Total Chaos

The world is full of unpredictable and ever-changing systems, including the weather, the stock market, and animal populations. In this module, you investigate chaos theory, a relatively new branch of mathematics that studies the behavior of dynamical systems.

Byron Anderson • Sandy Johnson • Lisa Wood

Total Chaos

Introduction

Imagine a video camera standing on a tripod with its lens facing a television. The camera records a picture of the television screen. The signal for that picture is fed back to the television, where it appears on the screen as an image.

The camera then records an image of this image, and transmits it back to the television, where it appears on screen, and the cycle begins again.

If this process—known as video feedback—continues indefinitely, what would you expect to see on the television screen?

Mathematics Note

Recursion is a repetitive process that uses the output from one sequence of operations or instructions as the input for the next iteration of the same operations.

In general, *iteration* means repetition. Recursion is an iterative process. However, an iterative process is not always recursive.

Figure **1** shows a diagram of a recursive process.

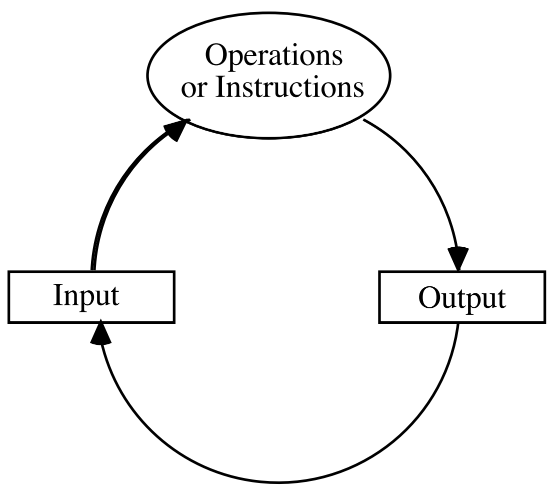


Figure 1: The recursive process

Each complete cycle of the process is a **stage**. For example, Figure **2** shows the result of the iterative process that occurs when an image is reflected between two parallel mirrors. Each individual image is a stage.

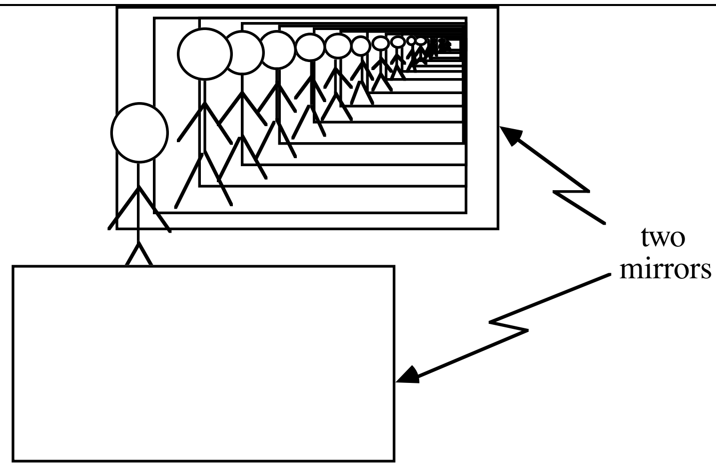


Figure 2: Two mirrors showing the iterative process

Discussion

a. Describe the input and output for video feedback.

b. Describe the process of video feedback.

c. How are the reflections in a set of parallel mirrors similar to the images formed by video feedback?

Activity 1

The word *fractal* was coined by mathematician Benoit Mandelbrot in 1975. He used the term to designate a set that may require three parameters to describe it: length, fractional (or fractal) dimension, and chance. Figure **3** shows an image of a Mandelbrot set, the fractal that bears his name.

Figure 3: A Mandelbrot set

Because of their complexity, fractals can be used to model the geometry of many natural objects, such as mountain ranges, shorelines, and the root systems of plants. For example, Figure **4** shows the image of a fern leaf defined by a fractal.

Like many other fractals, this one exhibits the property of **self-similarity**. In other words, a small portion of the set is similar to the whole set of which it is a part.

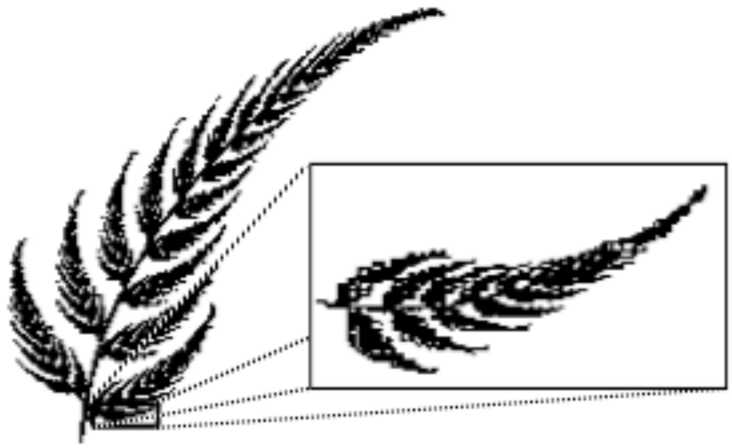


Figure 4: Fractal model of a fern leaf

Exploration

Though Mandelbrot was the first to publish a comprehensive theory of fractals, other mathematicians had been aware of their interesting geometry for many years. In this activity, you explore the initial stages of two classic fractals: the Koch curve, introduced by Helge von Koch in 1904, and the hat curve.

a. The shape of the Koch curve at its initial stage, or **stage 0**, is a line segment.

