Naturally Interesting

How much money would you have to invest now in order to be a millionaire at age 65? In this module, you use exponentials and logarithms to answer this question.

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Introduction

Banks and investment companies offer a variety of accounts to help customers reach their financial goals. These accounts may offer different rates of interest, based on the initial amount invested. How much money would a person need to invest today in order to be a millionaire at age 65? How long will it take an investment to double in value? These are questions financial advisers must be able to answer for their clients.

In banking, **principal** refers to the amount of money invested or loaned. **Interest** is the amount earned on invested money, or the fee charged for loaned money. The amount of interest received or paid depends on three quantities: principal, interest rate, and time. Interest also varies according to the method used to calculate it. In the following activities, you investigate how savings accounts earn money.

Mathematics Note

One method for determining the amount of interest earned or owed involves **simple interest**. In this case, interest is paid or charged only on the original principal. The formula for calculating simple interest, where I represents interest, P represents principal, r represents the interest rate per time period, and t represents the number of time periods, is shown below:

$$I = Prt$$

To use this formula, t must be expressed in the same units as time in the interest rate, r. For example, if the interest rate is 5% per year, then t must be expressed in years. If \$1000 is invested at an annual interest rate of 5% for 3 yr, the interest earned can be calculated as follows:

$$I = 1000(0.05)(3) = \$150$$

Discussion

a. What types of loans are available in your community? What are the current interest rates and terms available for these loans?

b. Describe the opportunities available for investing or saving money in your community. What are the current interest rates and terms available?

c. Consider an investment account that offers an annual interest rate of 10%. If you invest \$1000 in this account, the interest earned after 1 yr is \$100. If you reinvest the **account balance** (the original principal plus the \$100 interest earned) at the same rate, how much interest will you earn in the second year?

Activity 1

Most savings accounts pay interest not only on the original principal, but also on the interest earned and deposited in any previous time periods. This is an example of **compound interest**. Each time compound interest is calculated, the interest earned is added to the principal. This sum (the account balance) becomes the new principal for the next interest calculation.

Exploration 1

In this exploration, you develop a method for determining the balance of an account that earns compound interest.

a. Imagine that you have invested \$500 at a simple interest rate of 6% per year and plan to make no withdrawals for the next 20 yr.

Use the formula for simple interest to determine the account balance after 20 yr.

b. When interest is compounded annually, the interest earned each year is added to the account at the end of that year. Predict the account balance after 20 yr if interest is compounded annually.

c. To determine the actual balance of the account after 20 yr when interest is compounded annually, you must examine what happens to the account balance at the end of each year.

1. Determine the account balance at the end of the first year by adding the interest earned for 1 yr to the principal.
2. Using the account balance at the end of the first year as the new principal, determine the account balance at the end of the second year.

d. Use a spreadsheet to repeat the process described in Part **c** for each of the next 18 yr. Record your data in a table similar to Table **1** below.

Table 1: \$500 invested at 6% for 20 yr, compounded annually

Years (t)	Principal at Beginning of Year (\$)	Account Balance at End of Year (\$)
1	500	
2		
3		
⋮		
20		

e. Use the spreadsheet to investigate how account balances are affected by changes in the interest rate. Record your observations.

f. Let P_t represent the principal at the end of t years in an investment with an interest rate of 6% per year, compounded annually. Write an expression that describes P_t in terms of the principal for the previous year. (In other words, write a recursive formula for the account balance after t years.)

Mathematics Note

The principal at the end of each time period in an investment or savings account can be thought of as a sequence.

For example, consider an initial principal of \$1000 invested at an interest rate of 8% per year, compounded annually. Assuming that no withdrawals are made and any interest earned is deposited in the account, the following geometric sequence is formed, where P_0 represents the initial principal, P_1 represents the principal after 1 yr, and so on:

$$\begin{aligned} P_0 &= 1000 \\ P_1 &= 1080 \\ P_2 &= 1166.40 \\ &\vdots \\ P_{10} &\approx 2158.92 \end{aligned}$$

In this case, the account balance at the end of 10 yr is approximately \$2158.92.

Such a sequence can be defined recursively by the following formula:

$$P_t = P_{t-1} + r \bullet P_{t-1} = P_{t-1}(1+r)$$

where P_t is the principal at the end of t years, r is the annual interest rate, and P_{t-1} is the principal for the previous year.

