Functioning on a Path

In this module, you use your knowledge of polynomial and rational functions to play a mathematics game.

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Functioning on a Path

Introduction

In a mathematics video game called Gates, there are three different levels of play. In the first level, the object of the game is to determine the characteristics of a **continuous** function that passes through several "gates," represented by a pair of squares on the screen. For example, Figure **1** shows a first-level screen with a second-degree polynomial (or quadratic) function passing through five gates.

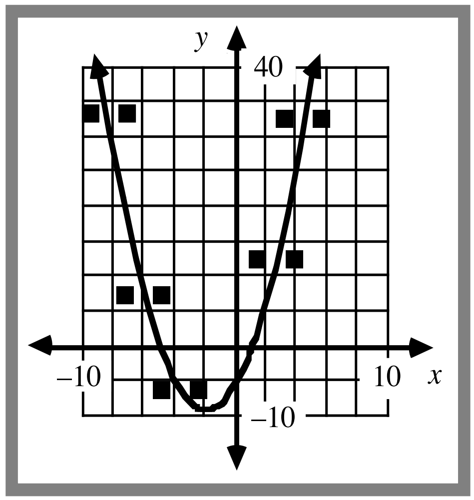


Figure 1: Quadratic function passing through gates

Mathematics Note

A function is **continuous** at a point c in its domain if the following conditions are met:

- the function is defined at c, or $f(c)$ exists
- the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
- the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$

A function is continuous over its domain if it is continuous at each point in its domain.

A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.

For example, a function is discontinuous at $x = c$ if the function is undefined at c, as shown in Figure **2**.

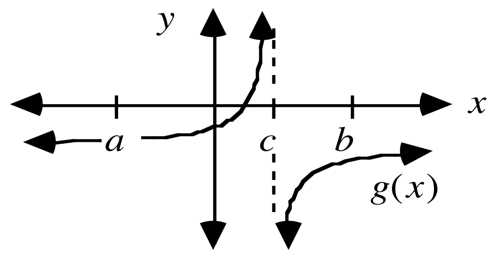


Figure 2: Graph of the discontinuous function $g(x)$

A function is also discontinuous at $x = c$ if the limit of the function does not exist at c, as shown in Figure **3**.

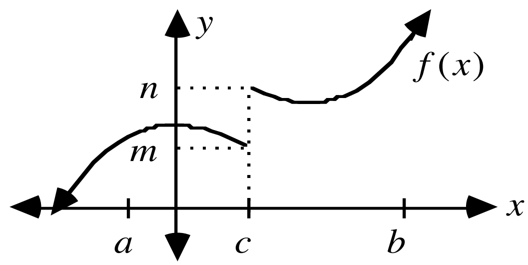


Figure 3: Jump discontinuity in the function $f(x)$

In this case, $f(x)$ approaches m as x approaches c from the left. As x approaches c from the right, $f(x)$ approaches n. Since $m \neq n$, the limit of $f(x)$ as x approaches c does not exist. This kind of discontinuity is referred to as a **jump discontinuity**.

A function is also discontinuous at $x = c$ if the value of the function at c does not equal the limit of the function as x approaches c, as shown in Figure **4**. In this case, the limit of $h(x)$ as x approaches c is m, while $h(c) = n$, and $m \neq n$.

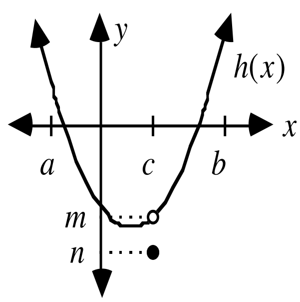


Figure 4: Hole in the graph of the function $h(x)$

Discussion

a. In the first level of Gates, players may use polynomial functions. Recall from the Level 4 module "Drafting and Polynomials" that a **polynomial function** can be written in the following general form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

where a_n is a real number and n is a non-negative integer.

Are the functions below polynomial functions? Explain your responses.

1. $f(x) = x^2 + \sqrt{3} \bullet x - 5$

2. $f(x) = x^2 + \dfrac{4}{x} - 5$

b. The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression. The coefficient of that variable is the **leading coefficient**. For a polynomial written in the general form in Part **a**, the degree is n and the leading coefficient is a_n.

Identify the degree and leading coefficient of the polynomial below.

$$x^2 + 5x^4 - 5x^5$$

c. Do you think that all polynomial functions are continuous? Explain your response.

d. The graph in Figure **1** is a second-degree polynomial function. It is only one of many polynomial functions that could be drawn through the desired gates. Describe how you could use polynomial regressions to determine three other continuous functions that pass through these gates.

Activity 1

Figure **1** showed a screen from the first level of a game in which players must identify a polynomial function that passes through gates. One way to determine such an equation involves the relationship among the degree of a polynomial, its roots (or zeros), and the characteristics of its graph.

Exploration 1

A completed screen from a game of Gates is shown in Figure **5**. The degree of the polynomial function that produced this path can be predicted from the characteristics of the graph. In this exploration, you examine polynomial functions of several different degrees and attempt to identify the characteristics of their graphs.

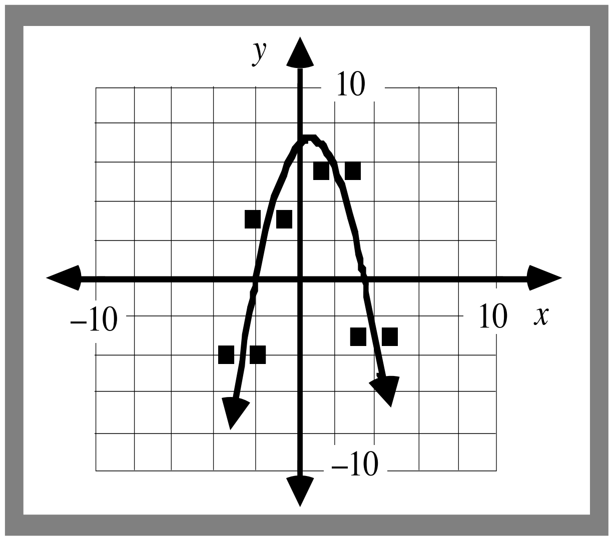


Figure 5: Screen in first level of Gates

a. Make a conjecture about the least possible degree of a polynomial function that could pass through the gates in Figure **5**.

b. Choose the coordinates of any three noncollinear points on a two-dimensional coordinate system.

c. Use a quadratic regression to determine an equation for a polynomial function passing through those three points.

d. A quadratic function can be written in the following general form:

$$q(x) = a_2x^2 + a_1x + a_0$$

1. Use the coordinates of the three points in Part **b** to write a system of equations involving a_2, a_1, and a_0.
2. Use technology to solve the system for a_2, a_1, and a_0.
3. Use the solutions to write a quadratic equation whose graph contains the original three points.

e. On the same coordinate system, plot the points selected in Part **b** and the function found in Part **d**.

