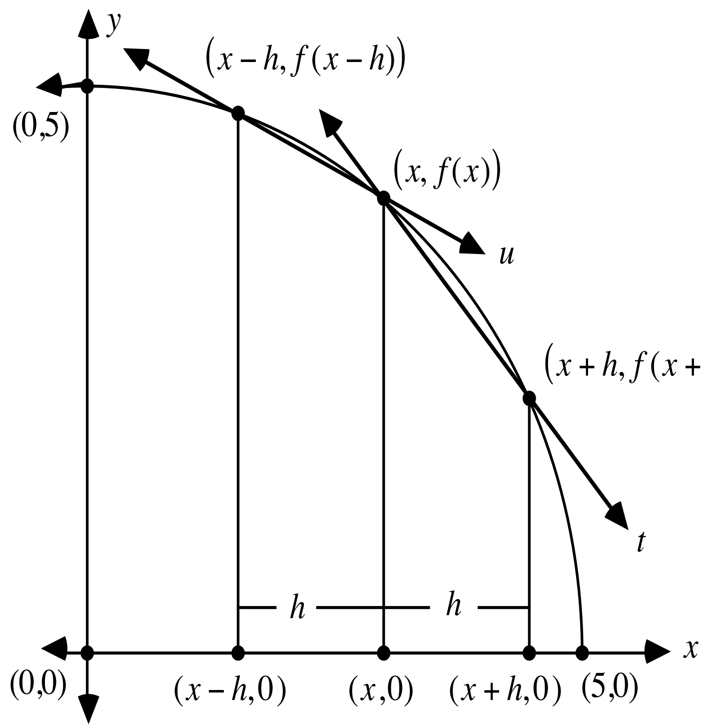

Figure 4: Approximating the slope of a tangent line

a.

1. Construct a circle with center at the origin of a two-dimensional coordinate system and a radius of 5 units.
2. As shown in Figure **4**, construct a point $(x, f(x))$ on the circle. Record its coordinates.
3. Construct a line perpendicular to the *x*-axis through $(x, f(x))$. Label the point of intersection with the axis $(x,0)$.

b. **1.** Construct a moveable point $(x+h,0)$ on the x-axis between $(x,0)$ and (5,0).

2. Construct the point $(x-h,0)$ by reflecting the point $(x+h,0)$ in the line created in Part **a**. (This guarantees that the distance from $(x+h,0)$ and $(x-h,0)$ to $(x,0)$ will be the same, h units.)

c. **1.** The point $(x+h, f(x+h))$ is the intersection of the circle with the line perpendicular to the x-axis passing through $(x+h,0)$. Construct the point $(x+h, f(x+h))$.

2. The point $(x-h, f(x-h))$ is the intersection of the circle with the line perpendicular to the x-axis passing through $(x-h,0)$. Construct the point $(x-h, f(x-h))$.

d. **1.** Construct two secant lines: one passing through $(x, f(x))$ and $(x+h, f(x+h))$, the other through $(x, f(x))$ and $(x-h, f(x-h))$.

2. Measure and record the slopes of the two secants.

e. To decrease the size of h, move the point $(x+h,0)$ toward the point $(x,0)$. As you move $(x+h,0)$, the point $(x-h,0)$ should move the same distance towards $(x,0)$.

As the points move closer together, the secant lines approach the tangent line through the point $(x, f(x))$. Observe the measures of the slopes of the secant lines as h approaches 0.

Use your construction to approximate the slope of the line tangent to the circle at $(x, f(x))$.

f. Use the construction to approximate the slopes of the lines tangent to three other points on the circle.

Discussion

a. If $f(x)$ represents distance and x represents time, how is the slope of a secant line related to average velocity?

Mathematics Note

Figure **5** shows a secant line passing through two points on the graph of a function $f(x)$. Note that *h* is depicted as a positive real number.

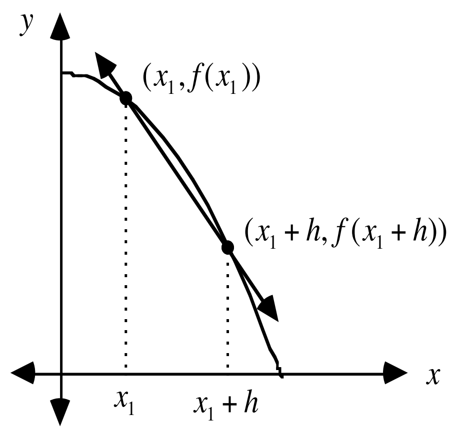

Figure 5: A secant to a curve

The slope of the secant line can be expressed as follows:

$$\frac{f(x_1 + h) - f(x_1)}{h}$$

By assigning a value to *h* that is close to 0, the point with coordinates $(x_1 + h, f(x_1 + h))$ can be moved very close to the point $(x_1, f(x_1))$. When this occurs, the slope of the secant line is a good approximation of the slope of a tangent line at $(x_1, f(x_1))$. A better approximation can be obtained by assigning *h* a value even closer to 0. This process can be repeated indefinitely and provides the basis for the following definition.

