Changing the Rules Changes the Game

In the past century, women's basketball has changed from a game of six-person teams and half-court play to a game of five-person teams and full-court play. In this module, you investigate how changing the rules of geometry affects the game of mathematics.

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Introduction

About 300 B.C., the Greek mathematician Euclid recorded a set of basic notions and **axioms** for geometry. Axioms are statements that are assumed to be true. Euclid's axioms described the properties of geometric figures and the relationships among them, including the concepts that a line is straight, that lines could be parallel, and that there is only one parallel line to a given line through a point not on the line (called the Parallel Postulate).

Since Euclid's era, his ideas about geometry have become a part of our everyday lives. But what would happen to a geometry if its basic notions were not those described by Euclid? For example, Euclid tacitly assumed that a line had infinitely many points. This is not the case in finite geometries.

To understand the coordinate system associated with a finite geometry, you must be able to perform arithmetic using a finite set of integers. One place to explore finite arithmetic systems is on the face of a clock. These arithmetic operations on a clock then can be linked to a non-Euclidean finite geometry. In this module, you investigate some properties of a finite geometry and a finite arithmetic.

Exploration

In **clock arithmetic**, an n-hour clock contains the digits 1, 2, 3, ..., n. In such a system, addition is accomplished by moving clockwise around the dial, while subtraction is accomplished by moving counterclockwise around the dial.

a. Draw a 12-hour clock face.

b.
1. Describe a method for representing integers greater than 12 on a 12-hour clock.

2. Describe a method for representing integers less than 1 on a 12-hour clock.

3. Use your method to determine the 12-hour clock values for 15 and –5.

c. To distinguish the symbols for operations in clock arithmetic from those used in real-number arithmetic, they are often drawn with circles around them. The symbol $\oplus$, for instance, indicates clock addition. On a 12-hour clock, 11 hours after 4 o'clock can be symbolized as $4 \oplus 11$, or 3 o'clock. Similarly, 3 hours before 2 o'clock can be written as $2 \ominus 3$, or 11 o'clock.

1. Describe a method to represent addition of integers on a 12-hour clock.

2. Describe a method to represent subtraction of integers on a 12-hour clock.

3. Use your method to determine the sum $8 \oplus 7$ and the difference $5 \ominus 10$.

d. In real-number arithmetic, 0 is the additive identity since, for any real number a, $a + 0 = 0 + a = a$.

An additive identity also exists for 12-hour clock arithmetic. Determine its value.

e. In real-number arithmetic, multiplication can be thought of as "multiple additions." This is also the case in 12-hour arithmetic. For example, the multiplication $7 \otimes 3$ can be considered as shown below:

$$\begin{aligned} 7 \otimes 3 &= (7 \oplus 7) \oplus 7 \\ &= 2 \oplus 7 \\ &= 9 \end{aligned}$$

1. Determine the value of $4 \otimes 2$ in 12-hour arithmetic.

2. Determine the value of $4 \otimes 5$ in 12-hour arithmetic.

Discussion

a. Compare the method you described for representing positive and negative integers on a 12-hour clock with others in your class.

b. What number is the additive identity for 12-hour arithmetic? Justify your response.

c. **1.** Two numbers are said to be additive inverses if their sum is the additive identity. Identify an additive inverse for each number in 12-hour arithmetic.

2. In real-number arithmetic, each number has exactly one additive inverse. Is the corresponding statement true in 12-hour arithmetic? Justify your answer.

d. In real-number arithmetic, 1 is the multiplicative identity since, for any real number a, $a \bullet 1 = 1 \bullet a = a$.

A multiplicative identity also exists for 12-hour arithmetic. What number do you think is this identity? Explain your response.

e. Two numbers are said to be multiplicative inverses if their product is the multiplicative identity. Do you think that each number in 12-hour arithmetic has a multiplicative inverse? If so, identify the multiplicative inverse for each number on a 12-hour clock. If not, describe a number that does not have a multiplicative inverse.

Activity 1

To explore other finite arithmetic systems, mathematicians developed **modular arithmetic**. Modular arithmetic can provide some basic tools for exploring finite geometries. In this activity, you examine modular arithmetic and determine some of its properties.

Mathematics Note

The modular arithmetic system of **modulo *n*** (or **mod *n***) contains the digits $0, 1, 2, 3, ..., n-1$. For example, a modulo 8 (or mod 8) clock contains the numbers 0, 1, 2, 3, 4, 5, 6, and 7.

Like clock arithmetic, modular arithmetic can be thought of as taking place on a circular dial, as shown in Figure **1** below.

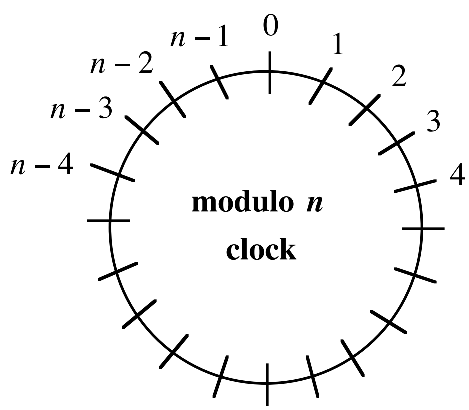

Figure 1: Mod *n* clock

Exploration 1

One way to visualize a modular arithmetic system is to consider a number line of integers "wrapped" around a mod *n* clock. Using this analogy, you can determine which modulo *n* values correspond with each integer on the number line.

