Ostriches Are Composed

Although ostrich ranching has flourished in South Africa since the mid-1800s, it has only recently become popular in North America. In this module, you use compositions of functions to investigate the business of raising big birds.

Todd Fife • John Gebert • Sue Moore

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Introduction

Sal and Guinn are partners in a new venture: an ostrich ranch. Although ostrich ranching has flourished in South Africa for over a century, it has only recently become popular in North America.

After months of research, Sal and Guinn buy several pairs of breeding ostriches. Shortly thereafter, the first chicks hatch. The two partners plan to market both unhatched eggs and adult birds. To help attract customers, they decide to come up with a catchy phrase for their new toll-free number—something like 1-800-OSTRICH or 1-800-BIG-BIRD. Unfortunately, both of these numbers are already in use. The telephone company assigns them the number 1-800-825-2445. With a little bit of creative thought, Sal and Guinn plan to use this number to remind their customers of ostriches.

Discussion

a. Using the keypad in Figure **1** for reference, what telephone numbers correspond to OSTRICH and BIG BIRD?

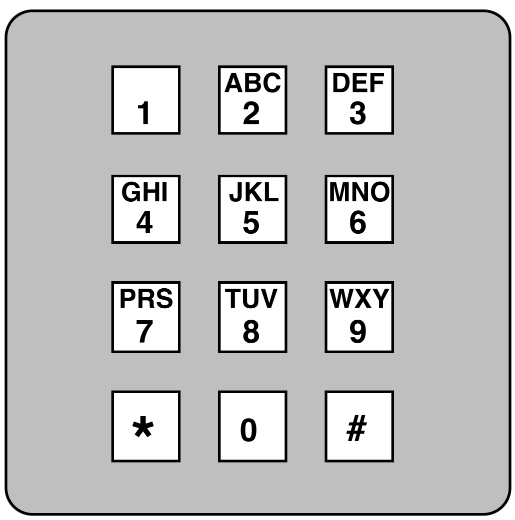


Figure 1: A telephone keypad

b. What catchy phrase might Sal and Guinn use for their assigned number, 1-800-825-2445?

c. Compare the processes you used to respond to Parts **a** and **b**.

d. Use ordered pairs to define the pairing of letters of the alphabet to numbers on the telephone keypad in Figure **1**.

e. Use ordered pairs to define the pairings of numbers to letters on the keypad.

f. Describe the differences between the pairings in Parts **d** and **e**.

Activity 1

Sal and Guinn are excited about the potential for profit in their new business. Ostriches are valued for beautiful feathers, leathery hides, and flavorful meat. Since Sal and Guinn plan to sell both unhatched eggs and adult birds, their income will depend on the number of breeding pairs in their flock. In turn, the number of birds they can keep will depend on the availability of such resources as fenced pasture and fresh water.

In applications such as ostrich ranching, it is often important to know how two quantities—such as profit and flock size—are related.

Exploration

Recall that a **relation** is a set of ordered pairs in which the **domain** is the set of first elements and the **range** is the set of second elements. A **function** is a relation from a domain to a range in which each element of the domain occurs in exactly one ordered pair. Both relations and functions are sometimes specified by a rule relating the domain and range.

Note: Functions often are written without listing a domain. In such cases, the domain is considered to contain all elements for which the function is mathematically meaningful. If the function is composed of numerical ordered pairs, the domain is typically either the set of real numbers or one of its subsets.

Determine a possible domain and range for each of the following. Use a graphing utility to check your results.

a. $f(x) = \sqrt{x-1}$

b. $g(x) = -|x|$

c. $h(x) = \sqrt{4-x^2}$

d. $k(x) = \dfrac{\sin x}{x-5}$

Discussion

a. Explain why the domains of some of the relations in the exploration are restricted.

b. How did your graphing utility indicate limitations in the domain for these relations?

c. Which of the relations in the exploration are functions?

d. Describe how a vertical line can be used to show that the graph of a relation does not represent a function.

e. Define a relation from a subset of the set of digits on a telephone keypad to the letters on a telephone keypad.

1. What is the domain of the relation?
2. What is the range of the relation?
3. Is this relation a function? Explain your response.

f. Consider the relation of the set of letters on a telephone keypad to the set of digits $\{0, 1, 2, \ldots, 9\}$.

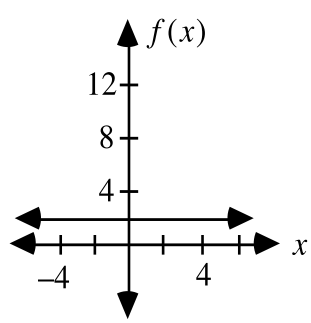
1. What is the domain of the relation?
2. What is the range of the relation?
3. Is this relation a function? Explain your response.

g. Describe a reasonable domain and range for a function Sal and Guinn might use to calculate their profit, if profit is a function of the number of ostriches in their flock. Justify your response.

Assignment

1.1 **a.** Identify the domain, range, and a possible rule for each of the following relations.

1. $h = \{(1,1), (1,-1), (4,2), (4,-2), (9,3), (9,-3), (16,4), (16,-4)\}$
2. $r = \{(1,1), (-2,-8), (3,27), (-4,-64)\}$
3.



b. Which of the relations in Part **a** are functions? Explain your response.

Mathematics Note

One way to represent a function is with a **set diagram**, using an arrow to represent the rule. The set diagram in Figure **2** illustrates the domain and range of the function *f* for $f(x) = x^2$. The domain is the set of real numbers; the range is the set of real numbers greater than or equal to 0.