The slope of the tangent line at $(x, f(x))$ is defined as:

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided that this limit exists. If this limit exists, it is the **derivative** of the function $f(x)$ at x and represents the **slope of the curve** at the point $(x, f(x))$. The derivative of *f* at x is denoted by $f'(x)$. The slope of the curve at a point is equal to the slope of the tangent at that point.

For example, consider an object dropped from a height of 10 m. The height of the object with respect to time is described by the function $f(t) = -4.9t^2 + 10$, where *t* represents time in seconds. In this case, the slope of the curve at the point $(t, f(t))$ represents the instantaneous velocity at *t*.

To find the function that describes instantaneous velocity with respect to time, you can determine the derivative $f'(t)$ as follows:

$$
\begin{aligned}
f'(t) &= \lim_{h\to 0}\left(\frac{f(t+h)-f(t)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{-4.9(t+h)^2+10-\left(-4.9t^2+10\right)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{-4.9t^2-9.8ht-4.9h^2+10+4.9t^2-10}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{-9.8ht-4.9h^2}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{h(-9.8t-4.9h)}{h}\right)\\
&= \lim_{h\to 0}(-9.8t-4.9h)\\
&= -9.8t
\end{aligned}
$$

b. On Earth, the height of a freely falling object after t sec can be described by the equation $h(t) = -4.9t^2 + v_0 t + h_0$, where v_0 is the object's initial velocity in the vertical direction, and h_0 is the object's initial height.

In this situation, what does the derivative $h'(t) = -9.8t + v_0$ represent?

c. Consider the linear function $f(x) = -2x + 3$. Explain why $f'(x) = -2$.

d. Figure **6** below shows a graph of height versus time for an object propelled upward with an initial velocity of 9.8 m/sec from a height of 196 m. Explain why $h'(t) = 0$ when the object reaches its highest point.

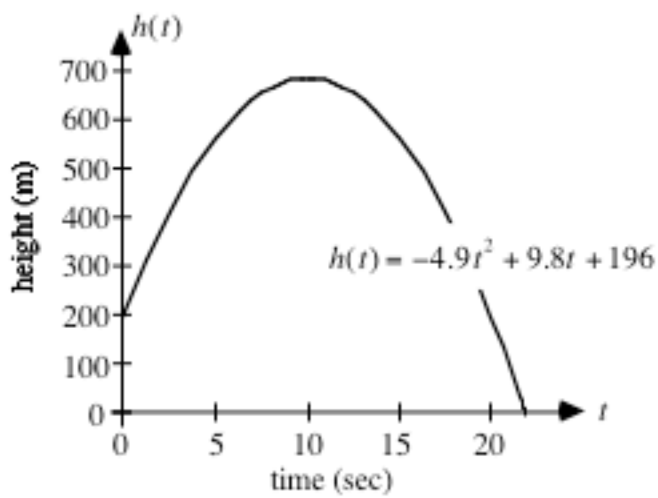

Figure 6: Graph of $h(t) = -4.9t^2 + 9.8t + 196$

Assignment

3.1 **a.** Use the definition given in the previous mathematics note to find the derivative of the linear equation $f(x) = 3x - 5$.

b. How is the derivative of $f(x)$ related to the slope of the line?

c. Compare the degree of $f(x)$ with the degree of its derivative $f'(x)$.

3.2 The regression equation $h(t) = -4.65t^2 - 0.14t + 1.82$ was used to model data collected in a ball-drop experiment.

a. Find its derivative $h'(t)$.

b. Graph $h(t)$ and $h'(t)$ on the same coordinate system.

c. Describe what the coordinates of corresponding points on each graph represent in terms of the falling ball.

d. Compare the degree of $h(t)$ to the degree of its derivative $h'(t)$.

3.3 Consider a quadratic function whose derivative is $f'(x) = 2x - 7$.

a. For what value of x does the slope of the graph of $f(x)$ equal 0?

b. Over what interval are the values of $f(x)$ increasing?

c. Over what interval are the values of $f(x)$ decreasing?

3.4 **a.** Use the definition given in the previous mathematics note to find the derivative of $f(x) = x^2$.

b. Use a symbolic manipulator to verify your response to Part **a**.

c. Find the slope of the graph of $f(x)$ at $x = -3$ and at $x = 15$.

d. For what value of x is the slope of the graph 0? What is the significance of the point that corresponds to this value of x?

3.5 Consider the function $f(x) = 2x^3 - 3x^2 + 4x + 2$.

a. To determine the derivative of this function, expand the expression below, then evaluate the limit of this expression as h approaches 0:

$$\frac{f(x+h) - f(x)}{h}$$

b. Use technology to find the derivative of $f(x)$ and compare it to the limit found in Part **a**.

c. Compare the degree of $f(x)$ to its derivative $f'(x)$.

3.6 **a.** The graph of a semicircle with center at the origin and radius 5 is defined by the function $f(x) = \sqrt{25 - x^2}$. Use technology to find the derivative of this function.

b. Evaluate the derivative at $x = 1.5$, $x = 2.5$, and $x = 3.5$.

c. Recall that a radius of a circle is perpendicular to the tangent at the point of tangency. On a coordinate plane, two lines are perpendicular if the product of their slopes is –1.