For example, given an initial principal $P_0 = \$345$ and an annual interest rate of 8%, the account balance at the end of 1 yr (P_1) can be determined as shown below (assuming that no withdrawals are made and any interest earned is deposited in the account):

$$P_1 = P_0 \bullet (1 + 0.08) = (345) \bullet (1 + 0.08) = \$372.60$$

g. To use the recursive formula given in the mathematics note, you must know the account balance in the previous year. An explicit formula, however, would allow you to find P_t without having to determine P_{t-1}.

1. Write P_1 in terms of P_0 (the original principal) and r (the annual interest rate).
2. Using substitution and the recursive formula, $P_t = P_{t-1}(1+r)$, determine an explicit formula for P_2, the principal at the end of 2 yr, in terms of P_0 and r.
3. Repeat Step **2** for P_3, the principal at the end of 3 yr.

h. Determine an explicit formula that could be used to find the account balance after t years (P_t) for an initial investment of P_0 at an annual interest rate of r, compounded annually.

i. Use your explicit formula to calculate the account balance, after 20 yr, of an investment of \$500 at an annual interest rate of 6%, compounded annually. Compare this value to the one you determined using the spreadsheet.

Discussion 1

a. What advantages are there to using an explicit formula for account balance rather than a recursive formula?

Mathematics Note

When interest is compounded annually, the yearly account balances that result can be thought of as a sequence defined explicitly by the following formula (assuming that no withdrawals are made and any interest earned is deposited in the account):

$$P_t = P_0(1+r)^t$$

where P_t is the account balance after t years, P_0 is the initial principal, r is the annual interest rate, and t is the time in years.

For example, given an initial principal of \$2000 and an annual interest rate of 4%, compounded annually, the account balance after 12 yr can be determined as follows:

$$\begin{aligned} P_{12} &= 2000(1+0.04)^{12} \\ &\approx 3202.06 \end{aligned}$$

b.

1. How do the terms in the formula $P_t = P_0(1+r)^t$ correspond with the terms of the following general formula for a geometric sequence?
$$g_n = g_1 r^{n-1}$$
2. How does $P_t = P_0(1+r)^t$ compare with the formula you wrote in Part **h** of Exploration **1**?

c. How will doubling the initial investment affect the account balance after 20 yr?

d. Describe two ways to determine the time required for a \$500 investment to double at an annual interest rate of 6%, compounded annually.

e. In previous modules, you modeled population growth with the equation $P_n = P_0(1+r)^n$, where P_n is the population after n time periods, P_0 is the initial population, and r is the growth rate per time period.

1. Compare the equation for population growth with the explicit formula for account balance.
2. What does the expression $(1+r)$ represent in each equation?

Exploration 2

In this exploration, you examine how the number of compoundings per year affects the amount of interest earned in an account.

a. Imagine that the \$500 invested in Exploration **1** is deposited in an account in which interest is compounded semiannually, or twice a year. Since the annual interest rate is 6%, the rate for each half year is $0.06/2$ or 3%.

1. What is the account balance after 2 compounding periods, or 1 yr?
2. Write an expression which describes the balance after 1 yr in terms of the original investment of \$500.
3. Repeat Steps **1** and **2** for the balance after 2 yr (4 compounding periods) and the balance after 3 yr (6 compounding periods).

b. Using an annual interest rate r and an initial principal of P_0, write a formula for P_n, the account balance after n compounding periods, when interest is compounded semiannually for t years.

c. Repeat Parts **a** and **b** for an account in which interest is compounded quarterly, or four times a year.

d. Using an annual interest rate r and an initial principal of P_0, write a general formula for the account balance after n time periods, when interest is compounded c times a year for t years.

e. The number of compounding periods per year affects the account balance at the end of the year. Investigate this effect by using a spreadsheet and your formula from Part **d** to complete Table **2** below.

Table 2: \$500 invested at 6%, with different compoundings

Initial Principal: \$500		
Annual Interest Rate: 6%		
Type of Compounding	**No. of Compoundings per Year**	**Account Balance (P_n)**
annually	1	\$530.00
semiannually		
quarterly		
monthly		
daily		
hourly		
by the minute		
by the second		

f. Predict the account balance in Table **2** after 1 yr if interest is compounded continuously.

g. Change the initial principal and the interest rate in your spreadsheet. For each change in principal or interest rate, observe how the balance is affected as the number of compoundings increases.