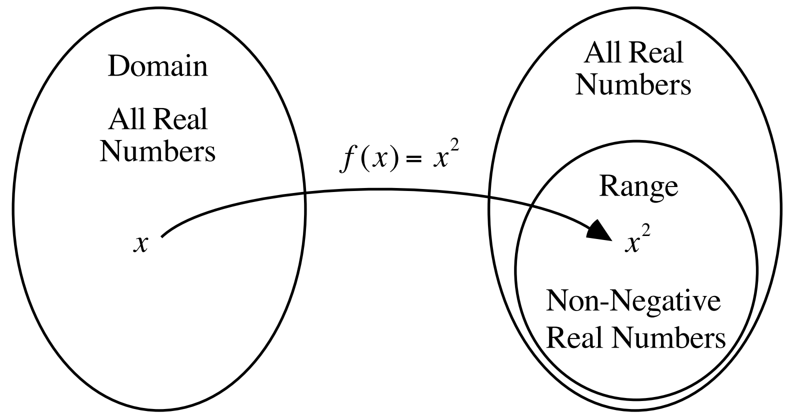


Figure 2: Set diagram illustrating the function $f(x) = x^2$

Another way to represent a function between two sets is a **mapping diagram**. Figure **3** shows a mapping diagram for the function $h(x) = x + 2$. Both the domain and range are the set of real numbers. The arrows indicate pairings of some elements in the domain with the corresponding elements in the range.

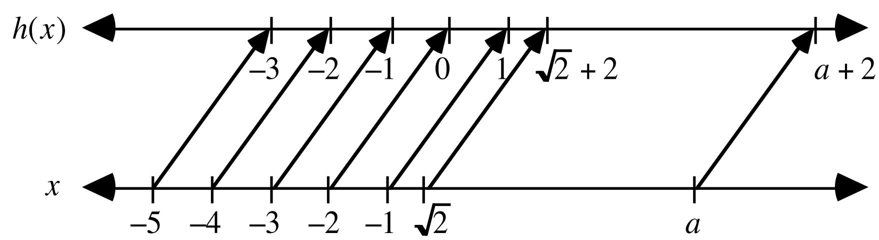


Figure 3: Mapping diagram illustrating the function $h(x) = x + 2$

1.2 Consider the telephone keypad shown in Figure **1**. Select a key that contains both letters and a number.

a. Consider the relationship between the number and the corresponding letters.

1. Draw a set diagram of this relationship.
2. Draw a mapping diagram of this relationship.
3. Write a description of the relationship.

b. Consider the relationship between the letters and the corresponding number. Repeat the steps in Part **a** for this relationship.

1.3 An ostrich can run at a rate of about 50 km/hr.

a. Using function notation, write an equation for the distance an ostrich can run in terms of time measured in minutes.

b. Determine a possible domain and range for the function.

c. Create a graph that could be used to estimate the distance an ostrich can run in a given time.

1.4 The domain of a function f is the set of real numbers. The function is defined by the rule $f(x) = 3x$.

a. What is the range of this function? Justify your response algebraically.

b. Given the restricted domain $[-1,1]$, find the corresponding range for this function.

c. Find the domain for which $f(x) = 3x$ results in the range $[-1,1]$.

1.5 Consider the function defined by the rule $f(x) = 3x - 2$, with domain $[-5,10]$, where x is a real number. Find the range of this function.

1.6 Write a function, giving its domain and range, in which each element of the range is paired with two elements of the domain.

1.7 For each of Parts **a–c** below, determine a rule for a function with the given domain and range. Illustrate each rule graphically.

a. domain: (∞,∞); range: $(-\infty,4]$

b. domain: $(-\infty,0)\cup(0,\infty)$; range: $(-\infty,0)\cup(0,\infty)$

c. domain: $[3,\infty)$; range: $[0,\infty)$

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1.8 At standard temperature and pressure, water boils at 100° C or 212° F. Likewise, water freezes at 0° C or 32° F. The relationship between temperatures measured in degrees Fahrenheit and Celsius is linear.

a. Write a function, $f(c)$, which would convert any given temperature in degrees Celsius to its corresponding temperature in degrees Fahrenheit.

b. Find the domain and range for the function.

1.9 The Greek mathematician Diophantus (ca. 250 A.D.) has sometimes been called the "father of algebra." A set of special equations bears his name. The solutions to a Diophantine equation consist of ordered pairs, each of whose elements are integers. For example, the solutions to the linear Diophantine equation $y = x$ is the set of ordered pairs below:

$$\{\ldots,(-3,-3),(-2,-2),(-1,-1)(0,0),(1,1),\ldots\}$$

Find the solutions, if any, to the following Diophantine equations:

a. $x + 5y = 11$

b. $3x + 6y = 71$

1.10 Identify an appropriate domain and range for each function described below.

a. The cost of producing *x* number of shirts by a clothing company is modeled by the function:

$$c(x) = 23 + 2x$$

b. The distance in meters that an object falls in *t* seconds can be modeled by the function:

$$d(t) = 4.9t^2$$

c. The quantity of radioactive carbon-14 remaining after *t* years is modeled by the following function, where $q_\circ$ is the initial quantity:

$$q(t) = q_\circ e^{-0.000121t}$$

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Activity 2

Sal and Guinn's ranch receives income from two sources: selling unhatched eggs and selling adult breeding pairs. Although the partners could determine a separate profit function for each of these income sources, they would like to simplify the process of projecting potential profits. In this activity, you examine how two functions can be combined into a single function.