1. Draw a line segment.
2. Divide the segment into thirds.
3. Draw an equilateral triangle whose base is the middle third of the segment.
4. Remove the base of the triangle to obtain the shape shown in Figure **5**. This is stage 1 of a Koch curve.

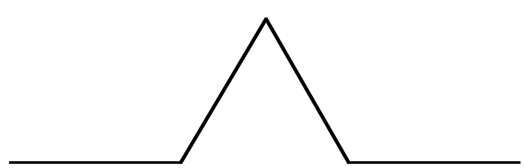


Figure 5: Stage 1 of a Koch curve

5. To produce stage 2 of a Koch curve, repeat the process described in Steps **2–4** on each line segment from stage 1.
6. Continue using the process described in Steps **2–4** on segments from previous stages until you reach stage 4.

b. Figure **6** shows the first three stages of another fractal, the hat curve. As in the Koch curve, stage 0 is a line segment. The shape displayed at stage 1 illustrates the iterative process that is performed on each line segment at each successive stage.

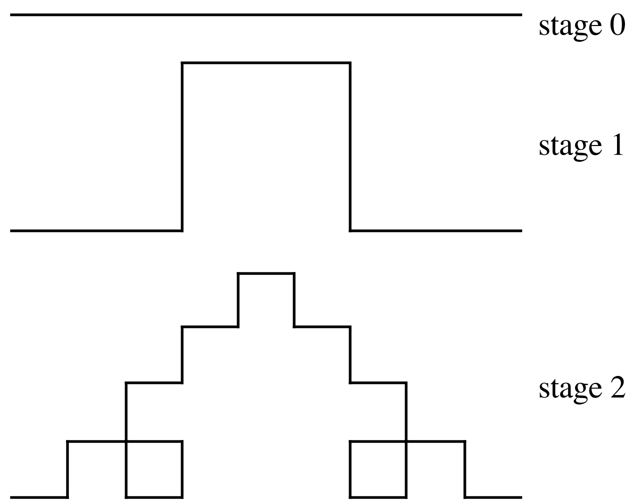


Figure 6: The first three stages of the hat curve

1. Use the pattern of the first three stages of the hat curve to describe the process used to draw stage 3.
2. To help confirm that the process you described is correct, draw stage 3 of the hat curve.

Discussion

a. As noted earlier, one feature common to many fractals is self-similarity under magnification. If your drawings of the Koch curve and the hat curve could be continued indefinitely, do you think the results would be self-similar? Explain your response.

b. As the number of stages increases without bound, describe the limit (if it exists) of the total length of each of the following:

1. a Koch curve
2. a hat curve.

c. How many times do you repeat the appropriate process to reach stage n of a fractal?

Assignment

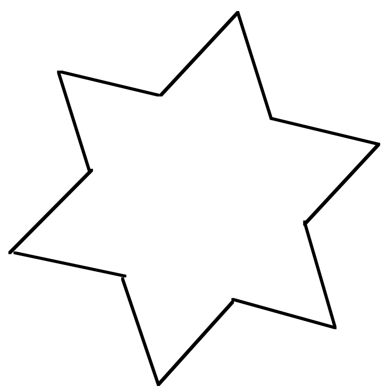
1.1 In the previous exploration, you constructed the first few stages of a Koch curve and a hat curve. Complete Parts **a** and **b** below for each of these fractals.

a. Assuming that stage 0 is 1 unit long, determine the lengths of stages 1–3 and record them in a table like the one below.

Stage	Length
0	1
1	
2	
3	

b. If possible, determine the length of the fractal after *n* stages.

1.2 Stage 0 of the Koch snowflake is an equilateral triangle. Stage 1 of the Koch snowflake is an equilateral triangle whose sides are stage 1 Koch curves, as shown below:

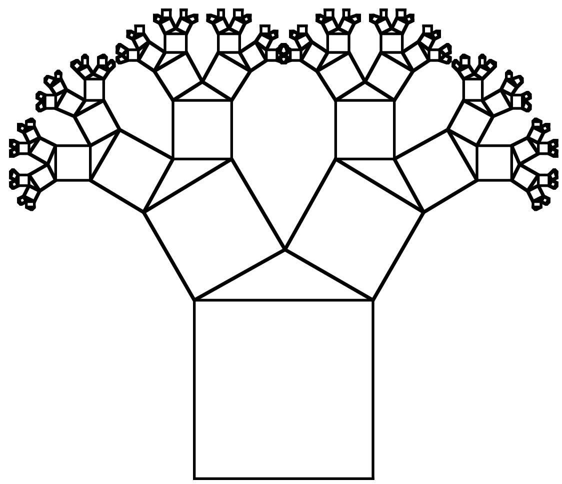


a. Make a sketch of stage 3 of the Koch snowflake.

b. As the number of stages increases without bound, what happens to the perimeter of the Koch snowflake? Explain your response.

c. As the number of stages increases without bound, what happens to the area of the Koch snowflake? Explain your response.