Mathematics Note

When compounding interest c times per year for t years, the formula for the account balance after n compounding periods is:

$$P_n = P_0\left(1+\frac{r}{c}\right)^n = P_0\left(1+\frac{r}{c}\right)^{ct}$$

where P_n represents the principal after n compounding periods, P_0 represents the initial principal, and r is the annual interest rate. Note that, in this formula, $n = ct$.

For example, consider an initial investment of \$1000 at an annual interest rate of 5%, compounded quarterly. Assuming that no withdrawals are made and any interest earned is deposited in the account, the principal after 3 yr (or $4 \bullet 3 = 12$ compounding periods) can be calculated as follows:

$$P_{12} = 1000\left(1+\frac{0.05}{4}\right)^{12} = \$1160.75$$

Discussion 2

a. To find the account balance after t years when interest is compounded annually, you used the formula $P_t = P_0(1+r)^t$. How does this formula differ from the one you wrote in Part **b** of Exploration **2**, when interest is compounded semiannually?

b. Describe what happens to the account balance in Table **2** as the number of compoundings per year increases.

c. How does increasing the number of compoundings per year appear to affect the total amount of interest earned after 1 yr?

d. As the number of compoundings increases, do you think that the sequence of account balances approaches a limit? Explain your response.

Assignment

1.1 Consider an initial investment of $700 at an annual interest rate of 7.5% for 1 yr. Assuming that no withdrawals are made and any interest earned is deposited in the account, determine the account balance when interest is compounded:

- **a.** annually
- **b.** quarterly
- **c.** monthly
- **d.** daily.

1.2 **a.** On his 18th birthday, a student invests $112.50 in an account with an annual interest rate of 9%, compounded annually. Assuming that he makes no withdrawals and any interest earned is deposited in the account, determine the account balance on his 65th birthday.

b. Repeat Part **a** for an account in which interest is compounded monthly.

1.3 **a.** Imagine that, on the day you were born, someone deposited $5000 in an account with an annual interest rate of 4.8%, compounded monthly. Determine how old you would be when the balance of the account is $20,000.

b. If the interest is compounded monthly, what annual interest rate would be required for the account to have a value of at least $30,000 on your 18th birthday?

1.4 **a.** Describe reasonable domains for r and c in the formula for account balance using compound interest (shown below).

$$P_n = P_0\left(1+\frac{r}{c}\right)^{ct},\quad n = ct$$

b. **1.** Choose values for P_0, c, and r and substitute these values into the equation.

2. To what family of functions does the equation belong?

c. **1.** Graph the equation from Part **b** as a function of t for 3 different values of r. Use the set of real numbers as the domain for t.

2. What effect does the magnitude of r have on the graphs?

1.5 As a financial advisor, you offer investment advice to your clients. One of your clients must decide whether to invest $1500 at an annual interest rate of 15%, compounded quarterly, or $1600 at an annual interest rate of 15.5%, compounded annually. Both investments have a 10-year term. Which one would you recommend? Explain your response.

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1.6 **a.** In 1991, China's estimated population was 1,151,300,000. The annual growth rate at that time was 1.4%. Assuming that this growth rate remains constant, write an equation that models China's population since 1991.

b. Use your model to estimate China's current population.

c. In 1991, India had a population of 859,200,000 and an annual growth rate of 4%. If this growth rate remains constant, predict the year in which India's population will surpass that of China.

1.7 Most durable goods, such as cars and computers, decrease in value over time. This is known as **depreciation**.

Consider a car which cost $17,000 new and loses 15% of its value each year.

a. Write both recursive and explicit formulas to represent the depreciation of this car.

b. What is the car's value after 5 yr?

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Activity 2

With the development of calculators and computers, the determination of compound interest has become quick and easy. This allows banks to compound interest on an account balance up to the instant in which it is withdrawn. This method of calculating interest, known as **compounding continuously**, means that the number of compoundings per year approaches **infinity**, denoted by ∞. Written as $+\infty$ or $-\infty$, this symbol also may be used to depict boundlessness in either a positive or a negative direction.

Exploration

As you saw in Activity **1**, the number of compoundings per year can affect the balance of a savings account. What happens to this balance when the number of compoundings increases without bound? In this exploration, you investigate what happens as c, the number of compoundings, changes for specific values of P_0, r, and t.

a. Consider an investment of \$1.00 at an annual interest rate of 100%, compounded continuously, for 1 yr. Predict the account balance at the end of the year.

b. Create a spreadsheet with columns similar to those in Table **3**. Use the formula for account balance when interest is compounded c times a year to complete the spreadsheet for an investment of \$1.00 at an annual interest rate of 100%.