Discussion 1

a. **1.** If a quadratic function models the graph in Figure **5**, what do you know about its leading coefficient? Why?

2. If any polynomial function with an even degree models the graph in Figure **5**, what do you know about its leading coefficient?

b. **1.** Why should you expect the functions found in Parts **c** and **d** of Exploration **1** to be equal?

2. Why might the actual equations be slightly different?

c. Considering the class results to Part **d**, do you think that any three noncollinear points can be contained in the graph of a quadratic polynomial?

Exploration 2

a. Sketch the graph of a quadratic function whose leading coefficient is positive.

b. Use your sketch to predict the maximum number of times function of this degree can intersect the x-axis.

Mathematics Note

The **end behavior** of the graph of a polynomial function describes the characteristics of the graph as $|x|$ approaches infinity.

For example, Figure **6** shows the graph of a fourth-degree polynomial function. As $|x|$ approaches infinity, $f(x)$ also increases without bound.

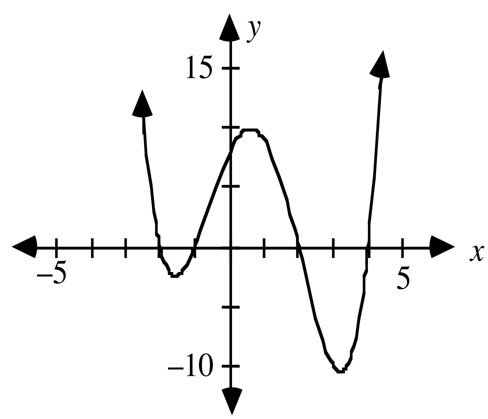


Figure 6: Graph of $f(x) = 0.5x^4 - 1.5x^3 - 4x^2 + 6x + 8$

c. Describe the end behavior of the graph you sketched in Part **a**.

d. Sketch graphs of quadratics that intersect the x-axis in 0, 1, … n times, where n is the number you identified in Part **b**.

e. Using a graphing utility, repeat Parts **a–d** for each of the following:

1. a third-degree polynomial function with four different coefficients and a positive leading coefficient
2. a fourth-degree polynomial function with five different coefficients and a positive leading coefficient.

Discussion 2

a. Recall from the Level 4 module "Can It," that the **absolute maximum** of a function is the greatest value of the range, while the **absolute minimum** is the least value of the range.

1. Of the functions that you graphed in Exploration **2**, which ones had an absolute maximum?
2. Which functions had an absolute minimum?

b. If the leading coefficients of the equations in Exploration **2** had been negative, how would this have affected the absolute maximums or minimums of the functions identified in Part **a**?

c.
1. Describe a function that has both an absolute maximum and an absolute minimum.
2. Describe some functions that have neither an absolute maximum nor an absolute minimum.

d. The x-coordinate of each point where the graph of a polynomial intersects the x-axis is a **zero** or a **root** of the polynomial. What do you think is the maximum number of zeros a polynomial of degree n can have? Explain your response.

e. Consider a function $f(x)$. What is the value of $f(x)$ when x is a root of the function?

f. Given the graph of a polynomial function with three zeros, what do you think is the least possible degree for the polynomial?

g. Recall that a polynomial function also can be written as follows, where each expression of the form $(x - a_i)$ represents a factor of the polynomial:

$$f(x) = (x - a_n)(x - a_{n-1})(x - a_{n-2})\cdots(x - a_0)$$

Consider a polynomial function with zeros of –6, 8, and 2. Write an equation for this function using the least possible degree.

h. How could you determine a polynomial of higher degree than the one in Part **g** that has exactly the same zeros?

Assignment

1.1 Consider a function that has exactly one zero. Identify the polynomial of least degree that could describe such a function. Justify your response with a graph and with a general equation for the polynomial.

1.2
- **a.** Sketch the graph of a function with an absolute maximum of 7 when $x = 2$.
- **b.** Do you think every quadratic function has either an absolute maximum or minimum? Explain your response.
- **c.** What can you tell about a quadratic function if its graph has an absolute maximum? an absolute minimum?
- **d.** Sketch a polynomial of degree greater than 2 that has an absolute maximum.
- **e.** Do you think that any polynomial of degree 3 has an absolute maximum or an absolute minimum? Explain your response.

1.3
- **a.** Figure **6** shows a graph of $f(x) = 0.5x^4 - 1.5x^3 - 4x^2 + 6x + 8$. Do you think this function has an absolute maximum or an absolute minimum? Explain your response.
- **b.** The "peaks" and "valleys" of the graph in Figure **6** might be used to identify "relative maximums" and "relative minimums." How would you define these terms?

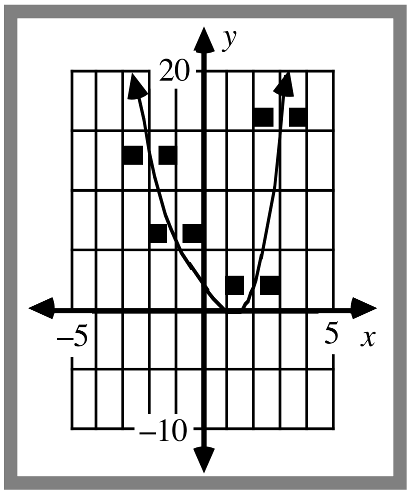
1.4 Consider the polynomial $f(x) = 3(x - 2)^2(x + 4)(x - \pi)$.
- **a.** Describe how to determine the zeros of this function.
- **b.** Identify the degree of this polynomial.
- **c.** Determine whether the polynomial has an absolute minimum or absolute maximum.
- **d.** Describe the end behavior of the polynomial.