Exploration 1

Sal and Guinn sell eggs for $500 each. The number of eggs that they sell each year depends on the number of adult breeding pairs in their flock. Each breeding pair produces an average of 40 viable eggs per year. Not all of the eggs are sold, however. Four of every 40 eggs are kept for hatching on the ranch. Annually, the ranch spends about $60,000 in overhead related to egg production.

Mature breeding pairs bring $10,000 per pair. Like the number of eggs sold, the number of breeding pairs sold is a function of the number of breeding pairs in the flock. Each breeding pair annually produces 4 eggs that are hatched on the ranch and raised to maturity. (Assume that 50% of the hatched chicks are male and 50% are female.) The ranch spends about $80,000 per year to maintain their adult ostrich flock.

a. **1.** Using the information given above, determine a function $e(x)$ that describes the annual profit on the sale of eggs, where x is the number of breeding pairs in the flock.

2. Determine another function $p(x)$ that describes the annual profit on the sale of breeding pairs, where x is the number of breeding pairs in the flock.

b. **1.** The size of Sal and Guinn's flock varies from 5 to 15 breeding pairs a year. Use appropriate technology to evaluate $e(x)$ for each element of this domain.

2. Evaluate $p(x)$ for each element of the domain described in Step **1**.

c. To find a function for the ranch's total profit, add the two functions in Part **a** to obtain $(e + p)$. Determine a rule for this new function.

d. Assuming the flock varies from 5 to 15 breeding pairs, determine the domain and range for $(e + p)$.

Discussion 1

a. Compare the profit functions e and p you obtained in Part **a** of Exploration **1** with those of others in the class.

b. Compare the rule you determined for $(e + p)$ with those of others in the class.

c. Describe how the range of the function $(e + p)$ was derived from the ranges of e and p.

d. Using the function $(e + p)$ as a model, can the total profit for any one year on Sal's and Guinn's ranch ever be exactly $200? Explain your response.

e. When adding functions, the distributive property of multiplication over addition allows like terms to be combined. For example, consider the sum of the functions $f(x) = 32x$ and $g(x) = 16x$. Since $32x$ and $16x$ have the common factor of x, the sum can be simplified using the distributive property of multiplication over addition as follows:

$$\begin{aligned} f(x) + g(x) &= 32x + 16x \\ &= (32 + 16)x \\ &= 48x \end{aligned}$$

Which of the following sums can be simplified using the distributive property?

1. $16x^2 + 16x^3$
2. $12\sqrt{x} + 3\sqrt{x}$
3. $\cos x + 5\cos x$
4. $2\sin x + \sin(2x)$

Exploration 2

In Exploration **1**, you added two functions together. In this exploration, you examine the results of some other operations on functions.

a. Consider the following functions:

$$f(x) = \sqrt{x}$$
$$g(x) = \sqrt{-x + 5}$$

1. Graph the two functions on the same coordinate system.
2. Determine the domain and range of each function.
3. Find the intersection of the domains of $f(x)$ and $g(x)$.

Mathematics Note

The function $(f + g)$ is defined by $(f + g)(x) = f(x) + g(x)$. Likewise, $(f - g)(x) = f(x) - g(x)$, and $(f \bullet g)(x) = f(x) \bullet g(x)$.

When adding, subtracting, or multiplying two or more functions, the operations are defined only for those values common to the domains of all the functions involved. For example, consider the function $f(x) = 2/x$, with a domain of $(-\infty, 0) \cup (0, \infty)$, and the function $g(x) = 5$, with a domain of $(-\infty, \infty)$. The domains have common values of $(-\infty, 0) \cup (0, \infty)$. Since addition, subtraction, or multiplication of the two functions is defined only for those common values of x, the domain of $(f + g)(x)$ is $(-\infty, 0) \cup (0, \infty)$.

b. Using the functions from Part **a**, graph each of the following on a separate coordinate system and determine its domain and range:

1. $(f+g)(x)$

2. $(f-g)(x)$

3. $(f \bullet g)(x)$

c. Functions may also be divided. When $f(x)$ is divided by $g(x)$, it can be represented as

$$\frac{f(x)}{g(x)}$$

or $(f/g)(x)$, where $g(x) \neq 0$.

1. Graph the function $(f/g)(x)$. Determine the domain and range.

2. Graph the function $(g/f)(x)$. Determine its domain and range.

d. Repeat Parts **a–c** for the functions $f(x)=2$ and $g(x)=\sin x$.

Discussion 2

a. Consider two functions: f and g. The domain of f is $\{1, 2, ..., 10\}$, while the domain of g is $\{-3, -2, ..., 5\}$. The range of each function is a subset of the real numbers. In this case, what is the domain of $(f+g)$? Explain your response.

b. In the function defined by f/g, why is it important that $g(x) \neq 0$?

c. Compare the domain and range of $(f/g)(x)$ and $(g/f)(x)$ when $f(x)=\sqrt{x}$ and $g(x)=\sqrt{-x+5}$.