Table 3: Account balances for different compoundings

No. of Compoundings per Year (c)	Account Balance at End of Year (\$)
1	
10	
100	
1000	
10,000	
100,000	
1,000,000	
10,000,000	
100,000,000	

c. As the number of compoundings per year increases, what happens to the sequence of account balances?

d. Since $P_0 = 1$, $r = 1$, and $t = 1$ in this situation, an explicit formula for the sequence of balances found in Table **3** is:

$$P_1 = 1\left(1+\frac{1}{c}\right)^{c \bullet 1} = \left(1+\frac{1}{c}\right)^{c}$$

In the context of this problem, c is a non-negative integer. However, this formula can be represented more generally as the function below:

$$y = \left(1+\frac{1}{x}\right)^{x}$$

What are the domain and range of this function?

e. Graph the function from Part **d**. As x increases without bound, what limiting value does the graph appear to approach?

Mathematics Note

The limit of the following expression, as n approaches infinity, is an irrational number approximately equal to 2.71828:

$$\left(1+\frac{1}{n}\right)^{n}$$

This irrational number, represented as ***e***, is sometimes called Euler's number in honor of Swiss mathematician Leonhard Euler. The value of e can be represented mathematically as shown below:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

Another way to describe e is as the infinite series below:

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

The value of e can also be derived from continued fractions as follows:

$$e = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\vdots}}}}}$$

f. In the formula for account balance after n compounding periods, $n = c \bullet t$, where c represents the number of compoundings per year and t represents time in years. Considering a period of 1 yr, therefore, $n = c \bullet 1 = c$. Given an initial principal of \$1.00, the formula for account balance can be written as follows:

$$P_n = \left(1 + \frac{r}{n}\right)^n$$

To investigate how a change in the value of r affects the limit of this expression, create and complete a spreadsheet with columns like those in Table **4** below.

Table 4: Balance in dollars for different interest rates

n	$P_n = \left(1 + \frac{1}{n}\right)^n$	$P_n = \left(1 + \frac{2}{n}\right)^n$	$P_n = \left(1 + \frac{3}{n}\right)^n$
1			
10			
100			
1000			
10,000			
100,000			
1,000,000			
10,000,000			
100,000,000			

g.
1. Calculate e^2. Compare it to the values in the spreadsheet in Part **f**.
2. Calculate e^3. Compare it to the values in the spreadsheet in Part **f**.

Discussion

a.
1. Describe the relationship between e^2 and the following expression:

$$\lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n$$

2. Describe the relationship between e^3 and the expression below:

$$\lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^n$$

3. What conjecture can you make about the value of this expression?

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$$

b. The value of n has no effect on the value of the constant 25 in the product below:

$$25\left(1+\frac{1}{n}\right)^n$$

In fact, the following equation is true:

$$\lim_{n\to\infty} 25\left(1+\frac{1}{n}\right)^n = 25\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right] = 25e$$

Use this fact to evaluate each of the expressions below:

1. $\lim_{n\to\infty} 10\left(1+\frac{2}{n}\right)^n$

2. $\lim_{n\to\infty} 20\left(1+\frac{3}{n}\right)^n$

3. $\lim_{n\to\infty} 8\left(1+\frac{r}{n}\right)^n$

4. $\lim_{n\to\infty} P_0\left(1+\frac{r}{n}\right)^n$

Mathematics Note

When compounding interest c times per year for t years, the formula for the account balance after n compounding periods, where $n = ct$, P_0 represents the initial principal, and r is the annual interest rate, is:

$$P_n = P_0\left(1+\frac{r}{c}\right)^{ct}$$

When the number of compoundings per year approaches infinity, then the interest is **compounded continuously**. In this case, the formula for account balance P can be written as follows:

$$P = \lim_{c\to\infty} P_0\left(1+\frac{r}{c}\right)^{ct} = P_0\left[\lim_{c\to\infty}\left(1+\frac{r}{c}\right)^c\right]^t = P_0e^{rt}$$

where P_0 represents the initial principal, r represents the annual interest rate, c represents the number of compoundings per year, and t represents number of years.