1.5
- **a.** Identify the general form of a polynomial function that has the least possible odd degree.
- **b.** Describe the end behavior of this type of polynomial.
- **c.** What do you think is true about the end behavior of any polynomial function with an odd degree? Explain your response.

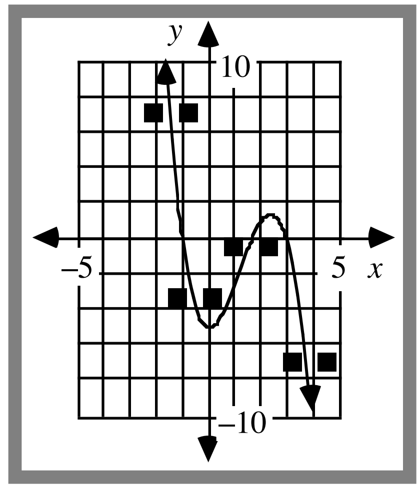
1.6 Given the second-degree polynomial $f(x) = x^2 + 5x + 6$, describe how you could find the exact values of its zeros.

1.7 Parts **a** and **b** below show two screens in the first level of Gates. Explain whether you think the degree of each polynomial shown is odd or even. Describe the characteristics that support your choice.

a.



b.



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1.8 Write an equation of a polynomial function whose graph contains the points with the following coordinates:

a. $(-8,-7)$ and $(5,10)$

b. $(0,0)$, $(-1,1)$, and $(-8,64)$

1.9 The data in the following table was collected during the flight of a model rocket. The values for time represent the number of seconds after the engine burned out. The values for distance represent height in meters. Find an equation that closely models this data and describe how you identified your model.

Time (sec)	Distance (m)
2	90.4
3	85.9
4	71.6
5	47.5
6	13.6

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Activity 2

In the second level of Gates, players use **piecewise functions** to describe graphs. Like any function, a piecewise function has a domain, a range, and a rule relating the two. In a piecewise function, however, different parts of the domain correspond with different rules.

For example, consider the absolute-value function, $f(x)=|x|$. In this function, the rule $f(x)=-x$ applies to the domain interval $(-\infty,0)$, while the rule $f(x)=x$ applies to the domain interval $[0,\infty)$. This can be written as shown below:

$$f(x)=\begin{cases}-x \text{ if } x\in(-\infty,0)\\ x \text{ if } x\in[0,\infty)\end{cases}$$

Exploration

In this exploration, you examine the use of piecewise functions to create successful functions for the game of Gates. For example, the screen in Figure **7** shows a piecewise function that passes through five gates.

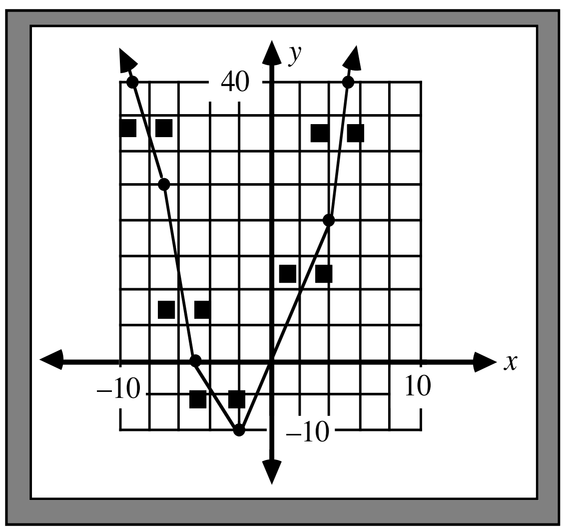


Figure 7: Screen with piecewise function

The graph of this function consists of two rays and three segments. The ray in the upper left-hand portion of the screen has its endpoint at $(-7,25)$ and contains the point $(-9,40)$. The ray in the upper right-hand portion of the screen has its endpoint at $(4,20)$ and contains the point $(5,40)$.

The three segments have the following pairs of endpoints: $(-7,25)$ and $(-5,0)$; $(-5,0)$ and $(-2,-10)$; and $(-2,-10)$ and $(4,20)$.

a. The equation of a ray or a segment can be found by determining the equation of the line that contains it and restricting the domain to an appropriate interval.

1. Write an equation for the ray that has its endpoint at $(-7,25)$ and contains the point $(-9,40)$. State the domain of the equation.
2. Write an equation for the segment with endpoints at $(-7,25)$ and $(-5,0)$. State the domain of the equation.
3. Write an equation for the segment with endpoints at $(-5,0)$ and $(-2,-10)$. State the domain of the equation.
4. Write an equation for the segment with endpoints at $(-2,-10)$ and $(4,20)$. State the domain of the equation.
5. Write an equation for the ray that has its endpoint at $(4,20)$ and contains the point $(5,40)$. State the domain of the equation.

b. Use the equations from Part **a** to write a piecewise function whose graph is the one shown in Figure **7**.

c. Use technology to graph the function defined in Part **b**.