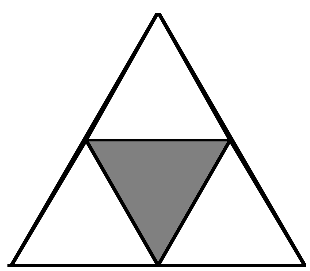
1.3 Describe the initial stage and iterative process used to generate the figure below.



1.4 Polish mathematician Waclaw Sierpinski (1882–1969) first described the fractal now known as Sierpinski's triangle. In the following assignment, you construct the initial stages of Sierpinski's triangle and discover one of its unusual properties.

a. **1.** Construct an equilateral triangle. This is stage 0 of the fractal.

2. Locate and mark the midpoints of each side of the triangle. Construct the line segments that connect the midpoints, creating four equilateral triangles. Shade the "middle" triangle to indicate that it should be removed from the picture. Your drawing should now resemble the figure below. This is stage 1 of the fractal.



3. To obtain the next stage in the fractal, repeat the process described in Step **2** on each remaining unshaded triangle.

4. Continue this iterative process through at least stage 3.

b. Write a sequence that describes the number of unshaded triangles at each stage of Sierpinski's triangle. Describe the limit (if any) to this sequence as the number of stages increases without bound.

c. Use a sequence to describe the perimeter of the fractal at each stage. What is the limit of this sequence as the number of stages increases without bound?

d. The area of the fractal at each stage is equal to the sum of the areas of the unshaded triangles. What happens to the area of the fractal as the number of stages increases without bound?

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1.5 **a.** Design a recursive process that could be used to create a fractal, beginning with a line segment.

b. Draw stage 3 of your fractal.

c. Determine the length of the fractal after *n* stages.

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Research Project

Figure **7** shows a portion of Pascal's triangle. Recall that each row of Pascal's triangle begins and ends with the term 1. All other terms are found by adding the term above and to the left to the term above and to the right. In row 4, for example, $4 = 1 + 3$ and $6 = 3 + 3$.

1	Row 0
1 1	Row 1
1 2 1	Row 2
1 3 3 1	Row 3
1 4 6 4 1	Row 4
1 5 10 10 5 1	Row 5

Figure 7: A portion of Pascal's triangle

Many computer-generated fractal images are obtained by systematically coloring a particular set. Using four copies of a template supplied by your teacher, explore the patterns you can obtain by coloring Pascal's triangle according to the instructions given in each of Parts **a–d**.

a. Shade cells that correspond with even numbers in Pascal's triangle one color. Shade cells that correspond with odd numbers in Pascal's triangle another color.

b. Shade cells that correspond with numbers in Pascal's triangle that are divisible by 3 one color. Shade cells that correspond with all other numbers in Pascal's triangle another color.

c. Shade cells that correspond with numbers in Pascal's triangle that are divisible by 5 one color. Shade cells that correspond with all other numbers in Pascal's triangle another color.

d. Shade cells that correspond with numbers in Pascal's triangle that are divisible by 9 one color. Shade cells that correspond with all other numbers in Pascal's triangle another color.

Describe your findings in a report, including a comparison of your results with the model of Sierpinski's triangle created in Problem **1.4**.

Activity 2

In Activity **1**, you used iterations of geometric processes to produce fractals. In this activity, you investigate the effects of iteration on functions.

Exploration 1

Iterating a function involves using the output of one calculation of the function as the input for the next calculation of the function. Iteration of a simple function can be used to generate a sequence such as x_0, $f(x_0)$, $f(f(x_0))$, …, where x_0 is the **initial value**. This type of iterative sequence is an **orbit** of x_0.

a. Using $f(x) = \sqrt{x}$ and an initial value of 1, calculate the first few terms of the orbit.

> **Mathematics Note**
>
> A **fixed point** *p* of a function $g(x)$ is a value such that $g(p) = p$.
>
> For example, $p = 3$ is a fixed point of $g(x) = (2/3)x + 1$ because $g(3) = 3$. Notice also that $g(g(3)) = 3$, $g(g(g(3))) = 3$, and so on.

b. **1.** Explain why 1 is a fixed point of $f(x) = \sqrt{x}$.

2. Identify any other fixed points of $f(x)$.

c. Using three different initial values greater than 1, list the first 15 terms of their respective orbits.

d. Repeat Part **c** for three other initial values between 0 and 1.

e. Compare the six orbits you found in Parts **c** and **d**.