For example, consider an initial investment of \$500 at an annual interest rate of 6%, compounded continuously. Assuming that no withdrawals are made and any interest earned is deposited in the account, the account balance after 5 yr can be calculated as follows:

$$P = 500e^{0.06 \cdot 5} = \$674.93$$

c. What can you conclude about an investment whose account balance is calculated by the equation below?

$$P = 750e^{0.01t}$$

Assignment

2.1. Imagine that you have deposited \$5000 in a savings account at an annual interest rate of 3%, compounded continuously. Assuming that you make no withdrawals and any interest earned is deposited in the account, how old will you be when the account balance is \$20,000?

2.2 Determine the value of a \$1000 investment at the end of 1 yr if the annual interest rate is 9% and interest is compounded:

a. annually

b. quarterly

c. monthly

d. daily

e. continuously.

2.3 **a.** Based on your responses to Problem **2.2**, find an annual interest rate that, when compounded annually, will produce the same balance after 1 yr as an annual interest rate of 9% compounded:

1. monthly

2. daily

3. continuously.

b. Describe how you determined your responses to Part **a**.

Business Note

To help consumers compare interest rates, banks often report **annual percentage yield (APY)** for savings accounts and **annual percentage rate (APR)** for loans. The APY or APR is the interest rate that, when compounded annually, will produce the same account balance as the advertised interest rate, which is typically compounded more often.

For example, the annual percentage yield of an initial investment of P_0 at an annual interest rate of 9%, compounded quarterly, can be found as follows:

$$\begin{aligned} P_0\left(1+\frac{r_{\text{APY}}}{1}\right)^1 &= P_0\left(1+\frac{0.09}{4}\right)^4 \\ \left(1+r_{\text{APY}}\right) &\approx 1.09308 \\ r_{\text{APY}} &\approx 0.09308 = 9.308\% \end{aligned}$$

This means that, for any given initial investment, an annual interest rate of 9%, compounded quarterly, produces the same account balance as an annual interest rate of 9.308%, compounded annually.

2.4 **a.** Consider an investment of \$1000 at an annual interest rate of 7.7%. Determine the annual percentage yield (APY) if interest is compounded:

1. quarterly
2. daily
3. hourly.

b. The APY reaches its maximum when interest is compounded continuously. Determine the maximum APY for an investment with an annual interest rate of 7.7%.

c. Write a formula for determining maximum APY.

2.5 The previous business note used an example to determine the annual percentage yield (APY) for a specific annual interest rate (r) and a given number of compoundings per year (c).

a. To determine the general relationship among APY, r, and c, solve the equation below for r_{APY}.

$$P_0\left(1+\frac{r}{c}\right)^c = P_0\left(1+\frac{r_{\text{APY}}}{1}\right)^1$$

b. As the number of compoundings per year increases without bound, what would you expect to find as a formula for r_{APY}?

c. How does the initial principal affect the relationship among APY, r, and c?

2.6 **a.** After 1 yr, will the account balance increase by a significant amount if interest is compounded every hour rather than every day? Use an example to support your response.

b. In general, what effect does increasing the number of compoundings per year have on account balance?

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2.7 One general equation used to model the growth or decay in a quantity is $N_t = N_0 e^{nt}$, where N_t represents the final amount, N_0 represents the initial amount, n represents some constant, and t represents time. When $n > 0$, the equation can be used to model growth; when $n < 0$, the equation can be used to model decay.

A population of bacteria has a constant n of 0.538 when t is measured in days. How many days will it take an initial population of 8 bacteria to increase to 320?

2.8 As mentioned in a previous mathematics note, e also can described using the infinite series shown below:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

or by using continued fractions as follows:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\vdots}}}}$$

Use both of these expressions to approximate the value of e to six decimal places.

2.9 In the Level 4 module "Nearly Normal," you learned that a normal probability distribution is symmetric about the mean and tapers to the left and right like a bell. A normal curve is defined by the following equation:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ and σ are the mean and standard deviation, respectively, of a normal distribution.

a. Select a value for μ, then choose several different values for σ and graph the resulting equations.

b. Select a value for σ, then choose several different values for μ and graph the resulting equations.

c. Use your graphs from Parts **a** and **b** to discuss the effects of μ and σ on a normal curve.

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Research Project

In his studies of infinite sets of numbers, Georg Cantor (1845–1918) developed **transfinite numbers**. A transfinite number is the cardinal number of an infinite set. Cantor also described a method for determining when one infinite set of numbers was larger than another by comparing their cardinal numbers. Write a report on transfinite numbers and their relationship to infinite sets.

Activity 3

In Activity **2**, you wrote equations for determining account balances and annual percentage yields in which a value for time was used as an exponent. In this activity, you use logarithms to determine the amount of time required for an investment to reach a particular amount.