Assignment

2.1 Consider the functions $f(x)=x^2+3x$ and $h(x)=x-2$.

a. Determine a rule for each of the following:

1. $f+h$

2. $f-h$

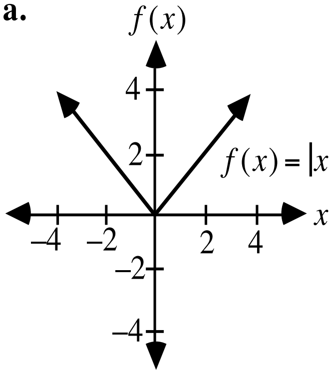
3. $f \bullet h$

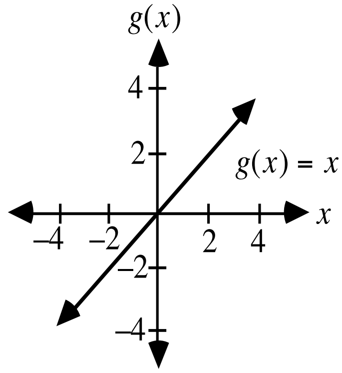
4. f/h

b. Graph each function in Part **a**. Use your graphs to identify the apparent domains and ranges.

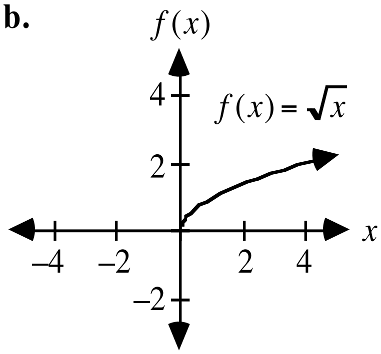
2.2 In each of Parts **a–c** below, determine the domains for which the functions $(f + g)(x)$, $(f - g)(x)$, $(f \bullet g)(x)$, $(f/g)(x)$, and $(g/f)(x)$ are defined.

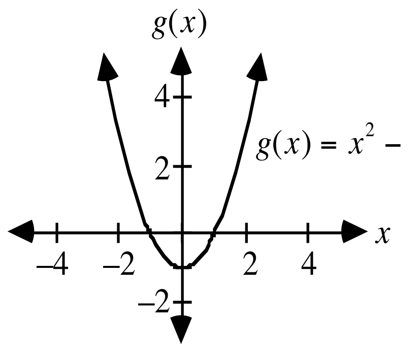
a.



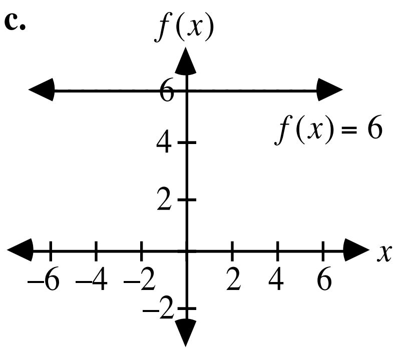


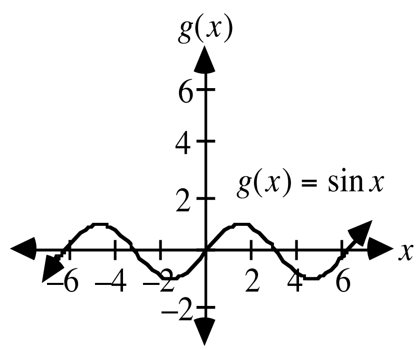
b.





c.





2.3 Use the following functions to complete Parts **a** and **b**.

$$f(x) = x + 5$$
$$g(x) = -2x - 3$$

a. Graph the three functions $(f - g)(x)$, $(f \bullet g)(x)$, and $(f/g)(x)$ on separate coordinate systems.

b. Determine a rule for each function in Part **a** and list its domain and range.

2.4 **a.** If $f(x) = \sin\ x$ and $g(x) = \cos\ x$, graph $(f(x))^2$ and $(g(x))^2$ on the same coordinate system. Determine the domain and range for $(f(x))^2$ and $(g(x))^2$.

b. Assume $(f(x))^2 = (\sin x)^2$ and $(g(x))^2 = (\cos x)^2$. Write an expression for $(f^2 + g^2)(x)$ and graph this new function. Determine the domain and range of this function.

c. Use the graph from Part **b** to write a simplified expression for $(f^2 + g^2)(x)$.

d. An equation involving trigonometric functions that is true for all real numbers in its domain is a **trigonometric identity**. Does the expression you wrote in Part **c** form a trigonometric identity? Explain your response.

e. Recall that the division of $\sin x$ by $\cos x$ results in $\tan x$. Using a graph, determine the domain and range of the tangent function.

f. Is the following expression a trigonometric identity? Explain your response.

$$\tan x = \frac{\sin x}{\cos x}$$

2.5 At another ostrich ranch, the owners sell either newly hatched chicks or breeding pairs. They charge $700 for each chick and have a total of $36,000 in annual overhead expenses related to the hatching operation. The ranch sells breeding pairs for $10,000 per pair and spends a total of $40,000 per year to maintain the adult ostrich flock.

a. **1.** Determine a function $c(x)$ to model the profit on the sale of ostrich chicks, where x represents the number of chicks sold.

2. Determine a function $p(y)$ to model the profit from selling breeding pairs, where y is the number of breeding pairs sold.

b. What do $c(5)$ and $p(5)$ represent?

c. Explain why it is not reasonable to add the two profit functions in this setting to form a new profit function, $(c + p)(x)$.

d. Consider the profit function $r(x,y) = c(x) + p(y)$ where x is the number of chicks sold and y is the number of breeding pairs sold.