Discussion 1

a. Given that *p* is a fixed point for a function, describe the orbit which uses *p* as the initial value.

b. What are the fixed points for $f(x) = -x^3$?

c. **1.** When an orbit approaches a fixed point, that fixed point is considered an **attractor**. Why is 1 an attractor for $f(x) = \sqrt{x}$?

2. Does 0 also appear to be an attractor for this function? Explain your response.

Exploration 2

In this exploration, you observe the behavior of orbits for several different functions.

a. Using $f(x) = \cos x$ and $x_0 = 2$, where x is an angle measure in radians, find the first 20 terms of the orbit.

b. Create a connected scatterplot of the term value versus the term number.

c. Experiment with different initial values for $f(x) = \cos x$. What changes, if any, do you observe in the term values of the orbits?

d. Does the orbit of $f(x) = \cos x$ appear to approach a limit? If so, use that value as the initial value. Record your results.

e. Identify all the fixed points of $f(x) = \cos x$. Demonstrate graphically that your list is complete.

f. Repeat Parts **a–d** using $f(x) = \sin x$.

g. Repeat Parts **a–d** using $f(x) = 0.5x + 5$.

h. Repeat Parts **a–d** using $f(x) = 2x + 3$.

Discussion 2

a. Which orbits in Exploration **2** have attractor points?

b. Which of the functions in Exploration **2** do not appear to approach a limit under iteration? Do these functions have fixed points?

c. Why must a function that has a fixed point intersect the line $y = x$?

d. Do you think that there are any functions which do not have a fixed point under iteration? Explain your response.

Assignment

2.1 **a.** Using the function below and an initial value of your choice, generate an orbit until an attractor can be identified.

$$f(x) = -\frac{1}{3}x + 12$$

b. Confirm your answer to Part **a** by finding the fixed point for $f(x)$ algebraically.

2.2 Consider the iteration of the function below with $x_0 = 2$:

$$f(x) = \frac{1}{3}x + b$$

a. Investigate the behavior of the orbits for four different values of b. You may wish to organize your data in a table like the one below.

Value of *b*	Orbit	Behavior of Orbit

b. What characteristic do the orbits in Part **a** have in common?

c. How is the limit of an orbit related to the value of b?

d. Confirm your response to Part **c** by finding an expression for the fixed point in terms of b.

2.3 Find an expression for the fixed point of the general linear function $f(x) = ax + b$, where a does not equal 0 or 1.

2.4 **a.** Find the fixed points with real values for the quadratic function $f(x) = ax^2 + b$, where $a \neq 0$.

b. Is it possible to determine a real fixed point for every value of a and b? Explain your response.

c. What restrictions are necessary on a and b in order to find real fixed points? Justify your response. As part of your justification, use graphs to show why these restrictions are necessary.

2.5 Find the fixed point of $f(x) = 19x + 6$.

2.6 Identify the fixed points of $f(x) = |x|$.

2.7 Determine the two complex fixed points of $f(x) = x^2 + 1 + i$.

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2.8 Describe how the balance in a savings account with an initial investment of $100 and an annual interest rate of 12% could be determined using an iterative process.

2.9 Determine a recursive formula for the following sequence:

$$4,\ \frac{1}{4},\ 4,\ \frac{1}{4},\ 4,\ \frac{1}{4},\ \ldots$$

2.10 Determine the fixed points for $f(x) = x!$

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Activity 3

In many cases, the graph of a function can help you identify its fixed points and characterize the behavior of its orbits. In this exploration, you investigate the different paths that occur for initial values near a fixed point.

Exploration

In this exploration, you examine four lines that have the same *y*-intercept but different slopes, then develop a conjecture about the behavior of orbits around fixed points. To create a geometric picture of the orbit of a linear equation, you graph the linear equation along with the equation $y = x$.

a. To produce a geometric picture of the orbit generated by $f(x) = 2x + 3$, complete the following steps on a sheet of graph paper.

1. Graph $f(x)$ and the line $g(x) = x$.
2. Identify the coordinates of the intersection of the two graphs, then select an initial value (x_0) greater than the *x*-coordinate of the intersection.
3. Draw a vertical line segment from x_0 on the *x*-axis to $f(x)$.
4. Draw a horizontal line segment from the intersection of the segment from the previous step and $f(x)$ to $g(x) = x$.
5. Draw a vertical line segment from the intersection of the segment from Step **4** and $y = x$ to $f(x)$.
6. Repeat the process described in Steps **4** and **5**.
7. Describe the path of the orbit, as represented by the segments, in relation to the fixed point.

b. Select an initial value less than the *x*-coordinate of the intersection of $f(x)$ and $y = x$. Repeat Steps **3–7** of Part **a**.

c. Use appropriate technology to repeat Parts **a** and **b** for each of the following functions:

1. $f(x) = 0.5x + 3$
2. $f(x) = -3x + 3$
3. $f(x) = -0.4x + 3$.