For example, the integer 0 on the number line corresponds with 0 on the modulo clock. Moving clockwise around the clock face corresponds to moving along the positive portion of a number line.

a. Sketch a circle on a sheet of paper. Mark and label the circle to form a modulo 5 clock.

b. **1.** Use your mod 5 clock from Part **a** to complete Table **1** for the integers 0 through 12.

Table 1: Integers and their corresponding mod 5 values

Integer on Number Line	Mod 5 Value
0	0
1	
2	
⋮	
12	

2. Moving counterclockwise around the modulo clock face corresponds to moving along the negative portion of a number line. Use this notion to complete Table **1** for the integers –1 through –12.

c. By examining the values in Table **1**, you should observe that the same mod 5 value corresponds with more than one integer.

1. To help visualize this relationship, create a scatterplot of the data in Table **1**. Represent the integers along the *x*-axis and the corresponding mod 5 values along the *y*-axis.

2. Use the scatterplot to identify all the integers from –12 to 12 that correspond with the same value in mod 5.

d. In Part **c**, you should have observed that 12 and –8 both correspond with 2 (mod 5). This fact can also be illustrated using the division algorithm. When using the division algorithm, the remainder must be a non-negative integer less than the divisor.

As shown below, for example, 12 and –8 both have a remainder of 2 when divided by 5.

$$\begin{array}{r} 2 \quad \text{R2} \\ 5\overline{)12} \\ \underline{-10} \\ 2 \end{array} \qquad \begin{array}{r} -2 \quad \text{R2} \\ 5\overline{)-8} \\ \underline{-(-10)} \\ 2 \end{array}$$

The division algorithm allows you to determine the mod 5 values that correspond with large integers.

Identify two integers with absolute values greater than 500, one positive and one negative, that correspond with the same mod 5 value.

Mathematics Note

In modulo n, two integers are **congruent** (symbolized by $\equiv$) if they have the same remainder when divided by n.

In mod 5, for example, the integers 12 and 2 are congruent because 12 divided by 5 and 2 divided by 5 both have a remainder of 2. This can be written symbolically as $12 \equiv 2 \pmod 5$.

Discussion 1

a. How is congruence illustrated on the scatterplot you created in Part **c** of Exploration **1**?

b. Describe the process you would follow when using the division algorithm to determine the congruent mod n value of a negative number.

c. Wrapping a number line of integers around the mod 5 clock can be thought of as a function. What are the domain and range of this function?

Exploration 2

In real-number arithmetic, the numbers 1 and 0 play special roles. In this exploration, you create addition and multiplication tables and use them to identify numbers that play similar roles in mod n arithmetic. You then use these numbers to solve some mod n equations.

a. Complete Table **2**, a table of addition facts for mod 5.

Table 2: Addition facts for modulo 5

+	**0**	**1**	**2**	**3**	**4**
0					
1		2			0
2					
3				1	
4					

b. Complete Table **3**, a table of multiplication facts for mod 5.

Table 3: Multiplication facts for modulo 5

×	**0**	**1**	**2**	**3**	**4**
0					
1		1			
2					3
3				4	
4					

c. Create a table of subtraction facts for mod 5. Each entry in the table should represent the row value minus the column value.

d. Determine the additive identity for mod 5.

e. Identify the additive inverse for each element in mod 5.

f. Determine the multiplicative identity for mod 5.

g. The multiplicative inverse of x is also referred to as the reciprocal of x. For the set of real numbers, $1/x$ is the multiplicative inverse of x, for $x \neq 0$, since

$$x \bullet \frac{1}{x} = \frac{1}{x} \bullet x = 1$$

The multiplicative inverse of any element x (other than the additive identity) can be denoted by x^{-1}.

Identify the multiplicative inverse for each element in mod 5.

Mathematics Note

Many of the properties of congruence are comparable to properties of equality.

The **substitution property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$ and $b \equiv c$, then $a \equiv c$.

The **addition property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$, then $a + c \equiv b + c$.

The **multiplication property of congruence** states that if a, b, and c are any real numbers with $a \equiv b$, then $a \bullet c \equiv b \bullet c$.

h. Congruences in mod n can be solved using methods similar to those used to solve algebraic equations involving real numbers. For example, the solution to the congruence $5x - 6 \equiv 4 \pmod 7$ is shown below.

$5x - 6 \equiv 4 \pmod 7$	given
$5x \equiv 4 + 6 \pmod 7$	addition property of congruence
$5x \equiv 3 \pmod 7$	definition of congruence mod 7
$3(5x) \equiv 3(3) \pmod 7$	multiplication property of congruence
$1x \equiv 2 \pmod 7$	definition of congruence mod 7
$x \equiv 2 \pmod 7$	multiplicative identity

The solution can be checked by substituting 2 into $5x - 6 \equiv 4 \pmod 7$. Since $5(2) - 6 \equiv 3 - 6 \equiv 4 \pmod 7$, 2 is a solution to the equation.