Discussion

a. Describe how you used technology to graph the piecewise function in Part **c** of the exploration.

b. Describe how you could show algebraically that each two consecutive parts of a continuous piecewise function contain a common point.

c. Explain why the function in Part **b** of the Exploration is continuous.

d. Consider the following:

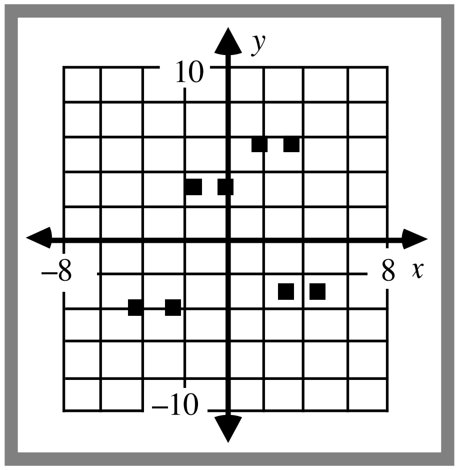
$$f(x) = \begin{cases} 1/x \text{ if } x \in (-\infty,0) \cup (0,+\infty) \\ 0 \text{ if } x = 0 \end{cases}$$

1. Is f a function?
2. Is f a piecewise function?
3. Is f continuous at 0?

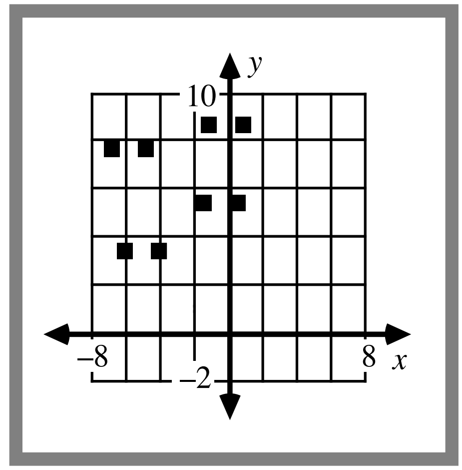
Assignment

2.1 For each screen shown in Parts **a–c** below, determine a continuous piecewise function that passes through all the gates.

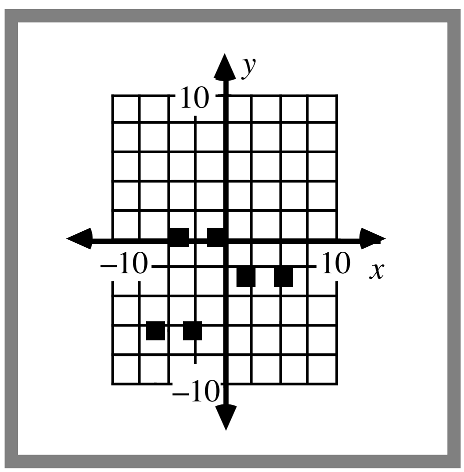
a.



b.

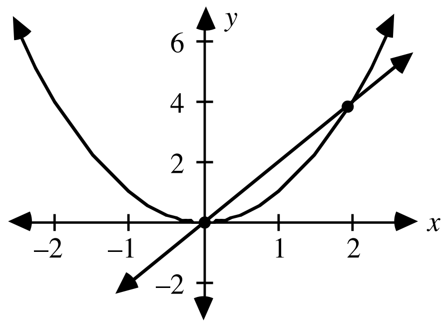


c.



2.2 The greatest integer function, $f(x) = [x]$, pairs every element x in the domain with the greatest integer less than or equal to x. Explain why this function can be considered a piecewise function.

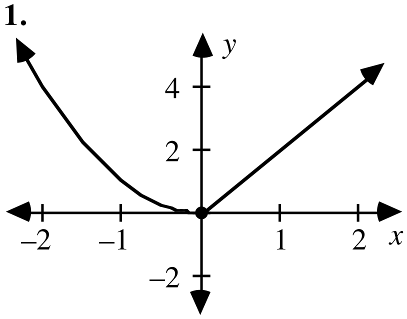
2.3 The graphs of the functions $f(x) = x^2$ and $g(x) = 2x$ are shown below.



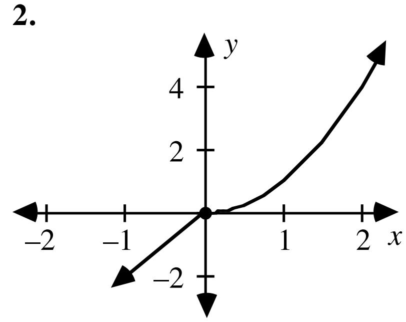
a. Determine the points of intersection of the two functions.

b. Each graph below shows a piecewise function that uses the rules for $f(x)$ and $g(x)$ over parts of its domain. Write a continuous piecewise function to describe each graph.

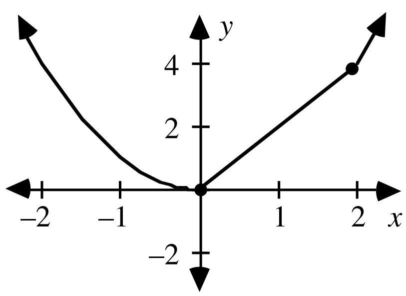
1.



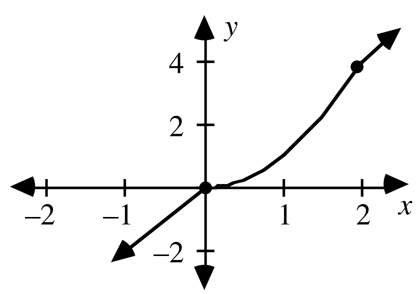
2.



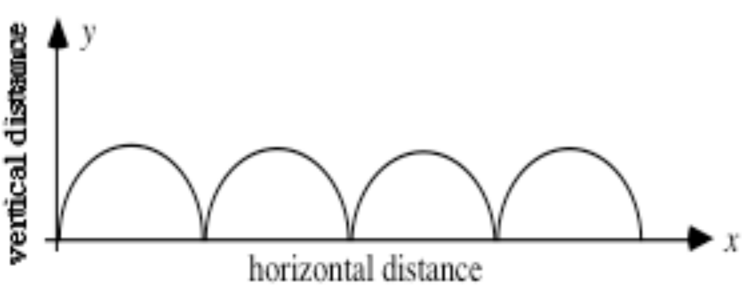
3.



4.



2.4 The graph below models the border to a flowerbed. Each piece of the border is made of a semicircular slab of concrete. Describe this curve using a piecewise function.