1. What does $r(10,8)$ represent?

2 Describe a possible domain for r. What is the corresponding range?

3. Is r a reasonable function to use for determining the ranch's total profit from the sale of chicks and breeding pairs? Explain your response.

2.6 Consider the functions $f(x) = 3x$, $g(x) = 1/(3x)$, and $h(x) = x - 3$ with their appropriate domains and ranges. Given these functions, find the simplest possible rule for each of the following and describe its domain.

a. $(f + g)(x)$

b. $(f - h)(x)$

c. $(f \bullet g)(x)$

d. $(h/g)(x)$

e. $(f/h)(x)$

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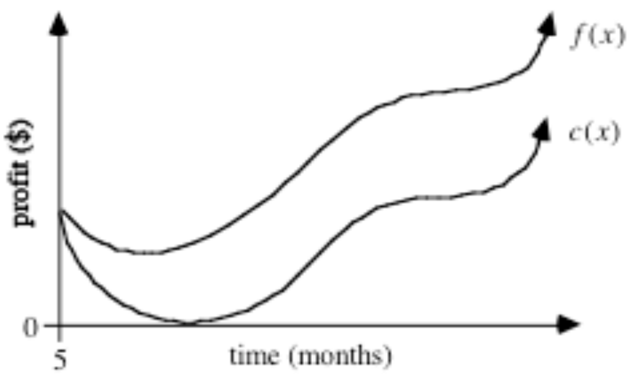
2.7 Since its fifth month in business, Bea's Beauty Salon has earned a profit in every month from at least one of two areas: product sales or customer services. The profit from customer services only can be described by the function $c(x)$, where x represents time in months. The total profit can be described by the function $f(x)$. The following diagram shows graphs of these two functions.



Sketch a graph of the function $p(x)$ that describes the profit from product sales only.

2.8 The function $h(t) = 2 + 15t - 4.9t^2$ models the height (in meters) after t sec of an object thrown straight up in the air at 15 m/sec from a distance of 2 m above the ground. This function can be thought of as the sum of three other functions: $s(t) = 2$, $p(t) = 15t$, and $g(t) = -4.9t^2$.

The function $s(t)$ describes the object's initial height (in meters). The function $p(t)$ describes the effect produced by the object's initial velocity. The function $g(t)$ describes the effect produced by the acceleration due to gravity.

a. Determine a function that models the height (in meters) after t sec of an object thrown straight down from a height of 50 m with an initial velocity of 10 m/sec.

b. How long will it take the object described in Part **a** to hit the ground?

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Activity 3

During their first year of operation, Sal and Guinn found that the size of their flock could be represented as a function of time. They also observed that the number of births at the ranch depended on the number of breeding pairs—more adult ostriches meant that more chicks were hatched. In this respect, the number of births could be thought of as a function of the number of adult ostriches.

Since the number of births is a function of the number of adult ostriches, and the number of adults is a function of time, it is also possible to consider the number of births as a function of time. The diagram in Figure **4** shows how this can be done by forming a **composite function** $f(g(x))$.

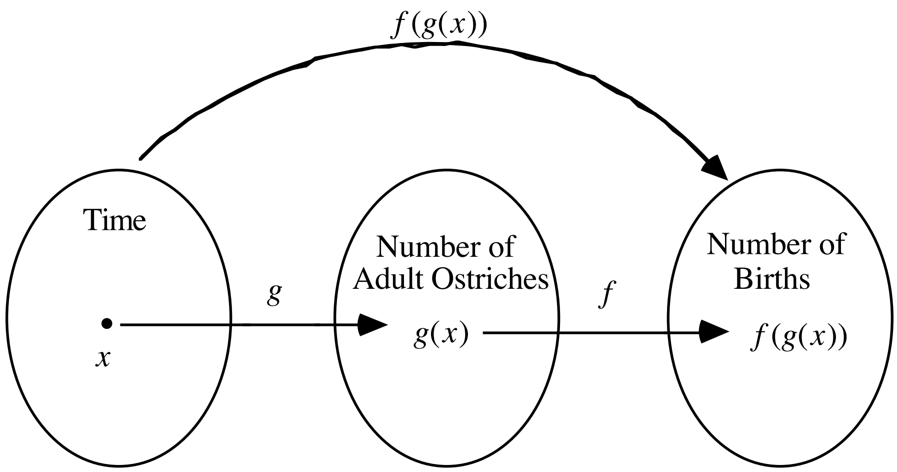

Figure 4: Diagram of a composition of functions

Mathematics Note

Given two functions *f* and *g*, the **composite function** $f \circ g$, read as "*f* composed with *g*" is defined as

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all values of *x* in the domain of *g* such that $g(x)$ is in the domain of *f*.

For example, if *g* is the set of ordered pairs {(1,3), (2,4), (5,11)}, its domain is {1,2,5}. If *f* is the set of ordered pairs {(3,8), (4,12), (7,13)}, then its domain is {3,4,7}. The domain of $f \circ g$ is {1, 2}. This is because {1,2} is part of the domain of *g* and $g(1) = 3$ and $g(2) = 4$ are values in the domain of *f*. Figure **5** illustrates these relationships using a mapping diagram.