Use the process described above to solve the equation $3x + 1 \equiv 2 \pmod 4$. Record the justification for each step in your solution.

Discussion 2

a. Which of the following modular operations are commutative?

1. addition

2. subtraction

3. multiplication

b. **1.** How can you define division in mod 5 arithmetic?

2. Is division commutative in mod 5 arithmetic? Explain your answer.

c. Consider the following congruence equation: $16 \bullet 6 \equiv x \ (\text{mod } 5)$. One way to determine a solution that is a mod 5 value is to multiply the two factors, then convert the product to mod 5.

Is the solution affected by converting both factors to mod 5 before multiplying?

d. For real numbers, addition and multiplication are both associative. This means that for any real numbers a, b, and c:

$$(a + b) + c = a + (b + c)$$

and

$$(a \bullet b) \bullet c = a \bullet (b \bullet c)$$

Are addition and multiplication associative in modular arithmetic?

e. For real numbers, multiplication is distributive over addition. In other words, for any real numbers a, b, and c:

$$a(b + c) = ab + ac$$

Do you think that multiplication (mod 5) is distributive over addition (mod 5)? Use an example to illustrate your response.

f. **1.** Why is there no multiplicative inverse for 2 (mod 6)?

2. Because there is no multiplicative inverse for 2 (mod 6), the equation $2x - 5 \equiv 4 \ (\text{mod } 6)$ has no solution. To verify that this is true, substitute each number in mod 6 into the equation.

g. **1.** Why is there no multiplicative inverse for 3 (mod 6)?

2. Although there is no multiplicative inverse for 3 (mod 6), the equation $3x \equiv 3 \ (\text{mod } 6)$ has three solutions: 1 (mod 6), 3 (mod 6), and 5 (mod 6).

Describe how you might find these solutions.

h. Describe some situations in which you might expect to use a modular arithmetic.

Assignment

1.1 Describe how to determine the number in mod 5 that is congruent to 33.

1.2 Calculate each of the following in mod 5:

a. $3+2$

b. $12 \bullet 8$

c. $13-20$

1.3 Complete the following addition and multiplication tables for mod 3.

+	**0**	**1**	**2**
0			
1			
2			

×	**0**	**1**	**2**
0			
1			
2			

1.4 **a.** Evaluate each of the following expressions.

1. $2+1 \pmod 3$

2. $2 \bullet 2 \pmod 3$

3. $16+9 \pmod 8$

4. $11 \bullet 7 \pmod{10}$

b. Find each of the following inverses.

1. the additive inverse of 3 (mod 5)

2. the additive inverse of 2 (mod 3)

3. the multiplicative inverse of 3 (mod 5)

4. the multiplicative inverse of 0 (mod 5)

1.5 **a.** What is the additive identity in mod 3?

b. Find the additive inverse in mod 3 for each of the following: 0, 1, and 2.

c. Recall that **proof by exhaustion** is the process of examining all possibilities to prove a statement. Use proof by exhaustion to show that 1 is the multiplicative identity for mod 3.

d. Prove or disprove the statement: "Every element in mod 3 has a multiplicative inverse."

1.6 Division in modular arithmetic may be defined as follows: $a \div b \equiv c \pmod n$ if and only if $b \bullet c \equiv a \pmod n$. Use this definition to find each of the following:

a. $1 \div 2 \pmod 5$

b. $3 \div 2 \pmod 5$

c. $4 \div 0 \pmod 5$

1.7 Prove that 2 does not have a multiplicative inverse in mod 4.

1.8 Solve for x in each of the following.

a. $x + 3 \equiv 1 \pmod{7}$

b. $4x \equiv 1 \pmod{7}$

c. $3x - 5 \equiv 4 \pmod{6}$

d. $2x + 1 \equiv 0 \pmod{3}$

e. $x^2 \equiv 1 \pmod{3}$

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1.9 Solve each of the following equations:

a. $11x - 6 \equiv 8 \pmod{13}$

b. $3x + 4 \equiv 5 \pmod{11}$

1.10 A monitoring device uses 0.5 m of paper per hour. Each roll of paper is 200 m long. If a new roll is installed at 9:00 A.M., at what time will the device run out of paper?

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Research Project

One common application of modular arithmetic is the Universal Product Code (UPC) found on nearly every consumer product. Each UPC bar code represents a 12-digit number.

Figure **2** shows a typical UPC bar code and number. The first number on the left (0) identifies the product. The last number on the right (8) is the check digit. To make certain that each code is read correctly, bar-code readers (such as those at supermarket cash registers) use an algorithm to perform an internal check.

0 75520 57070 8

Figure 2: A UPC bar code

For this research project, find an algorithm that performs a check on a bar code. Write an explanation of the algorithm. Collect some samples of UPC bar codes and verify that the algorithm works. Then create one valid and one invalid UPC bar code of your own.