Exploration 1

In the Level 4 module "Log Jam," you investigated **common logarithms**, base-10 logarithms which can be written either as $\log_{10} x$ or $\log x$. In this exploration, you examine logarithms that have bases other than 10.

a. Complete Table **5** below, which relates corresponding exponential and logarithmic equations.

Table 5: Logarithmic and exponential equations

Logarithmic Equation	Related Exponential Equation
$\log_2 8 = 3$	$2^3 = 8$
$\log_4 16 = 2$	
	$6^4 = 1296$
$\log_{1.5} 3.375 = 3$	
	$0.8^2 = 0.64$
$\log_{0.81} 0.9 = 0.5$	

b. Recall from the Level 4 module "Log Jam," that $y = \log_a x$ is equivalent to $a^y = x$. Knowing the relationship between *a* (the base) and *x* for various values of *y* can help you determine an unknown base of a logarithm.

1. Select a value for *x* greater than 1.
2. Using the value for *x* from Step **1**, determine the value of *a* in the equation $a^y = x$ when $y = 1$.
3. Compare the value of *a* to the value of *x*.
4. Repeat Steps **2** and **3** for several values of *y* greater than 1.
5. Repeat Steps **2** and **3** for several values of *y* less than 1.

c. Table **6** shows the logarithms, using two unknown rational bases *a* and *b*, for various values of *x*. Use your results from Part **b** to help determine the approximate value of each base.

Table 6: Two logarithms of *x*

x	$\log_a x$	$\log_b x$
2	0.431	0.333
3		0.528
4	0.861	
5		0.774
6	1.113	0.862
7		0.936
8	1.292	
9		
10		
11	1.490	1.153
12	1.544	1.195

Given the logarithmic equation $\log_{10} 10 = 1$, the related exponential equation is $10^1 = 10$. Write similar equations for $\log_a x$ and $\log_b x$ in Table **6**.

d. Most calculators and computers offer a feature for determining the **natural logarithm** of a number, denoted as $\mathbf{ln}\boldsymbol{x}$.

Determine the natural logarithm of various values of *x*. Use your results to approximate the value of the base of the natural logarithm to five decimal places.

Discussion 1

a. In Part **a** of Exploration **1**, you found that $x < a$ when $\log_a x < 1$ and $x > a$ when $\log_a x > 1$. Explain why this must be true.

b. What value did you determine for the approximate base of the natural logarithm?

Mathematics Note

Logarithms with base e are referred to as **natural logarithms**. The natural log of x is denoted by $\ln x$, where $x > 0$. The equation $\ln x = y$ is true if $e^y = x$.

For example, $\ln 7 \approx 1.9$. The related exponential equation is $e^{1.9} \approx 7$.

c. Describe how you might estimate each of the following:

1. $\log_7 52$
2. $\log_{19.7} 18$
3. $\ln 2$
4. $\ln 20$ (Hint: $e^3 \approx 20$.)

d. Why does the definition of natural logarithms given in the previous mathematics note restrict x to values greater than 0?

Exploration 2

In this exploration, you use natural logarithms to determine the time required for an account to reach a desired balance when interest is compounded continuously.

a. Consider an initial investment of \$500 at an annual interest rate of 6%, compounded continuously. Write a function that describes the account balance after t years.

b. In the module "Log Jam," you used the properties of logarithms to solve equations such as the one below for x.

$$y = 3 \bullet 10^x$$
$$y/3 = 10^x$$
$$\log(y/3) = \log 10^x$$
$$\log(y/3) = x$$

Use natural logarithms and the properties of logarithms to solve the equation in Part **a** for t.

Mathematics Note

The properties that are true for $\log_b x$ also are true for $\ln x$. Therefore, for $b > 0$, $b \neq 1$, $x > 0$, and $y > 0$:

- $\log_b b = 1$ and $\ln e = 1$
- $\log_b b^x = x$ and $\ln e^x = x$
- $\log_b x^y = y \log_b x$ and $\ln x^y = y \ln x$
- $\log_b (xy) = \log_b x + \log_b y$ and $\ln (xy) = \ln x + \ln y$
- $\log_b (x/y) = \log_b x - \log_b y$ and $\ln (x/y) = \ln x - \ln y$
- $b^{\log_b x} = x$ and $e^{\ln x} = x$

c. Use your response to Part **b** to determine the approximate number of years required for the account balance to reach each of the following amounts:

1. \$1660.10
2. \$3024.80

Discussion 2

a. In the example given in Part **b** of Exploration **2**, the equation $y = 3 \bullet 10^x$ is solved for x. Describe how the properties of logarithms were used to solve this equation.

b. Explain how you used the properties of logarithms to solve the equation in Part **a** of Exploration **2** for t.

c. In Part **c** of Exploration **2**, you used natural logarithms to determine that an initial investment of \$500 at an annual interest rate of 6%, compounded continuously, would require approximately 20 yr to reach a balance of \$1660.10.