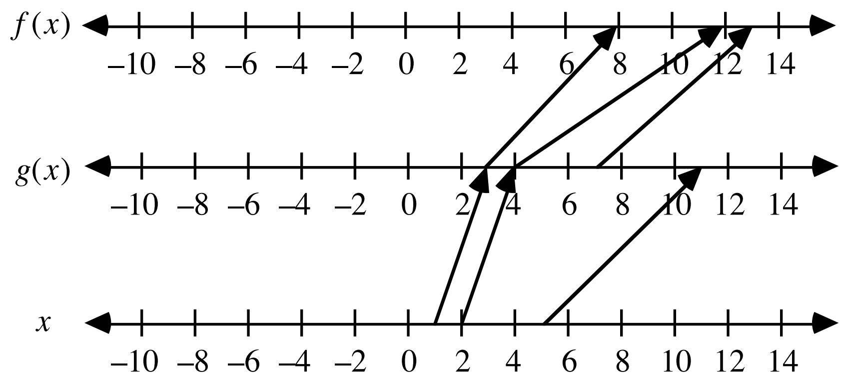

Figure 5: Mapping diagram of $f(g(x))$

Notice in the mapping diagram in Figure **5** that $g(5) = 11$. Since 11 is not in the domain of *f*, however, 5 is not in the domain of $f \circ g$. Figure **6** shows a simplified version of the mapping diagram for $(f \circ g)(x)$.

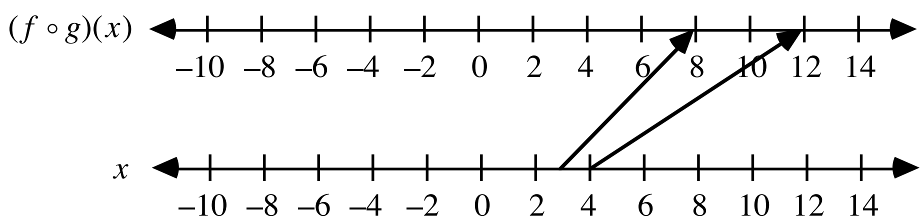

Figure 6: Mapping diagram of $(f \circ g)$

From Figure **6**, the domain of $f \circ g$ is $\{1,2\}$ and the range is $\{8,12\}$.

Exploration

In this exploration, you investigate the composition of functions using $f(x) = 3x$ and $g(x) = x^2$ and the domain $\{-5, -4, -3, ..., 5\}$.

a. **1.** Determine the range for $g(x)$.

2. Find the range for $f(g(x))$ by completing a mapping diagram like the one shown in Figure **7**.

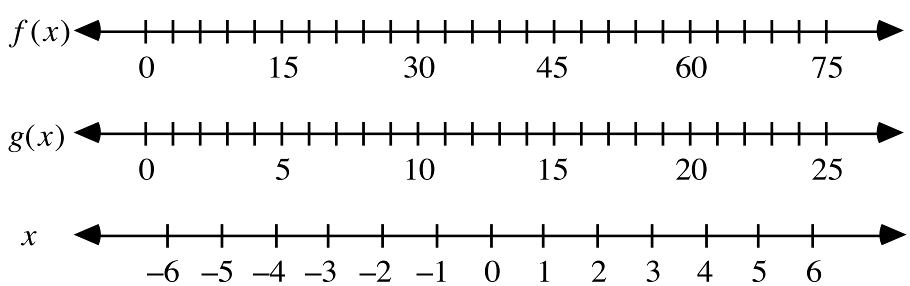

Figure 7: Incomplete mapping diagram for $f(g(x))$

3. Determine a rule for $f(g(x))$. State the domain and range for $f(g(x))$.

b.

1. Determine the range for $f(x)$.
2. Find the range for $g(f(x))$ by completing a mapping diagram.
3. Determine a rule for $g(f(x))$. State the domain and range for $g(f(x))$.

Discussion

a. Consider the two functions $q(x) = 3x + 2$ and $f(x) = x^2$.

1. Describe what $q(a)$ indicates.
2. Describe what $f(q(a))$ indicates.

b. How do the ranges of $f(g(x))$ and $g(f(x))$ from the exploration compare?

c. Does composition of functions appear to be commutative? Explain your response.

d. If f is the set of ordered pairs $\{(3,5), (4,6), (7,13)\}$ and g is the set $\{(5,10), (6,14), (9,15)\}$, what ordered pairs would be in $g(f(x))$? Explain your response.

e. Consider the functions $f(x) = \sqrt{x+5}$ and $g(x) = x^3 + 3$.

1. Describe the domain and range for $f(g(x))$.
2. How does the process of finding $g(f(2))$ differ from the process of finding $f(g(2))$?
3. Does the domain of $f \circ g$ include all the values in the domain of $g(x)$? Explain your response.
4. Does the domain of $g \circ f$ include all the values in the domain of $f(x)$? Explain your response.

Assignment

3.1 Suppose an ostrich ranch keeps 4 eggs for hatching each year and sells the rest. The number of eggs for sale can be modeled by the function $h(x) = x - 4$, where x represents the total egg production that year. Since about 20% of each year's eggs are lost to breakage, the number of unbroken eggs can be modeled by $g(x) = 0.8x$.

a. If there are 40 eggs in a given year, which of the following correctly describes the number of unbroken eggs that are not kept by the ranch for hatching: $h(g(40))$ or $g(h(40))$?

b. Use your answer from Part **a** to help show that composition of functions is not commutative.

3.2 Given the functions $f(x) = x^2$ and $g(x) = \sin x$, what are the domain and range of the composite function $(f \circ g)(x)$?