Activity 2

In Activity **1**, you examined some of the basic principles of modular arithmetic. In this activity, you investigate a finite geometry coordinatized with a modular arithmetic system.

Mathematics Note

Finite geometries are unlike traditional Euclidean geometry because they use only finite numbers of points.

For example, one finite geometry is based on a modulo 3 arithmetic system. In this system, each point of a lattice has coordinates (x, y) where x and y are elements of the set $\{0, 1, 2\}$. Figure **3** shows the nine-point lattice used to construct this finite geometry.

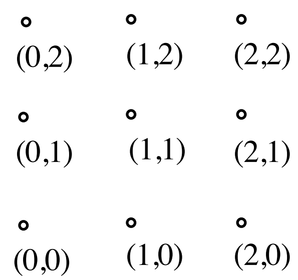


Figure 3: Coordinatized nine-point lattice

In this geometry, a line is defined as the set of all points that satisfies a mod n equation of the form $Ax + By + C = 0 \pmod{3}$ where A, B, and C are elements of the set $\{0, 1, 2\}$ with A and B not both 0. For example, one line is identified with the equation $x + y + 0 = 0 \pmod{3}$ or $x + y = 0 \pmod{3}$. This line contains only the points with coordinates (0,0), (2,1), and (1,2). A graph of this line is shown in Figure **4**.

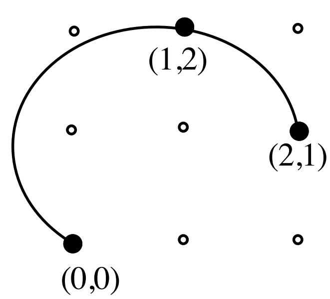


Figure 4: Graph of the line $x + y = 0 \pmod{3}$

Although Figure **4** shows the three points on the line connected by an arc, the line contains only those three points. There are no other coordinates that satisfy the equation. **Note:** For the remainder of this module, mod n equations will be written with equals signs rather than congruence signs.

Discussion 1

a. Compare the characteristics of a line in Euclidean geometry with the characteristics of a line in the finite geometry described in the previous mathematics note.

b. How do these characteristics of a line compare with its characteristics in spherical geometry?

c.
1. How might a triangle be defined in this nine-point geometry?
2. Give an example of a triangle that satisfies your definition.
3. How does your definition compare with the definition of a triangle in Euclidean geometry?

Exploration

In this exploration, you continue to investigate the nine-point geometry described in the mathematics note.

a. Determine the number of possible equations of the form $Ax + By + C = 0 \pmod 3$. List each of these equations.

b. Determine the coordinates of all the points in the nine-point geometry that satisfy each equation identified in Part **a**.

c. Graph each of the equations on a copy of the lattice template (available from your teacher). Connect each set of points in the solution with segments or arcs.

d.
1. It is possible for more than one equation to define the same line. Identify the equations of the unique lines in this nine-point system.
2. Label the points in a nine-point lattice *A* through *I*, as shown in Figure **5** below.

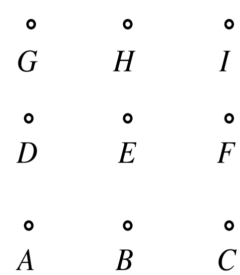


Figure 5: A nine-point lattice

3. Record both the coordinates and the letters that correspond with the points which satisfy each unique line. **Note:** Save this information for use in Problem **2.2**.

e. Determine if each of the following properties of lines in traditional Euclidean geometry is true in this nine-point geometry.

1. Two points determine a unique line.
2. If two distinct lines contain a common point, they contain exactly one common point.

f. Lines in a plane are **parallel** if they have either no points in common or all points in common. Are there parallel lines in this geometry? If so, identify them.

g. According to the Parallel Postulate (mentioned in the introduction), there is exactly one line parallel to a given line through a point not on that line.

1. Select a line in the nine-point geometry and a point not on the line. Determine if the parallel postulate is true for your selections.
2. Repeat Step **1** for each point not on the line until you have checked all appropriate points.
3. Select another line in the finite geometry and repeat Steps **1** and **2**.
4. Repeat Step **3** until all lines have been checked.

Discussion 2

a. What patterns do you observe among the values of A, B, and C in the equations of parallel lines?

Mathematics Note

In the nine-point geometry, a line given by $Ax + By + C = 0 \pmod 3$, where $B \neq 0$, may be expressed in the form $y = mx + b \pmod 3$ where m and b are elements of the set $\{0, 1, 2\}$ and m represents the **slope** of the line.