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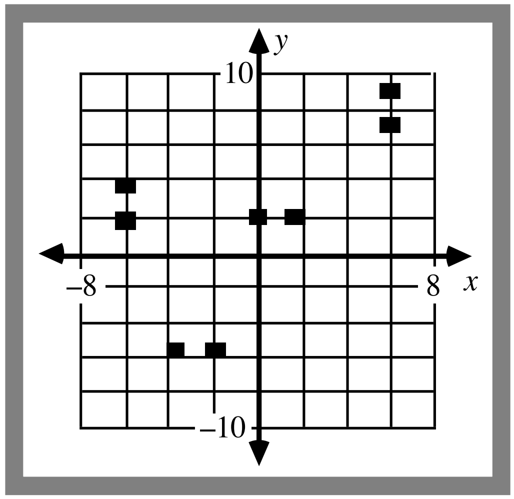
2.5 Use piecewise functions to describe a graph that passes through the gates in the following screen.



2.6. Recall that the height of a freely falling object can be described by the following function:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0$$

where g represents the acceleration due to gravity $\left(9.8 \text{ m/sec}^2\right)$, t is time in sec, v_0 is the object's initial velocity, and h_0 is its initial height.

a. Consider a ball dropped from a height of 1 m. On each successive bounce, it rises to two-thirds the height of the previous bounce. Describe the graph that you think would model the first three bounces of the ball. Justify your response.

b. The ball hits the ground at approximately 0.45 sec, 1.19 sec, 1.79 sec, and 2.29 sec. Determine a piecewise function that models the first three bounces of the ball.

c. Create a graph of your piecewise function.

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Activity 3

In the previous activities, you investigated some characteristics of polynomial functions and piecewise functions while playing the first two levels of Gates. In this activity, you examine the graphs of yet another type of function. Your discoveries should help you develop a strategy for the next level of Gates.

Exploration 1

Figure **8** shows a screen from the third level of Gates. The line represents the graph of the polynomial function below:

$$f(x) = \frac{1}{3}x + 5$$

Your challenge is to alter the function so that its graph passes through all four gates.

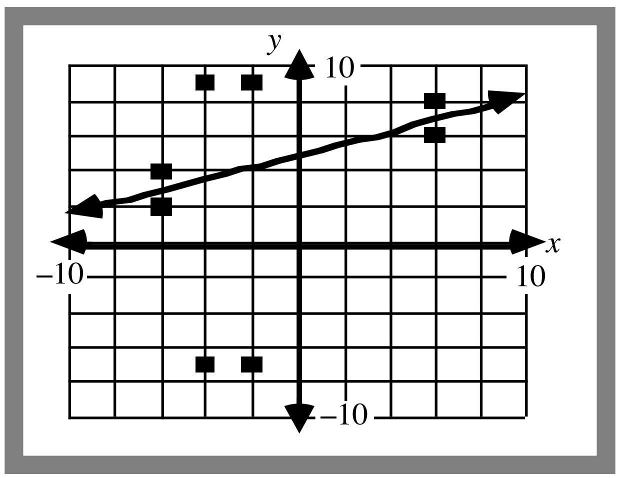


Figure 8: Screen in the third level of Gates

One way to accomplish this task involves rational functions. In this exploration, you examine how the graph of a linear function is affected by the addition of a rational expression.

Recall from the Level 4 module "Big Business" that a **rational function** can be written in the following general form:

$$r(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomial functions and $g(x) \neq 0$.

a. Any rational function also can be expressed in the form below, where $q(x)$ and $h(x)$ are polynomial functions such that $q(x)$ is the quotient and $h(x)$ is the remainder when $f(x)$ is divided by $g(x)$:

$$r(x) = q(x) + \frac{h(x)}{g(x)}$$

1. Choose a polynomial function $q(x)$ of degree 1.
2. Choose a polynomial function $g(x)$ of degree 1.

b.

1. Create a rational function $r(x)$ by adding the expression $1/g(x)$ to $q(x)$ as follows:

$$r(x) = q(x) + \frac{1}{g(x)}$$

2. Determine the domain of $r(x)$.
3. Recall that a discontinuity occurs in a rational function at any point where the value of the function is undefined. Predict the behavior of the graph of $r(x)$ near its point of discontinuity.

Mathematics Note

An **asymptote** to a curve is a line such that the distance from a point *P* on the curve to the line approaches zero as the distance from *P* to the origin increases without bound, where *P* is on a suitable part of the curve.

For example, Figure **9** shows a graph of the function $f(x) = \log x$. In this case, the suitable part of the curve lies below the *x*-axis. As a point *P* moves farther away from the origin on this part of the curve, the distance between *P* and the *y*-axis approaches 0. Therefore, the line $x = 0$ is an asymptote for the curve.

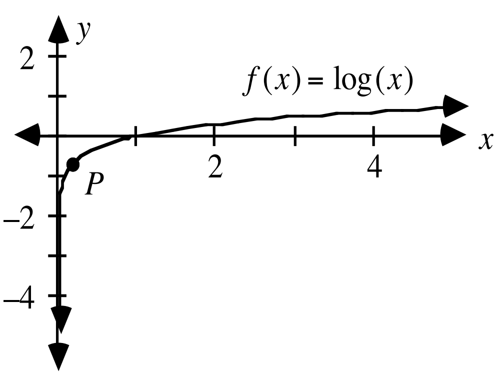


Figure 9: Graph of $f(x) = \log(x)$

c. Graph $r(x)$ and $q(x)$ on the same coordinate system. Use an interval of the domain that includes the point where the graph is discontinuous.

d. Determine what happens to the values of the functions $q(x)$, $r(x)$, and $1/g(x)$ as each of the following occurs:

1. x approaches the point of discontinuity
2. $|x|$ increases without bound.

e. Select a new function $q(x)$ with degree 2 and a new function $g(x)$ with degree 1. Repeat Parts **b–d**.

f. Select a new function $q(x)$ with degree 3 and a new function $g(x)$ with degree 1. Repeat Parts **b–d**.