Discussion

a. In Part **a** of the exploration, what process is illustrated by drawing a vertical segment to $f(x)$, followed by a horizontal one to the line $y = x$?

b. Describe any similarities you observed in the paths of the orbits for $f(x) = 2x + 3$ and $f(x) = 0.5x + 3$.

c. Describe any similarities you observed in the paths of the orbits for $f(x) = -3x + 3$ and $f(x) = -0.4x + 3$.

d.
1. Which of the functions in the exploration generated orbits that appear to approach a fixed point?
2. What differences did you observe in the paths of the orbits?
3. Compare the coefficients of the functions whose orbits approach fixed points.

e.
1. Which of the functions in the exploration generated orbits that moved away from a fixed point?
2. What differences did you observe in the paths of the orbits?
3. Compare the coefficients of the functions whose orbits do not approach fixed points.

Mathematics Note

A fixed point that is approached by an orbit in the plane is an **attractor**. A fixed point from which an orbit in the plane moves away is a **repeller**.

Web plots model the paths created by a function's orbit. If a fixed point is an attractor, then the path may approach it in either a **staircase** or a **spiral**. If a fixed point is a repeller, then the path may move away from it in either a staircase or a spiral.

Figure **8** shows an example of a path that approaches an attractor in a spiral.

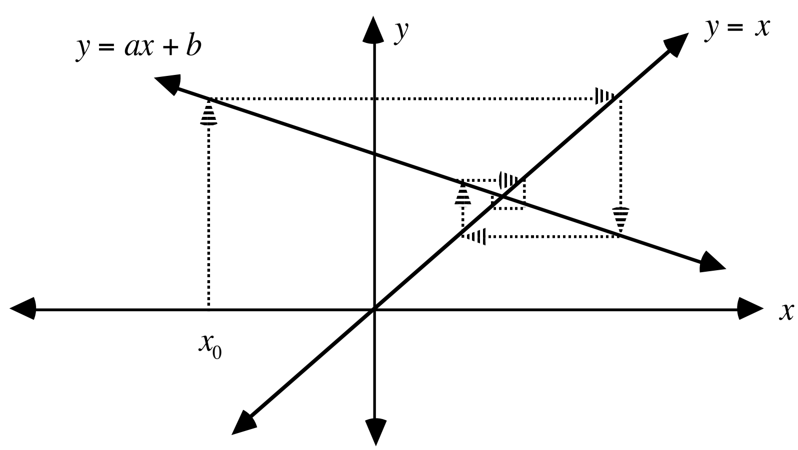

Figure 8: A web plot that spirals in

Figure **9** shows an example of a path that moves away from a repeller in a staircase.

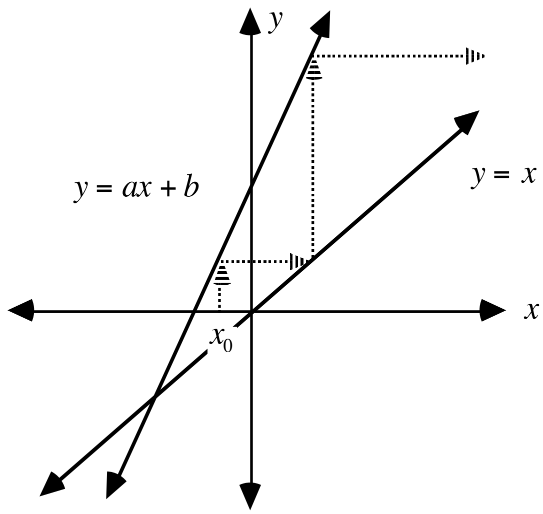

Figure 9: A web plot that staircases out

f. Characterize each fixed point for the orbits in the exploration as an attractor or repeller. Justify your responses.

g. In Exploration **2** of Activity **2**, you observed the behavior of orbits for four different functions and identified those that appeared to approach a limit. How do your observations in that exploration relate to the presence of repellers and attractors?

h. What is the fixed point of the function $f(x) = ax + 3$, where $a \neq 0$?

i. Predict the orbits of $f(x) = ax + 3$ for each of the following:

1. $a < -1$

2. $-1 < a < 1$

3. $a > 1$

Assignment

3.1. Consider the function

$$f(x) = -\frac{1}{3}x + 12$$

a. Find the fixed point of the function.

b. Is this fixed point an attractor or a repeller? Justify your response.

3.2 **a.** Use a spreadsheet or graphing utility to investigate the behavior of the orbits of $f(x) = -1x + b$ for at least three different values of b. For each value of b, experiment with at least two different initial values. Record your results in a table like the one below.

Value of b	**Initial Value x_0**	**Orbit**

b. Write a summary of your findings, including a description of the fixed point and how it was determined, and a description of the orbits for various values of b.

Be sure to mention any common characteristics of the orbits, as well as any differences they exhibit when compared to orbits of a function of the form $f(x) = ax + b$, where $a \neq -1$.

Mathematics Note

An orbit is **periodic** if it eventually cycles between two or more values. If the orbit cycles between n different values, the orbit has a cycle of period n.