For example, consider the line defined by the equation $1x + 2y + 1 = 0 \pmod 3$. This equation may be rewritten using mod 3 arithmetic as follows:

$$\begin{aligned} 1x + 2y + 1 &= 0 \pmod 3 \\ 2y &= -1x + -1 \\ 2(2y) &= 2(-1x + -1) \\ 1y &= -2x + -2 \end{aligned}$$

However, –2 may be rewritten as 1 in mod 3 since $-2 \equiv 1 \pmod 3$. Therefore,

$$\begin{aligned} 1y &= 1x + 1 \\ y &= x + 1 \end{aligned}$$

In this case, the slope of the line is 1.

b. When the equation for a line in the form $Ax + By + C = 0 \pmod 3$ is rewritten in slope-intercept form, it becomes:

$$y = -\frac{A}{B}x - \frac{C}{B} \pmod 3$$

1. Describe the slope of the line when $A = 0$.
2. Describe the slope of the line when $B = 0$.

c. In a real-number coordinate plane, the slope of a line can be found using the coordinates of two points on the line. Is it possible to determine the slope of a line in the nine-point geometry using the coordinates of two points on the line? Justify your response.

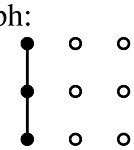
d. Compare the slope of a line in Euclidean geometry to the slope of a line in nine-point geometry.

Assignment

2.1 Write each of the distinct equations found in the exploration in the form $y = mx + b \pmod 3$ or $x = a \pmod 3$.

2.2 Use the equations in Problem **2.1** and the information you recorded in Part **d** of the exploration to complete the chart supplied by your teacher. The following diagram shows one completed cell in the chart. **Note:** Save this chart for use throughout the remainder of this module.

a.
(0,0), (0,1), (0,2)
Equation:
$x = 0$
Graph:



2.3 Consider a line in the nine-point geometry that contains the point (0,1) and has a slope of 1.

a. Write an equation for the line.

b. Use the completed chart from Problem **2.2** to verify your equation from Part **a** and identify the other points on the line.

2.4 How many triangles are there in the nine-point geometry? Explain your response.

2.5 Each row of the chart in Problem **2.2** contains three lines. By considering them in pairs, prove that the lines in each row are parallel. Recall that in coordinate geometry, two lines are parallel if they either have the same slope or both have undefined slopes and are vertical.

2.6 In Euclidean geometry, two lines with non-zero slopes are perpendicular when the product of their slopes is -1. A line with an undefined slope is perpendicular to a line with a slope of 0.

a. Consider two perpendicular lines with non-zero slopes in the nine-point geometry. If a comparable definition of perpendicular lines is true, what must be the product of their slopes?

b. Identify all pairs of perpendicular lines in the nine-point geometry.

c. In Euclidean geometry, the following properties involving perpendicular lines in a plane are true.

1. At a point on a line, there is exactly one line perpendicular to the given line.
2. From a point not on a line, there is exactly one line perpendicular to the given line.
3. Two lines perpendicular to the same line are parallel.

Determine if these properties are true in the nine-point geometry by considering every possible case.

2.7 Consider the lines defined by the following mod 3 equations:

$$y = 2$$

$$y = 2x$$

a. Find the intersection, if any, of these two lines.

b. Repeat Part **a** for the mod 3 equations below:

$$y = x$$

$$y = 2x$$

2.8 Is every pair of intersecting lines in the nine-point geometry perpendicular? Explain your response using a proof.

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2.9 Consider a geometry based on a modulo 4 number system in which each point of a lattice has coordinates (x,y) where x and y are elements of the set $\{0, 1, 2, 3\}$. In this geometry, a line is defined as the set of all points on the lattice that satisfies a mod 4 equation of the form $Ax + By + C = 0$.

a. Construct a lattice and graph the equation $y = 2x + 3$ (mod 4).

b. Write the equation (in mod 4) of a line parallel to the line given in Part **a** and containing point (0,0).

c. Find the equation (in mod 4) of a line perpendicular to the line given in Part **a** and containing point (0,0).

2.10 Does every pair of perpendicular lines in the nine-point geometry intersect? Verify your response using proof by exhaustion.

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Research Project

A triangle with two sides perpendicular is a **right triangle**. By this definition, the points with coordinates (0,0), (0,2), and (2,1) in the nine-point geometry determine a right triangle. This fact can be proven as described below.

The slope of the line through the points with coordinates (0,0) and (2,1) can be calculated as follows:

$$\frac{1-0}{2-0}=\frac{1}{2}$$

Since $1 \div 2 \equiv 2 \pmod 3$, the slope of the line is 2. Similarly, the slope of the line through (0,2) and (2,1) is 1. The product of the two slopes is 2.

As mentioned in Problem **2.6**, two lines with non-zero slopes are perpendicular when the product of their slopes is –1. Since –1 is congruent to 2 (mod 3), the lines are perpendicular. Therefore, the triangle is a right triangle.

A picture of the triangle formed by (0,0), (0,2), and (2,1) is shown in Figure **6**.

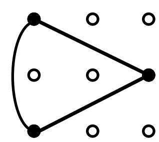

Figure 6: A right triangle in the nine-point geometry

How many other right triangles can be formed in the nine-point geometry?

Activity 3

In Activity **2**, you investigated a finite geometry using algebra and the coordinates of points. In this activity, you explore a finite geometry as an **axiomatic system**. In other words, you use undefined terms, definitions, axioms, and proven theorems to describe the system.