Discussion 1

a. Describe the graphs of $r(x)$ and $q(x)$ as x approaches the point of discontinuity. Why does this behavior occur?

b. How does the addition of the rational expression $1/g(x)$ affect the graph of a polynomial function $q(x)$?

c. Describe a rational function that has more than one vertical asymptote.

d. Suppose that, in Part **b** of Exploration **1**, you had added the additive inverse of $1/g(x)$ to $q(x)$. How would this have affected the graph of the resulting function $r(x)$?

e. What rational expression could you add to the following polynomial function in order to create a path that passes through all four gates in Figure **8**?

$$f(x) = \frac{1}{3}x + 5$$

f. Consider the rational function

$$r(x) = f(x) + \frac{1}{g(x)}$$

1. Describe the graph of $r(x)$ when $g(x)$ is a linear function.
2. How does the end behavior of $r(x)$ compare to that of $f(x)$?

g. Consider a rational function written in the following general form, where $q(x)$, $h(x)$, and $g(x)$ are all polynomial functions and $g(x) \neq 0$:

$$r(x) = q(x) + \frac{h(x)}{g(x)}$$

Describe how you could rewrite this function in the form below:

$$r(x) = \frac{f(x)}{g(x)}$$

Exploration 2

Figure **10** shows another screen in the third level of Gates. To complete this screen, you must alter the function $f(x) = 2$ so that its graph passes through all three gates without running into the brick wall.

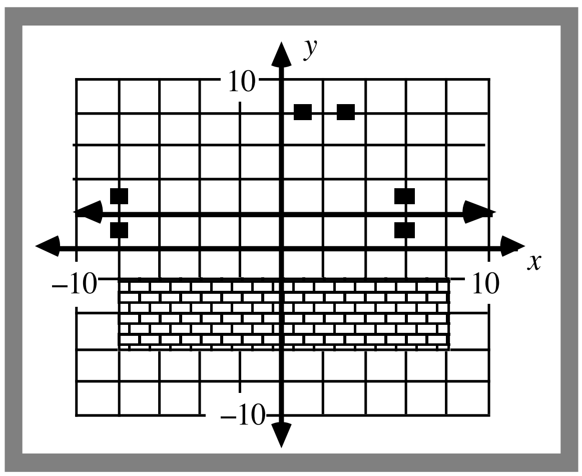


Figure 10: Screen in third level of Gates

As you discovered in Exploration **1**, a rational function of the form below has the same end behavior as $f(x)$, when $g(x)$ is a first-degree polynomial.

$$r(x) = f(x) + \frac{1}{g(x)}$$

In this exploration, you experiment with other degrees for the denominator of the added rational expression.

a. Select a first-degree polynomial $k(x)$ and a constant function $h(x)$. Graph each of the following pairs of functions on a different set of axes. Note any similarities or differences between the two graphs, including their end behaviors.

1. $f(x) = h(x) + \dfrac{1}{k(x)}$ and $g(x) = h(x) + \dfrac{1}{(k(x))^2}$

2. $f(x) = h(x) + \dfrac{1}{(k(x))^3}$ and $g(x) = h(x) + \dfrac{1}{(k(x))^4}$

b. **1.** Compare the graphs of the two functions below for a positive value of *n,* where *n* is an integer other than 1.

$$f(x) = h(x) + \frac{1}{(k(x))^2} \text{ and } g(x) = h(x) + \frac{n}{(k(x))^2}$$

2. Repeat Step **1** for several different positive values of *n*.

3. Repeat Step **1** for several different negative values of *n*.

Discussion 2

a. What similarities or differences did you observe in the graphs of the two functions below?

$$f(x) = h(x) + \frac{1}{k(x)} \text{ and } g(x) = h(x) + \frac{1}{(k(x))^2}$$

b. **1.** How did the graphs of the following two functions compare when *n* was a positive integer?

$$f(x) = h(x) + \frac{1}{(k(x))^2} \text{ and } g(x) = h(x) + \frac{n}{(k(x))^2}$$

2. How did the graphs compare when *n* was a negative integer?

c. Consider the following two rational functions:

$$f(x) = 3 + \frac{1}{(x+6)^2} \text{ and } g(x) = 3 + \frac{-1}{(x+6)^2}$$

1. As $|x|$ increases without bound, the values of both $f(x)$ and $g(x)$ approach 3. Explain why this occurs.

2. As *x* approaches –6, however, $f(x)$ approaches $+\infty$ while $g(x)$ approaches $-\infty$. Explain why this occurs.

d. Consider a function of the form below, where c is a constant and n is a positive integer:

$$f(x) = h(x) + \frac{c}{\left(k(x)\right)^n}$$

1. Describe how raising the denominator of the added rational expression from an odd power to an even power affects the graph of the resulting function.
2. Describe how changing the numerator of the added rational expression affects the graph of the resulting function.

e.

1. What rational expression could you add to $f(x) = 2$ to create a graph that passes through all the gates in Figure **10** but misses the brick wall?
2. Are there other rational expressions that would accomplish this task? Explain your response.

f. Consider a rational function written in the form below, where $g(x) \neq 0$:

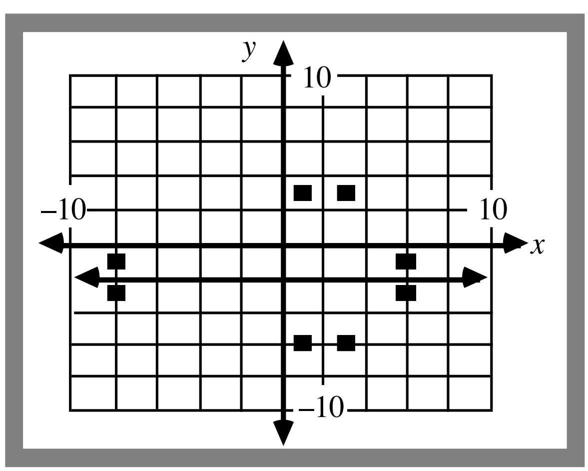
$$r(x) = \frac{f(x)}{g(x)}$$

Describe how you could express this function in the following form, where $q(x)$ is a polynomial function and the degree of $h(x)$ is less than or equal to the degree of $g(x)$:

$$r(x) = q(x) + \frac{h(x)}{g(x)}$$

Assignment

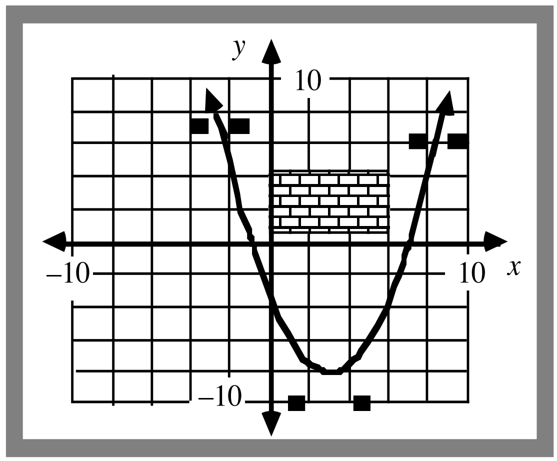
3.1 The figure below shows a screen in the third level of Gates. Add a rational expression to $f(x) = -2$ so that the graph of the resulting function passes through all four gates. Identify the domain and range of your function.