For example, given $f(x) = -x + 5$ and the initial value $x_0 = -2$, the orbit is $-2, 7, -2, 7, \ldots$. Since the orbit cycles between two different values, –2 and 7, the orbit has a cycle of period 2. In fact, every orbit for this function—with the exception of those with the fixed point as the initial value—has a cycle of period 2.

3.3 In a paragraph, characterize the fixed point of the general linear function $f(x) = ax + b$, where $a \neq 0$, as an attractor, a repeller, or neither for various values of a.

3.4 Use the function $f(x) = a/x$, where $a \neq 0$, $x \neq 0$, and $x_0 = 1$, to complete Parts **a–d**.

a. Use a spreadsheet to find the orbits for several values of a in the interval $-5 < a < 5$.

b. Find the fixed point(s) algebraically.

c. For what values of a are the terms of the orbit identical?

d. For what values of a are the orbits periodic? Describe the period for each of these orbits.

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3.5 Consider the function $f(x) = 16/x^3$, where $x \neq 0$.

a. Find the fixed points algebraically. Hint: There are two real solutions and two imaginary solutions.

b. Verify that all four solutions are fixed points.

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Activity 4

Dynamical systems are systems that change over time. Such systems often involve iteration. The study of the behavior of dynamical systems under repeated iteration is known as **chaos theory**.

Making predictions about complicated systems can be a difficult, if not impossible, task. In this activity, you examine the orbits of several nonlinear functions which become unpredictable and chaotic.

Exploration

In this exploration, you use the parent function $f(x) = ax(1 - x)$, where $a \neq 0$, to investigate the iteration of selected nonlinear functions.

a. To generate orbits for the function $f(x) = ax(1 - x)$, where $a \neq 0$, create a spreadsheet with headings like those in Table **1**. Design your spreadsheet so that changing either the value of *a* or the initial value results in updated values for orbit and output.

Table 1: Sequence of terms for iteration on $f(x) = ax(1 - x)$

a	2	
x_0	0.1	
Term No.	**Orbit**	**Output**
1		0.1800
2	0.1800	0.2952
⋮	⋮	⋮

b. **1.** Determine the orbit of the function when $a = 2$ using $x_0 = 0.1$. Graph the function and create a web plot.

2. Complete Table **2** below. Record the fixed points, the number and values of the attractors, and the behavior of the orbits (spiral in, staircase out, and so on).

Table 2: Orbits for the function $f(x) = ax(1 - x)$

a	Fixed Point(s)	Number of Attractors	Value of Attractor(s)	Behavior of Orbit
2.0	0, 0.5000	1	0.5000	staircase in
2.2				
2.4				
2.6				

c. Use your table to make predictions about the number of attractors, the values of the attractors, and the behavior of the orbits as a increases.

d. To test your predictions, extend Table **2** to include three additional values of a: 2.8, 3, and 3.2.

e. Summarize the results of your tests from Part **d**.

f. Re-evaluate your predictions about the number of attractors, the values of the attractors, and the behavior of the orbits as a increases.

g. To test your revised predictions, extend Table **2** to include three additional values for a: 3.4, 3.6, and 3.8.

h. Summarize the results of your tests from Part **g**.

i. What do you expect to happen when the value of a is 4? Test your conjecture.

j. The orbit generated by the function $f(x) = 4x(1 - x)$ becomes stable when the initial value is 0, 0.5, 0.75, or 1. Graph the orbits that are generated for values of x_0 slightly greater than or less than each of these values. Describe any changes that occur in the graphs.

Mathematics Note

Dynamical systems may be considered **chaotic** if the following criteria are met.

- There is sensitive dependence on initial conditions. In other words, small changes can create vast differences in long-term behavior.
- The system appears to exhibit random behavior.

For example, the orbit of $0 < x_0 < 1$ for the function $f(x) = 4x(1 - x)$ displays chaotic behavior when $x_0 \neq 0.75$ or $x_0 \neq 0.5$ (the fixed points). When $x_0 = 0.75$, the orbit is stable: 0.75, 0.75, 0.75, 0.75, 0.75, 0.75, 0.75, … . With only a slight change, however, to $x_0 = 0.76$, the orbit is 0.76, 0.730, 0.789, 0.666, 0.890, 0.391, 0.952, 0.182, 0.596, … . This reveals both the system's sensitivity to initial conditions, and the appearance of random behavior.

Discussion

a. Were your predictions from Part **e** of the exploration supported or refuted by your tests? How did this affect your next predictions?

b. How did changing the value of a change the number of attractors?

c. How did the paths of the orbits change as the value of a increased?

d. In Part **j** of the exploration, how did small changes in the initial value affect the orbit generated by $f(x) = 4x(1 - x)$?

e. The motion of clouds in the sky is an example of chaotic behavior in a dynamical system. Describe other dynamical systems that display chaotic behavior.