Mathematics Note

An **axiomatic system** is a mathematical system that contains:

- **undefined terms** (terms assumed without definition)
- **definitions** (terms defined using undefined terms and other definitions)
- **axioms** (rules assumed to be true that describe relationships among terms)
- **theorems** (statements proven true using logic)

In Euclidean geometry, for example, both *line* and *point* are undefined terms. The statement, "A line extends indefinitely in two directions" is an axiom, since it is assumed to be true. The statement, "The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs" is a theorem since it can be proven.

Exploration

Gino Fano was one of the first mathematicians to study a finite geometry. In 1892, he built a geometry to satisfy the following five axioms, leaving the terms *point*, *line*, and *on* undefined. (In these axioms, the words *contains* and *has* also are undefined terms.)

1. There exists at least one line.

2. Every line has exactly three points.

3. Not all points are on the same line.

4. For any two points, there exists exactly one line that contains both of them.

5f. Every two different lines have at least one point in common. (The *f* in 5*f* stands for Fano.)

In this exploration, you use Fano's axioms to deduce the properties of his geometry.

a. Considering only Axioms **1** and **2**, determine the minimum number of points in this geometry. Hint: Start listing points by designating each one in order with the letters A, B, C, and so on.

b. Considering only Axioms **1–3**, determine the minimum number of points in this geometry.

c. Now consider all five of Fano's axioms. Determine the number of points and the number of lines in this geometry.

d. Draw a model of Fano's geometry using the number of points and lines from Part **c**.

e. By changing Fano's fifth axiom, John Wesley Young developed another finite geometry (referred to as "Young's geometry" in this module). Young's fifth axiom reads as follows:

> **5*y*.** For each line l and each point B not on line l, point B is on one line that does not contain any other points from line l. (The y in 5y stands for Young.)

Young's geometry has nine points. Use a lattice similar to the one shown in Figure **7** to draw a model of Young's geometry.

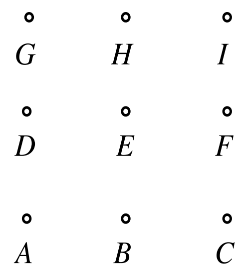

Figure 7: A nine-point lattice

Discussion

a. In Fano's geometry, does Axiom **1** tell you that there are any points on a line? Does it give any hint about how a line might look?

b. What does it mean to say that Fano's five statements are axioms?

c. In either Fano's or Young's geometry, is it possible for two distinct lines to contain the same two points? Explain your response.

d. How does Young's geometry compare to Fano's geometry?

Mathematics Note

The process of **deductive reasoning** begins with a hypothesis, then uses a logical sequence of valid arguments to reach a conclusion.

In mathematical proofs by deductive reasoning, each argument is typically supported by an axiom, definition, or previously proven theorem. A **direct proof** makes direct use of the hypothesis to arrive at the conclusion.

For example, consider the following statement: "In Fano's geometry, each point on a line is a member of at least three lines." To prove this statement using a direct proof, it should first be restated in if-then form: "If a point is on a line, then it is a member of at least three lines."

Assuming that the hypothesis, "If a point is on a line," is true, it can be symbolized as follows: Point *B* is on one line, l_1.

By Axiom **2**, l_1 must contain two other points: *A* and *C*.

By Axiom **3**, there must exist a point *D* not on l_1.

By Axiom **4**, there must be a line through points *B* and *D*: l_2.

From Part **c** of Discussion **1**, both *A* and *B* cannot be on l_2. Similarly, both *A* and *C* cannot be on l_2. By Axiom **2**, however, l_2 must contain one other point: *E*.

By Axiom **4**, there exists a line, l_3, that contains *A* and *D*.

To satisfy Axiom **2**, l_3 also must contain a third point, *F*.

According to Axiom **4**, another line, l_4, must contain *B* and *F*.

Lines l_1, l_2, and l_4 all contain *B*. Therefore, point *B* is contained in at least three lines.

In conclusion, the following statement is true in Fano's geometry. "If a point is on a line, then it is a member of at least three lines."

e. How does a theorem differ from an axiom?

f. The points and lines used to prove the statement in the mathematics note were given arbitrary names. Describe how this helps prove that the theorem is true for all points and lines in Fano's geometry.

Assignment

3.1 Consider the following theorem in Young's geometry: "For any point, there is a line not containing it."

a. Rewrite this theorem as an if-then statement. Identify the hypothesis and the conclusion.

b. When considering this theorem given any point *D* and a line *m*, there are two possible cases. If *D* is not on line *m*, then there is nothing to prove. If *D* is on line *m*, the theorem can be proved by Steps **1–3** below. Give a reason for each step in this proof.

1. Point *D* is on line *m*.
2. There exists a point *E* not on line *m*.
3. Through *E* there is a line *l* not containing any points of line *m*.
4. In conclusion, given any point, there is a line not containing it.

3.2 For two different lines to be parallel, they must not intersect. In other words, the two lines must have no points in common. Prove that there are no parallel lines in Fano's geometry. Begin your proof with the hypothesis "Lines l_1 and l_2 are different lines in Fano's geometry." Conclude your proof with the statement, "Line l_1 is not parallel to l_2."