Use these facts to demonstrate that the derivative is the slope of the tangent at the three points described in Part **b**.

3.7 **a.** Write a function $f(r)$ that describes the volume of a cylinder in terms of the radius r when its height is twice the radius.

b. Graph the function found in Part **a**.

c. What does $f'(r)$ represent?

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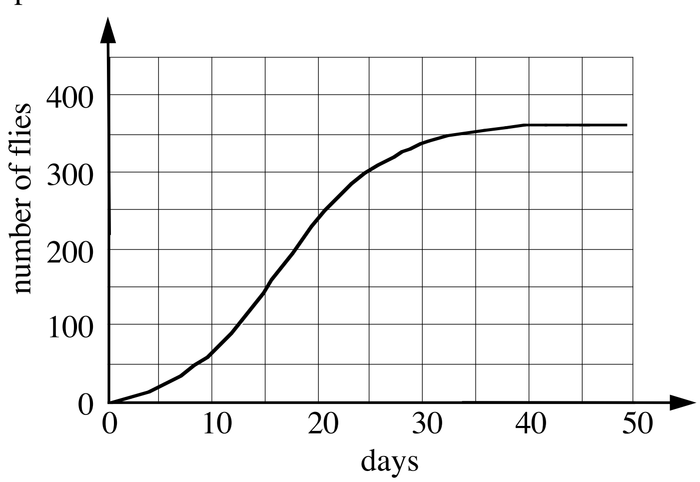
3.8 Consider the general linear equation $f(x) = mx + b$.

a. Use the definition of a derivative to find $f'(x)$.

b. What is the mathematical meaning of $f'(x)$ for a linear equation?

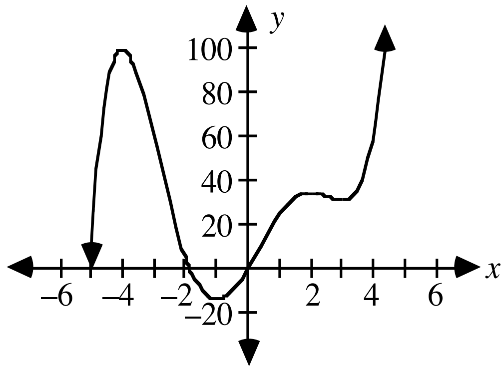
3.9 The following graph models the number of fruit flies in a laboratory population over time.



a. Determine the average rate of change in the fruit fly population from day 15 to day 35.

b. Determine the instantaneous rate of change in the population on day 25.

3.10 Consider the graph of the function $f(x) = 0.2x^5 - 5x^3 + 5x^2 + 24x$:



a. Describe the intervals on the graph where the slopes of the tangents to the curve are:

1. positive

2. negative.

b. Use the intervals you described in Part **a** to predict the points for which the derivative of the function is 0.

c. Use technology to determine the derivative of the function, then test your predictions from Part **b**.

d. Describe the significance of the points where the slope of the tangent to the curve is 0 in terms of the graph of the function.

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Summary Assessment

Understanding rates of change is important in business and economics. To most companies, for example, the bottom line means profit. The relationship between profit (P), sales or revenue (R), and costs (C) can be described by the equation $P = R - C$. For manufacturers, these quantities can be described by functions in terms of the number of units (x) produced and sold.

The derivative of a profit function determines the instantaneous rate of change in the profit for a given number of units. This is known as *marginal profit*. Similarly, the derivative of a cost function determines marginal cost, the instantaneous rate of change in the cost for a given number of units. In the same manner, the derivative of the revenue function determines the marginal revenue.

1. Consider a company that makes only one product. The cost of producing x of these items can be described by the function $C(x) = x^2 + \$80,000$ for $x > 0$.

- **a.** The company sells each item it produces for \$750. Write an expression for R in terms of x.
- **b.** Given that $P = R - C$, write an expression for P in terms of x.

2.

- **a.** Graph $P(x)$, $R(x)$, and $C(x)$ on the same coordinate system.
- **b.** Explain what the graphs show about the relationship between profit, revenue, and cost for the company.

3. Determine the equations for the marginal profit, the marginal revenue, and the marginal cost.

4. Since the derivative of a cost function for a particular x is the instantaneous rate of change in the cost, the marginal cost approximates the additional cost of producing one more item $(x + 1)$. If the company has already produced 200 items, what is the additional cost of producing the 201st item?

5. Determine the number of items for which the company will receive the maximum profit. Compare the marginal revenue and marginal cost for this number of items.

6. Imagine that you are a business consultant. How many items would you advise the company to produce? Explain your response.

Module Summary

- A **secant** is a line that intersects a curve but is not tangent to it.
- The **derivative** of a function at the point $(x, f(x))$, denoted by $f'(x)$, is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  This value is the slope of the tangent line to the function at $(x, f(x))$ and represents the instantaneous rate of change in the function with respect to x.

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