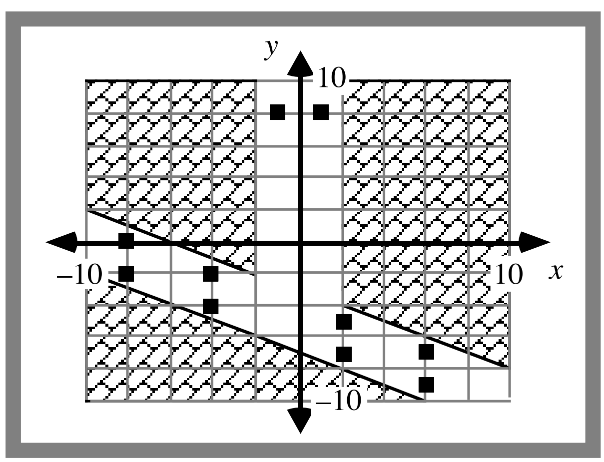
3.2 The following screen in the third level of Gates includes three gates and a brick wall. The curve shown on the screen represents a graph of the function

$$f(x) = \frac{1}{2}x^2 - 3x - \frac{7}{2}$$

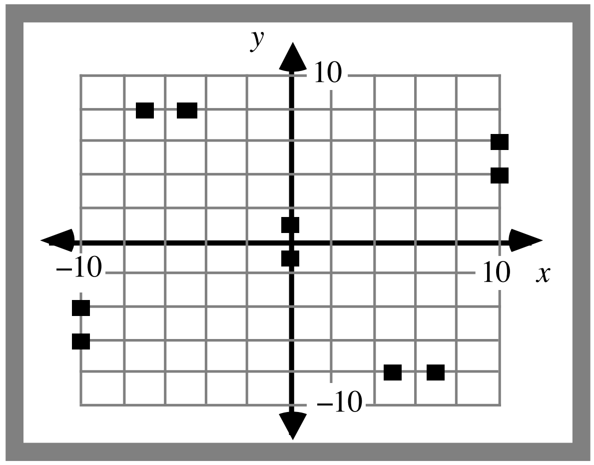
Add a rational expression to $f(x)$ so that the graph of the resulting function passes through all three gates while avoiding the brick wall. Identify the domain and range of your function.



3.3 The figure below shows another screen in the third level of Gates. Determine a rational function that passes through all five gates without touching the brick walls.



3.4 **a.** Create a rational function that passes through all the gates shown on the screen below.



b. Create a continuous piecewise function that passes through all the gates shown on the screen in Part **a**.

c. Compare the domains and ranges of the two functions created in Parts **a** and **b**.

3.5 A screen in the third level of Gates has six gates and a brick wall. The locations of the gates are defined by the following pairs of points: (–7,–1) and (–5,–1); (–5,0) and (–3,0); (–3,1) and (–1,1); (1,–2) and (3,–2); (5,5) and (7,5); (7,6) and (9,6).

The region occupied by the brick wall is defined by the following constraints: $1 \leq x \leq 3$ and $4 \leq y \leq 6$.

a. Use a graphing utility to recreate this screen.

b. Find a rational function whose graph passes through all the gates without hitting the brick wall.

3.6 Create a screen for the third level of Gates that includes five gates and two brick walls. Find a rational function whose graph passes through all the gates without hitting the walls. Identify the domain and range of your function.

3.7 Rational functions are generally written as the quotient of two polynomials. In this activity, however, you wrote rational functions in a form that makes it easier to determine the locations of any vertical asymptotes. Use a symbolic manipulator to convert each of the following functions to the form $r(x) = f(x)/q(x)$. Test your answers by graphing both forms of each equation on the same coordinate system.

a. $h(x) = 5x^2 + x - 10 + \dfrac{3}{x-7}$

b. $g(x) = -11x + 4 + \dfrac{-11}{3x+2}$

c. $q(x) = x^3 + x^2 + \dfrac{3}{x+1}$

* * * * *

3.8 Write each of the following rational functions as a polynomial expression plus a rational expression. Describe where asymptotic behavior might occur in the graph of each function.

a. $h(x) = \dfrac{3x^2 + 29x - 39}{x+11}$

b. $g(x) = \dfrac{-16x^3 + 40x^2 - 8x + 18}{2x-5}$

c. $q(x) = \dfrac{x^5 - 7x^4 - x^3 + 4x^2 + 25x - 31}{x-7}$

3.9 A cattle rancher would like to create a rectangular corral with an area of 150 m^2 using the least possible amount of fencing materials.

a. Let x represent the width of the rectangle. Write a rational function that describes the perimeter of the rectangle in terms of x.

b. Graph the function and determine the location of any asymptotes that occur. If an asymptote occurs, describe what its location represents in this situation.

c. Considering the context, identify a reasonable domain and range for the function.