3.3 In Young's geometry, prove that every point is contained in at least four lines. Draw a sketch to support your proof.

3.4 In Young's geometry, prove that given any line, there is a different line parallel to it.

3.5 How would your model of Young's geometry change if a line contained four points?

3.6 Consider the points in Young's geometry with coordinates (0,0) and (1,1). Do you think that the Pythagorean theorem could be used to find the distance between these two points? Explain your response. If so, can the distance be expressed in mod 3?

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3.7 Use a direct proof to prove the following statement: "If n is an even number, then n^2 is an even number."

3.8 Use a direct proof to prove the following statement: "If n is an integer, then $n^3 - n$ is even." Hint: you will need to prove two cases, one in which n is even and one in which n is odd.

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Activity 4

In Activity **3**, you examined some properties of Fano's and Young's geometries using proofs by exhaustion and direct proofs. In this activity, you use **indirect proofs** to continue your investigations of these two geometries.

Exploration

Indirect proofs are based on the notion of *reductio ad absurdum*, or "reduction to the absurd." In an indirect proof, the property to be proven true is assumed to be false. From this assumption, statements are argued logically with supporting reasons (axioms, definitions, and proven theorems) until a contradiction to either a known fact or an assumption is reached. If a contradiction can be reached, then the assumption must be false. Therefore, the original statement is true.

For example, consider a number n that is an even perfect square. In the following exploration, you prove that the square root of n also is even using an indirect proof.

a. Suppose that m is the square root in question—in other words, that $m^2 = n$. Assume that m is not even. Use an algebraic equation to express m in terms of a, another natural number. Your equation should show that m is indeed odd.

b. Square both sides of the equation from Part **a**. Is the result an even or an odd natural number?

c.
1. Can the statement that n is an even perfect square and the result in Part **b** both be true?
2. Do you believe that the following statement is true: "The square root of an even perfect square also is even"? Explain your response.

Discussion

a. What is the negation of the statement: "The square root of 2 is an irrational number"?

b. In general, the negation of a statement "if p, then q" is "p and not q." How do truth tables verify this relationship?

Mathematics Note

In an **indirect proof**, a statement is proven true by proving that its negation cannot be true.

For example, consider this statement in Young's nine-point geometry: "If a line intersects one of two parallel lines, then it intersects the other." This statement may be rewritten as follows: "If line l_1 is parallel to l_2, and lines l_2 and l_3 each contain point B, then lines l_1 and l_3 intersect."

To prove this statement using an indirect proof, assume that line l_1 is parallel to l_2, that lines l_2 and l_3 each contain point B, and that lines l_3 and l_1 do not intersect. A sketch of this situation is shown in Figure **8**. (The three points in each line are connected for organizational purposes.)

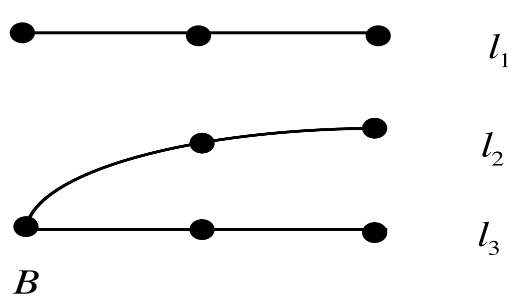

Figure 8: Sketch created using an assumption

If l_3 and l_1 do not intersect, then l_3 and l_1 have no points in common because of the meaning of non-intersecting.

Line l_1 is parallel to l_2, by the hypothesis.

Point *B* is on l_3 and l_2, by the hypothesis.

Now there are two lines, l_3 and l_2, that both contain point *B* and both are parallel to l_1. This contradicts Axiom **5y**, which states that through a point *B* not on a line *l*, there is exactly one line that has no points in common with the given line.

Therefore, the assumption that lines l_3 and l_1 do not intersect must be false because this would mean there could be more than one line that has no point in common with the given line. Consequently, lines l_3 and l_1 intersect.

In conclusion, the following statement is true: "If a line intersects one of two parallel lines, then it intersects the other."

Assignment

4.1 Consider the following theorem in Young's nine-point geometry: "If two lines intersect, then they intersect in exactly one point."

- **a.** Identify the hypothesis and the conclusion in this statement.
- **b.** To prove this statement using an indirect proof, what must you assume to be true?
- **c.** Sketch a picture of the situation that includes your assumption.
- **d.** Which axiom does your assumption contradict?
- **e.** What does this contradiction indicate about your assumption?
- **f.** What can you now conclude about the theorem to be proved? Explain your response.

4.2 Prove indirectly the following statement in Young's geometry: "If two lines are parallel to the same line, then they are parallel to each other." Hint: Use the theorem proven in the mathematics note.

4.3 Prove that every two different lines in Fano's seven-point geometry have exactly one point in common. Begin your proof with the hypothesis that l_1 and l_2 are different lines. Conclude your proof with the statement that lines l_1 and l_2 have exactly one point in common.