Assignment

4.1 Consider the function $f(x) = x^2 + c$ and the initial value $x_0 = 0$. Analyze the orbits that occur given the following values of c: $0.1, -0.5, -0.8, -1.3, -1.37, -1.4, -1.76, -1.77, -1.95$.

Summarize the results of your analysis. If any of the orbits are periodic, describe the period and the attractors.

4.2 Using the function $f(x) = ax(1 - x)$, where $a \neq 0$, and an initial value of 0.1, find values of a in the interval $2.5 \leq a \leq 4.0$ (to the nearest hundredth) that result in each of the long-term behaviors in the table below.

Long-term Behavior	***a***
fixed point	
cycle of period 2	
cycle of period 3	
cycle of period 4	
cycle of period 5	
chaos	

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4.3 Consider the function $f(x) = x^2 + (-1 + 0i)$. Determine the behavior of the orbit when the initial value is $0 + 0i$.

4.4 Consider the function $f(x) = x^2 + (-1 - 1i)$. Determine the behavior of the orbit when the initial value is $0 + 0i$.

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Activity 5

The population of a given species in a particular habitat can be thought of as a dynamical system. In the mid-nineteenth century, Belgian mathematician Pierre François Verhulst developed a formula that could be used to model population dynamics. In Verhulst's formula, the growth rate of a population is proportional to the difference between its present size and the **carrying capacity** of the environment, where carrying capacity is the number of individuals that a given habitat can support.

In this activity, you use one version of Verhulst's formula to model population dynamics as an iterative process.

Science Note

In a very simple model of population growth, the population for the following year can be determined by the formula,

$$P_{t+1} = P_t + (P_t \bullet r)$$

where P_t is the population after t years and r is the annual percentage change in the population.

According to this model, populations that have positive growth rates could continue to increase indefinitely, which is unrealistic. A more complex model of population dynamics takes the carrying capacity of the habitat into account. In Verhulst's formula, for example, the growth rate depends on the percentage of the carrying capacity C unaccounted for by the current population:

$$\frac{C - P_t}{C}$$

This modified equation for population size—sometimes referred to as the **logistic model**—can be expressed as follows:

$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C}\right)$$

Exploration

To explore the application of the logistic model to population dynamics, imagine a population of 50 kangaroos in an Australian wildlife preserve. Considering the available land, food, and water, wildlife biologists have determined that the carrying capacity of the preserve is 150 kangaroos. In the past year, the annual growth rate r for the population was 1.9.

a. Using the logistic model and appropriate technology, predict the population of kangaroos in the preserve for each of the next 20 years.

b. Use the results of the iterative process from Part **a** to create a graph of population size versus time.

c. Investigate how the predicted behavior of the population is affected by changing the initial number of kangaroos, while maintaining the same growth rate of 1.9. Include some values greater than the carrying capacity in your experiment. Describe any changes you observe in the graphs of population size versus time.

d. Repeat Parts **a** and **b** for an initial population of 50 kangaroos and a growth rate of 2.1. Record any changes you observe in the graph of population size versus time.

e. Repeat Part **c** using a growth rate of 2.1. Describe any changes you observe in the graphs of population size versus time.

Discussion

a. What patterns did you observe in your graph from Part **b** of the exploration?

b. What factors might contribute to alternating high and low values in an actual animal population?

c. In Part **c** of the exploration, how did the predicted long-term behavior of the kangaroo population change for different values of the initial population?

d. How did the results you described in Part **c** differ from the predicted long-term behavior of the kangaroo population in Parts **d** and **e** of the exploration?

Assignment

5.1 Show that one of the fixed points for any population described by the logistic model is the carrying capacity.

5.2 The wildlife preserve described in the exploration also has a population of 125 wombats. The annual growth rate for this population is 1.25. Biologists believe that the preserve's carrying capacity is 500 wombats.

a. Using the logistic model, predict the population of wombats for each of the next 10 years and create a graph of population size versus time.

b. Would you describe the predicted behavior of the population as fixed, periodic, or chaotic? Explain your response.

5.3 **a.** Using each of the following annual growth rates, create a graph of population versus time over 30 years for an initial population of 125 wombats and a carrying capacity of 500 wombats.

For each growth rate, characterize the predicted behavior of the population as fixed, periodic, or chaotic.

1. $r = 2.25$
2. $r = 2.5$
3. $r = 2.55$
4. $r = 2.57$
5. $r = 2.75$

b. If the logistic model is accurate, what can you conclude about population dynamics from your findings in Part **a**?

5.4 Suppose that wildlife managers want to transplant some additional wombats to the preserve. Assume that the carrying capacity is still 500 and the annual growth rate is 1.25.

Using the logistic model, what is the largest possible initial population which does not result in extinction after 1 year?