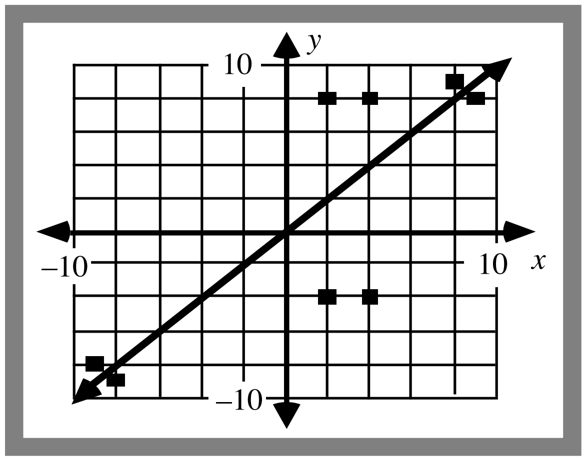
d. What is the minimum value for the perimeter of the rectangular corral? What are the corresponding values for the width and length of the corral?

e. Describe the shape of the rectangle that minimizes the perimeter for an area of 150 m^2. Do you think that this shape will minimize the perimeter for any rectangular region of a given area? Use examples to support your response.

* * * * * * * * * *

Summary Assessment

1. **a.** The figure below shows a screen in the third level of Gates. The line represents the graph of the polynomial function $f(x) = x$. Add a rational expression to $f(x) = x$ so that the graph of the resulting function passes through all four gates.

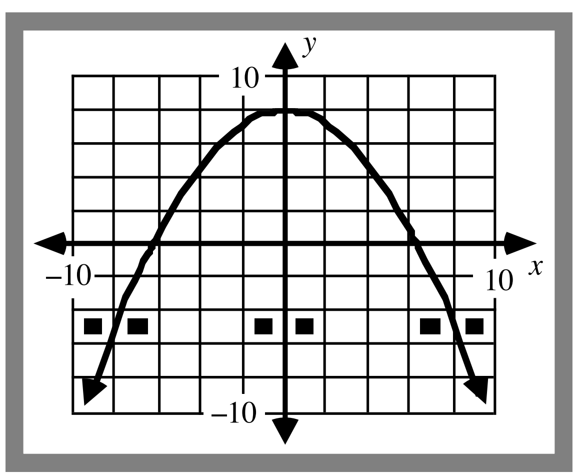


b. Write your function from Part **a** in the general form of a rational function, shown below:

$$r(x) = \frac{f(x)}{g(x)}$$

c. Use piecewise functions to create a graph that passes through the four gates shown in Part **a**.

2. The figure below shows another screen in the third level of Gates.



a. The curve shown represents the graph of the function below:

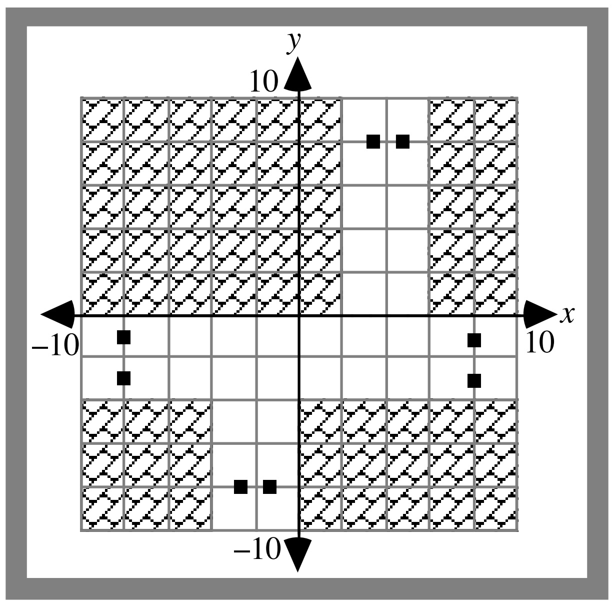
$$f(x) = -\frac{1}{5}x^2 + 8$$

Does this function have an absolute maximum or minimum? Justify your response.

b. Add a rational expression to $f(x)$ so that the graph of the resulting function passes through all the gates.

c. Write the new function in the general form of a rational function and identify its domain and range.

3. The figure below shows a screen in the third level of Gates. Determine a function whose graph passes through all four gates without touching the brick walls.



Module Summary

- A function is **continuous** at a point c in its domain if the following conditions are met:
  1. the function is defined at c, or $f(c)$ exists
  2. the limit of the function exists at c, or $\lim_{x \to c} f(x)$ exists
  3. the two values listed above are equal, or $f(c) = \lim_{x \to c} f(x)$.
- A function is continuous over its domain if it is continuous at each point in its domain.
- A function is **discontinuous** at a point if it does not meet all the conditions for continuity at that point.
- A **polynomial function** is a function of the form below, where a_n is a real number and n is a non-negative integer.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

- The **degree** of a polynomial is equal to the greatest exponent of the variable in the expression. The coefficient of that variable in the expression is the **leading coefficient**.
- The **end behavior** of the graph of a polynomial function describes the characteristics of the graph as $|x|$ approaches infinity.
- The **absolute maximum** of a function is the greatest value of the range, while the **absolute minimum** is the least value of the range.
- The x-coordinate of each point where the graph of a polynomial intersects the x-axis is a **zero** or a **root** of the polynomial.
- As $|x|$ approaches infinity, the graph of a polynomial function of even degree has the same behavior at both ends. As $|x|$ approaches infinity, the graph of a polynomial function of odd degree has opposite behavior at each end.
- In a **piecewise function**, different parts of the domain correspond with different rules.
- A **rational function** is a function of the form

$$r(x) = \frac{f(x)}{g(x)}$$

  where $f(x)$ and $g(x)$ are polynomial functions and $g(x) \neq 0$.

- An **asymptote** to a curve is a line such that the distance from a point *P* on the curve to the line approaches zero as the distance from *P* to the origin increases without bound, where *P* is on a suitable part of the curve.
- The graph of a rational function of the form

$$r(x) = f(x) + \frac{g(x)}{h(x)}$$

  where $h(x) \neq 0$ and the degree of $g(x)$ is less than the degree of $h(x)$, approaches the graph of $f(x)$ as $|x|$ increases without bound. This function also may be asymptotic to the vertical line $x = k$, where $h(k) = 0$, or it may have a point of discontinuity at $x = k$.

Selected References

Demana, F., B. K. Waits, and S. R. Clemens. *College Algebra & Trigonometry*. New York: Addison-Wesley, 1992.

Faires, J. D., and B. T. Faires. *Calculus and Analytic Geometry*. Boston: Prindle, Weber & Schmidt, 1983.