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4.4 The following paragraph provides an indirect proof of the statement, "The $\sqrt{2}$ is an irrational number." Describe the contradiction which shows that the assumption must be false.

Assume $\sqrt{2}$ is a rational number. This means that $\sqrt{2} = p/q$ where p and q are whole numbers and p/q is in lowest terms. Square both sides of the equation, as shown below:

$$\left(\sqrt{2}\right)^2 = \left(p/q\right)^2$$

$$2q^2 = p^2$$

Since p^2 is even, p must be even. Since p is even, it can be written in the form $p = 2r$, where r is a whole number. By substitution,

$$(2r)^2 = 2q^2$$

$$4r^2 = 2q^2$$

$$2r^2 = q^2$$

Since q^2 is even, q also must be even. Therefore, the square root of 2 must be irrational.

4.5 Use an indirect proof to prove the following: "If a cash register contains $1.45 in nickels and dimes, there must be an odd number of nickels."

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Summary Assessment

1. **a.** Construct addition and multiplication tables for mod 4.

b. Use these tables to describe the existence of additive inverses, an additive identity, multiplicative inverses, and a multiplicative identity in mod 4. Use specific examples in your response.

2. Consider a four-point geometry that has the following three axioms.

- There are exactly four points.
- Through any two points there is exactly one line.
- Given two points there is exactly one line containing them.

In this geometry, the terms *point*, *line*, and *contains* are undefined.

a. Draw a model to represent this geometry.

b. Create a coordinate system in mod 2 for this geometry.

c. Construct addition and multiplication tables for mod 2.

d. Find the equations for all distinct lines in this geometry.

e. If parallel lines are defined as having no points in common, prove that this system has at least three pairs of parallel lines.

f. Use a direct proof to prove that any point in the system is contained in at least three lines.

3. Use an indirect proof to show that if n is an integer and n^2 is odd, then n is odd.

Module Summary

- In **clock arithmetic**, an *n*-hour clock contains the digits 1, 2, 3, ..., *n*. In such a system, addition is accomplished by moving clockwise around the dial, while subtraction is accomplished by moving counterclockwise around the dial.
- To distinguish the symbols for operations in clock arithmetic from those used in real-number arithmetic, they are often drawn with circles around them. The symbol $\oplus$, for instance, indicates addition.
- A modular arithmetic system of **modulo *n*** (or **mod *n***) contains the digits 0, 1, 2, 3, ..., $n-1$. In such a system, addition and subtraction are accomplished in a manner similar to clock arithmetic.
- In modulo *n*, two numbers are **congruent** if they have the same remainder when divided by *n*. The symbol $\equiv$ denotes congruence.
- Given a set and the operation of addition defined on that set, an **additive identity** is the unique element *a* of the set such that when *a* is added to any element *x*, the result is that element *x*. In other words, $x + a = a + x = x$.
- Two elements whose sum is the additive identity are **additive inverses**. In other words, *b* is an additive inverse of *x* if $x + b = b + x = a$. The additive inverse of *x* is denoted by $-x$.
- Given a set and the operation of multiplication defined on the set, a **multiplicative identity** is the unique element *c* of the set such that when any element *x* is multiplied by *c*, the result is that element *x*. In other words, $x \bullet c = c \bullet x = x$.
- Two elements whose product is the multiplicative identity are **multiplicative inverses**. In other words, *d* is the multiplicative inverse of *x* if $x \bullet d = d \bullet x = c$. The multiplicative inverse of any element *x* (other than the additive identity) can be denoted by x^{-1}. The multiplicative inverse of *x* is also referred to as the reciprocal of *x*.
- The **substitution property of congruence** states that if *a*, *b*, and *c* are any real numbers with $a \equiv b$ and $b \equiv c$, then $a \equiv c$.
- The **addition property of congruence** states that if *a*, *b*, and *c* are any real numbers with $a = b$, then $a + c \equiv b + c$.
- The **multiplication property of congruence** states that if *a*, *b*, and *c* are any real numbers with $a = b$, then $a \bullet c \equiv b \bullet c$.

- An **axiomatic system** is a mathematical system that contains:
    - **undefined terms** (terms assumed without definition)
    - **definitions** (terms defined using undefined terms and other definitions)
    - **axioms** (rules assumed to be true that describe relationships among terms)
    - **theorems** (statements proven true using logic).
- A **finite geometry** is an axiomatic system which, unlike traditional Euclidean geometry, uses a finite number of points.
- The process of **deductive reasoning** begins with a hypothesis, then uses a logical sequence of valid arguments to reach a conclusion.

  In mathematical proofs by deductive reasoning, each argument is typically supported by an axiom, definition, or previously proven theorem. A **direct proof** makes direct use of the hypothesis to arrive at the conclusion.
- In an **indirect proof**, the property to be proven true is assumed to be false. From this assumption, statements are argued logically with supporting reasons (axioms, definitions, and proven theorems) until a contradiction to either a known fact or an assumption is reached. If a contradiction can be reached, then the assumption must be false. Therefore, the original statement is true.

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