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5.5 The following equation provides a possible model of the price of gasoline, where p_n is the price on day n, $D(p_n)$ is a function that models the demand for gas, and $S(p_n)$ is a function that models the supply of gas:

$$p_{n+1} = p_n + k(D(p_n) - S(p_n))$$

In this case, the constant k is some positive value.

a. When the supply and demand are equal, the price of the gasoline is the **equilibrium price**. Assume that the equations below represent the supply and demand functions, respectively:

$$S(p) = 900p - 500$$
$$D(p) = 1000 - 100p$$

Find the equilibrium price. Hint: This is the price when $S(p) = D(p)$.

b. Over time, the current market price of gasoline approaches the equilibrium price. Use the equation $p_{n+1} = p_n + k\big(D(p_n) - S(p_n)\big)$, with $k = 0.0001$ and $p_1 = \$1.80$, to determine when the market price will equal the equilibrium price.

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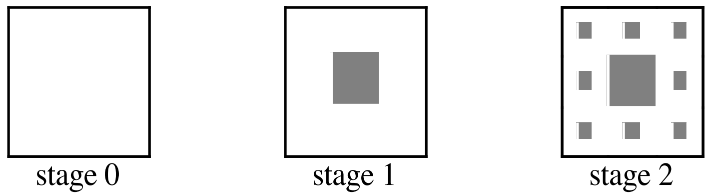
Summary Assessment

1. **a.** Describe the iterative process used to create the following stages of a fractal known as Sierpinski's carpet.



b. Recreate Sierpinski's carpet, showing at least stage 3.

c. Find the limit of the area of the shaded region of the fractal.

2. Design your own model of a fractal. Include a description of the iterative process you used to create your model, as well as a drawing of at least its first three stages.

3. Imagine that 50 koala bears are currently living in a habitat with a carrying capacity of 400 koala bears. Using the logistic model below—where P_t is the population after t years, r is the annual percentage change in the population, and C is the carrying capacity—create several graphs of predicted population size versus time.

$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C} \right)$$

Find at least one value for r that illustrates each of the following types of long-term behavior: fixed, periodic, and chaotic.

4. The iterative process has been a useful mathematical tool for many centuries. The iteration of simple arithmetic operations, for example, can often provide a good approximation of a more complicated operation. This type of iteration was especially useful prior to the development of the electronic calculator.

The function below is a combination of three simple arithmetic operations: multiplication, addition, and division.

$$f(x) = 0.5\left(\frac{a}{x} + x\right)$$

Iteration of this function can be used to approximate a more complicated function performed on an integer a. To investigate the operation this function approximates, complete Parts **a–g**.

a. Using each of the integers from 1 to 10 for a and an initial value of 0.1 for x, create the orbit generated by $f(x)$ until you can determine its limit, if one exists. If a limit exists, how does its value compare to the value of a?

b. When used in an iterative process, what operation does $f(x)$ appear to approximate?

c. Create the orbits generated by $f(x)$ using three different negative integers for a. Describe your results and explain their significance.

d. Predict what the function $g(x)$ below will approximate when used in an iterative process.

$$g(x) = 0.5\left(\frac{a}{x^2} + x\right)$$

e. Test your prediction from Part **d** using various integers for a and describe the results.

f. Repeat Parts **d** and **e** for the following function:

$$h(x) = 0.5\left(\frac{a}{x^3} + x\right)$$

g. Describe the possible uses of functions of the form below:

$$h(x) = 0.5\left(\frac{a}{x^n} + x\right)$$

Module Summary

- **Recursion** is a repetitive process that uses the output from one sequence of operations or instructions as the input for the next iteration of the same operations. Each complete cycle of the process is a **stage**.
- If a figure contains smaller replicas of the whole, then the figure is **self-similar**. Many fractals are self-similar.
- An **orbit** of a function *f* is an iterative sequence:

$$x_0 ,\ f(x_0) ,\ f\big(f(x_0)\big), \ldots$$

  where x_0 is the **initial value**.
- A **fixed point** *p* of a function $f(x)$ is a value such that $f(p) = p$.
- A fixed point that is approached by an orbit in the plane is an **attractor**. A fixed point from which an orbit in the plane moves away is a **repeller**.
- **Web plots** model the paths created by a function's orbit.
- If a fixed point is an attractor, then a path may approach it in either a **staircase** or a **spiral**. If a fixed point is a repeller, then a path may move away from it in either a staircase or a spiral.
- An orbit is **periodic** if it eventually cycles between two or more values. If the orbit cycles between *n* different values, the orbit has a cycle of period *n*.
- **Dynamical systems** are systems that change over time. Such systems often involve iteration. The study of the behavior of dynamical systems under repeated iteration is known as **chaos theory**.
- A dynamical system may be considered **chaotic** if the following criteria are met:
  1. There is sensitive dependence on initial conditions. In other words, small changes can create vast differences in long-term behavior.
  2. The system appears to exhibit random behavior.
- The **logistic model** for population growth can be expressed as follows:

$$P_{t+1} = P_t + P_t \bullet r \bullet \left(\frac{C - P_t}{C} \right)$$

  where P_t is the population at time *t*, *r* is the growth rate of the population, and *C* is the carrying capacity of the environment.

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