3.3 Consider the functions $h(x) = x^2$ and $k(x) = \sqrt{x}$.

a. Find the domain and range of each of the following compositions:

1. $(h \circ k)(x)$

2. $(k \circ h)(x)$

b. Write each of the composite functions below in simplified form:

1. $(h \circ k)(x)$

2. $(k \circ h)(x)$

c. Explain why the compositions in Part **b** result in different equations.

3.4 Given the functions $g(x) = x + 4$ and $f(x) = 4x$, simplify the following:

$$c(x) = \frac{f(g(x)) - f(x)}{4}$$

3.5 Sal and Guinn have agreed to invest 20% of the profit from their ostrich operation in a retirement fund. The remaining amount will be divided evenly between them as salary.

a. Determine a function $s(x)$, where x represents annual profit, that can be used to determine the salary for each partner.

b. In the exploration in Activity **2**, you described Sal and Guinn's annual profit using the function $n(x) = 38{,}000x - 140{,}000$, where x represents the number of breeding pairs in the flock. Use composition of functions to determine the annual salary for each partner if the ranch maintains a flock of 7 breeding pairs.

c. Determine a function that could be used to calculate the partners' annual salaries given the number of breeding pairs.

3.6 Sal and Guinn employ high school students to clean the ostrich barn. The amount that each student earns depends on the number of hours worked. The amount that each saves for college depends on the total earnings.

a. Choose an appropriate wage for a student. Express earnings as a function of the number of hours worked.

b. Select a realistic proportion of earnings for a student to save for college. Express the amount saved as a function of earnings.

c. Use composition of functions to describe the amount saved for college as a function of the number of hours worked.

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3.7 Consider the functions $g(x) = \sqrt{16 - x^2}$ and $h(x) = 5x - 1$.

a. Identify the domain and range of each function.

b. Write each of the composite functions below in simplified form:

1. $(g \circ h)(x)$

2. $(h \circ g)(x)$

c. Find the domain and range of each of the following compositions:

1. $(g \circ h)(x)$

2. $(h \circ g)(x)$

3.8 To complete Parts **a** and **b** below, consider the functions, $f(x) = x - 9$, $g(x) = \sqrt{x} + 5$, and $h(x) = x^2$.

a. Find $g\big(f(h(7))\big)$, $f\big(h(g(7))\big)$, and $g\big(h(f(7))\big)$.

b. Determine a rule for $g\big(f(h(x))\big)$ and identify the domain.

3.9 Consider a function *m* which connects each child with his or her mother.

a. If the domain is the set of all children, describe a possible range for *m*.

b. Describe a possible range for $m \circ m$.

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Activity 4

Although ostriches can survive for days without water, they prefer to drink and bathe frequently. On Sal and Guinn's ranch, each ostrich consumes about 8 L of water per day. The ranch faces a possible drought this summer. Should Sal and Guinn reduce the size of their flock or should they start looking for more water?

The way in which the partners approach this problem depends on their plans for the business. If they would like to maintain a certain number of birds, they can determine the total amount of water required to sustain a flock of that size. Once that amount is known, they can estimate the volume of water the ranch can provide during a drought, then acquire any additional water needed.

On the other hand, if Sal and Guinn want to use only the available water, they can determine the number of ostriches this supply will maintain and reduce their flock accordingly. The functions involved in analyzing these two different approaches are **inverses** of each other.

Mathematics Note

The **inverse** of a relation results when the elements in each ordered pair of the relation are interchanged.

For example, the relation {(0,2), (1,3), (4,–2), (–3,–2)} has an inverse relation {(2,0), (3,1), (–2,4), (–2,–3)}. The domain of the original relation becomes the range of the inverse, while the range of the original relation becomes the domain of the inverse.

If a relation is a function, and its inverse is also a function, the inverse is an **inverse function**. The inverse function of $f(x)$ is denoted by $f^{-1}(x)$, often shortened to f^{-1}.

For example, consider the function $f = \{(1,5),(2,4),(-1,0)\}$. Its inverse is the function $f^{-1} = \{(5,1),(4,2),(0,-1)\}$.

Exploration 1

a. Use technology to create a scatterplot of the relation f and its inverse using the domains and ranges given in Table **1**. To simplify comparisons, make the length of one unit on the x-axis the same as the length of one unit on the y-axis.

Table 1: Relation f and the inverse of f

Relation	Domain	Range
$f(x) = 2x - 5$	{–2, –1, 0, ..., 10}	{–9,–7,–5, ...,15}
Inverse of f	range of f	domain of f

b. Determine the point of intersection of the graph of f and the graph of its inverse.

c. Determine the equation of the line containing the origin and the point identified in Part **b**. Plot this line on the same graph as in Part **a**.

d. Describe any special relationships you observe among the graphs of f, its inverse, and the line graphed in Part **c**.

e. Create another linear function $g(x)$ and repeat Parts **a–d**. You may wish to choose a different domain for $g(x)$.

f. Repeat Part **a** using the quadratic function $h(x) = x^2 + 2$. You may wish to choose a different domain for $h(x)$.

g. Plot the line you found in Part **c** on the graph from Part **f**. Describe any special relationships you observe among the graphs of h, its inverse, and this line.

Discussion 1

a. Describe the graphs of the inverses of *f*, *g*, and *h*.

b. Is each inverse also a function? Explain your responses.

c. What relationships are there among the graphs of each function, its inverse, and the graph of the line $y = x$? Explain your response.