It also is possible to determine this solution using common logs, as shown below.

$$1660.10 = 500e^{0.06t}$$

$$1660.10/500 = e^{0.06t}$$

$$\log(1660.10/500) = \log e^{0.06t}$$

$$\log(1660.10/500) = 0.06t \log e$$

$$\frac{\log(1660.10/500)}{0.06 \log e} = t$$

1. What advantages are there to using natural logs to solve this equation for t?

2. Consider another equation in which the variable to be solved for is an exponent. Do you think it would be possible to solve this equation using logarithms of any base?

Assignment

3.1. **a.** Using natural logarithms, convert each of the following equations from exponential form to logarithmic form.

1. $e^5 = x$
2. $e^0 = 1$
3. $e^{0.06x} = 3$

b. Solve $e^{0.06x} = 3$ for x.

3.2 Convert each of the following equations from logarithmic to exponential form.

a. $\ln x = 2$

b. $\ln e = 1$

c $\ln(y/750) = 0.05x$

3.3 Imagine that a young child invests 100 pennies in an account which compounds interest continuously. Using the equation $P_0 e^{rt} = 2P_0$, determine how long it will take the child's initial investment to double at each of the following annual interest rates.

a. 6%

b. 8%

c. 10%

3.4 **a.** To help pay for their newborn child's future education, two parents decide to open a savings account. They make an initial deposit of \$1275 at an annual interest rate of 7%, compounded quarterly.

1. How long will it take for the initial deposit to double?
2. How long will it take for the initial deposit to triple?
3. If the parents make no further deposits or withdrawals, what will the account balance be when the child is ready to enter college?

b. Repeat Part **a** for an initial deposit of \$1275 at an annual interest rate of 7%, compounded continuously.

c. Compare your responses to Part **b** with your responses to Part **a**. Describe any differences you observe.

3.5 Consider an initial investment of P_0 at an annual interest rate of r, compounded continuously.

a. Write an equation that describes the value of the investment after t years.

b. Solve the equation in Part **a** for t.

3.6 LaSasha wants to purchase a new stereo system. The one she has selected costs $715. At the moment, however, she has only $500 available to spend. While exploring her options, LaSasha examines an investment account that offers an annual interest rate of 8%, compounded continuously. Use natural logarithms to complete Parts **a–c** below.

a. If LaSasha decides to invest in this account, how long will it take for the account balance to reach $715?

b. LaSasha would like to buy the stereo system within 6 months. What annual interest rate would she have to earn to make this possible?

c. Is it reasonable to expect an interest rate of this size?

* * * * *

3.7 At last count, the population of Central City was 410,000. City planners expect the population to increase at a rate of 4.25% each year.

a. Write a function to model the number of years t it will take for the city to grow to a given population p.

b. The city prefers to employ one law enforcement officer for every 1000 people. If its growth rate remains constant, determine when the city will need to employ each of the following numbers of officers:

1. 500

2. 1000

3.8 **a.** Describe how different values of k affect the graphs of the following equations:

1. $y = \ln x + k$

2. $y = \ln x + \ln e^k$

b. Repeat Part **a** for the equations below.

1. $y = k \ln x$

2. $y = \ln x^k$

c. **1.** Describe a relationship between the pair of equations in Part **a**.

2. Describe a relationship between the pair of equations in Part **b**.

d. Use laws of exponents to support your responses to Part **c**.

* * * * * * * * * *

Summary Assessment

1. As part of his savings plan, Vonzel invested \$5000 in a one-year certificate of deposit (CD) at an annual interest rate of 7%, compounded daily. When Vonzel told his Aunt Theresa about his investment, she advised him to withdraw the money. Another bank in town, she said, advertises the same interest rate, compounded continuously.

   a. Since Vonzel's bank charges a \$150 penalty for early withdrawal, he decided not to move the money. Did Vonzel make the right decision? Explain your response.

   b. How long will it take Vonzel's CD to earn \$150 (the cost of the penalty) in interest?

   c. The total value of the certificates of deposit at each bank is \$5 million. In this situation, how much more does it cost a bank to pay interest compounded continuously rather than daily?

2. Shortly after buying the \$5000 certificate of deposit, Vonzel purchases a \$1400 stereo system with his credit card. His credit card company charges interest at an annual rate of 13%, compounded daily. Another company has offered him a credit card with the same annual interest rate, compounded monthly. The annual fee for the new card is \$55; the annual fee for his current card fee is \$50. Should Vonzel change credit cards? Explain your response.

3. Vonzel's credit card company offers a no-minimum payment option to some customers with excellent credit ratings. Using this option, customers may carry any balance due until the end of the next month. The interest charged on the balance, however, continues to be compounded daily.

   Due to some unforeseen expenses, Vonzel can't afford to pay his \$1400 credit bill. Should he withdraw his \$5000 certificate of deposit, pay the \$150 penalty, then use some of the remaining cash to pay his credit card bill? Explain your response.

Module Summary

- **Principal** is the amount of money invested or loaned.
- **Interest** is the amount earned on invested money, or the fee charged for loaned money.
- The formula for calculating **simple interest**, where I represents interest, P represents principal, r represents the interest rate per time period, and t represents the number of time periods is shown below:

$$I = Prt$$

- The principal at the end of each time period in an investment or savings account can be thought of as a sequence. Assuming that no withdrawals are made and any interest earned is deposited in the account, such a sequence can be defined recursively by the following formula:

$$P_t = P_{t-1} + r \bullet P_{t-1} = P_{t-1}(1 + r)$$

  where P_t is the principal at the end of t years, r is the annual interest rate, and P_{t-1} is the principal for the previous year.

- When interest is compounded annually, the yearly account balances that result can be thought of as a sequence defined explicitly by the following formula (assuming that no withdrawals are made and any interest earned is deposited in the account):

$$P_t = P_0(1 + r)^t$$

  where P_t is the account balance after t years, P_0 is the initial principal, r is the annual interest rate, and t is the time in years.

- When compounding interest c times per year for t years, the formula for the account balance after n compounding periods is:

$$P_n = P_0\left(1 + \frac{r}{c}\right)^n = P_0\left(1 + \frac{r}{c}\right)^{ct}$$

  where P_n represents the principal after n compounding periods, P_0 represents the initial principal, and r is the annual interest rate. Note that, in this formula, $n = ct$.

- **Infinity**, represented by the symbol ∞, depicts an unlimited quantity or an amount larger than any fixed value. Written as $+\infty$ or $-\infty$, it also may be used to depict quantities that extend without bound in either a positive or a negative direction.

- The value of e can be represented mathematically as shown below:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$$

- When the number of compoundings per year approaches infinity, then the interest is **compounded continuously**. In this case, the formula for account balance P can be written as follows:

$$P=\lim_{c\to\infty}P_0\left(1+\frac{r}{c}\right)^{ct}=P_0\left[\lim_{c\to\infty}\left(1+\frac{r}{c}\right)^{c}\right]^{t}=P_0e^{rt}$$

  where P_0 represents the initial principal, r represents the annual interest rate, c represents the number of compoundings per year, and t represents number of years.

- To help consumers compare interest rates, banks often report **annual percentage yield (APY)** for savings accounts and **annual percentage rate (APR)** for loans. The APY or APR is the interest rate that, when compounded annually, will produce the same account balance as the advertised interest rate, which is typically compounded more often.

- Logarithms with base e are referred to as **natural logarithms**. The natural log of x is denoted by $\ln x$ where $x>0$. The equation $\ln x=y$ is true if $e^y=x$.

- The properties that are true for $\log_b x$ also are true for $\ln x$. Therefore, for $b>0$, $b\neq 1$, $x>0$, and $y>0$:

  - $\log_b b=1$ and $\ln e=1$
  - $\log_b b^x=x$ and $\ln e^x=x$
  - $\log_b x^y=y\log_b x$ and $\ln x^y=y\ln x$
  - $\log_b(xy)=\log_b x+\log_b y$ and $\ln(xy)=\ln x+\ln y$
  - $\log_b(x/y)=\log_b x-\log_b y$ and $\ln(x/y)=\ln x-\ln y$
  - $b^{\log_b x}=x$ and $e^{\ln x}=x$

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