Exploration 2

Each morning when Guinn turns the ostrich chicks into the fenced pasture, she opens the pasture gate, then drives the chicks out of the barn. In the evening, she drives the chicks back into the barn, then closes the gate. In one sense, her chores in the evening represent the "inverse" of her chores in the morning. In the evening, Guinn undoes her actions of the morning and the chicks are returned to the barn.

a. A similar process can be used to determine the equation that represents the inverse of a function, such as $f(x) = 2x + 8$. In this case, the function multiplies a domain value by 2, then adds 8 to the product. The equation that represents the inverse of $f(x)$ undoes these actions, first by subtracting 8, then by dividing the result by 2:

$$y = \frac{x - 8}{2}$$

Since this equation defines a function, it can be denoted as follows:

$$f^{-1}(x) = \frac{x - 8}{2}$$

Use this method to determine an equation that represents the inverse of each of the following relations. Describe any restrictions on the domain of each inverse, and determine whether or not each is a function.

1. $f(x) = 3x - 1$
2. $g(x) = \sqrt{x + 7}$
3. $h(x) = 9x^2$
4. $k(x) = x^3 - 8$

b. Use a symbolic manipulator to check your work in Part **a**. **Note:** Many symbolic manipulators require a function to be denoted as $y = f(x)$. Since the inverse relation of a set of ordered pairs (x,y) is a set of ordered pairs (y, x), *x* and *y* should be interchanged to obtain the rule for the inverse. This new equation should then be solved for *y*, subject to any necessary restrictions.

c. When two functions f and g are inverses of each other, $(f \circ g)(x) = (g \circ f)(x) = x$. Use this definition to verify that each inverse you obtained in Part **a** is correct.

Discussion 2

a. Which of the relations in Part **a** of Exploration **2** have inverses that are not functions? Explain your response.

b. The graphs in Figure **8** below show two relations and their inverses. In each case, the graph of the relation is indicated by a solid curve, while the graph of the inverse is indicated by a dotted curve.

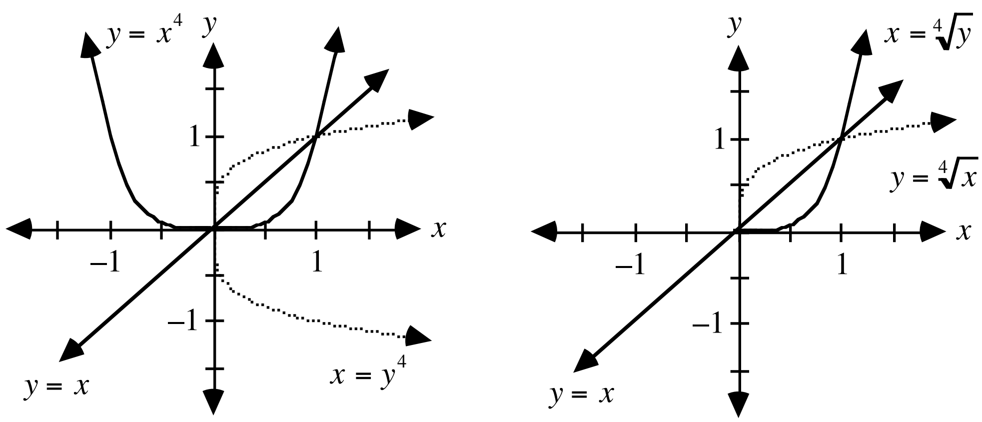


Figure 8: Two relations and their inverses

1. Describe the relationship between the line $y = x$ and the graphs in Figure **8**.
2. Explain why the graphs in Figures **8a** and **b** are identical in the first quadrant but not in the second and fourth quadrants.
3. If a horizontal line intersects the graph of a function in more than one point, then its inverse is not a function. Use this notion to explain whether or not the inverse of each relation in Figure **8** is a function.

Mathematics Note

A **one-to-one function** is a function such that each element in the range corresponds to a unique element of the domain. In other words, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. One-to-one functions are important because they are the only functions whose inverses are also functions.

For example, the one-to-one function $f = \{(-3,-13), (-1,-7), (1,-1), (3,5), (5,11)\}$ has the inverse function $f^{-1} = \{(-13,-3), (-7,-1), (-1,1), (5,3), (11,5)\}$. The function $g = \{(-2,4), (-1,1), (1,1), (2,4)\}$, however, is not a one-to-one function. Its inverse is $\{(4,-2), (1,-1), (1,1), (4,2)\}$, which is not a function.

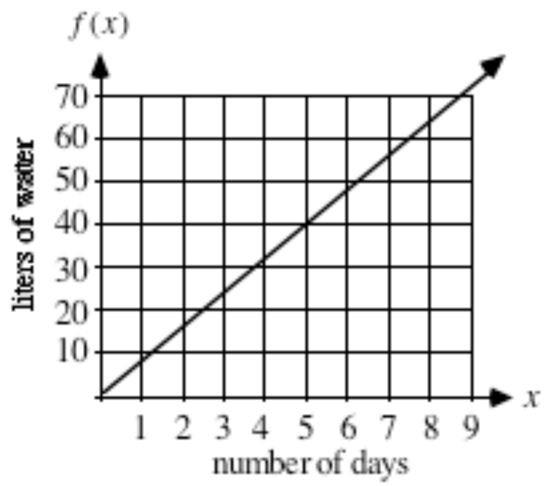
c. Use ordered pairs to illustrate why the function $h(x) = 9x^2$ is not a one-to-one function.

d. Explain why a horizontal line test can be used to determine when a graph is not a one-to-one function.

Assignment

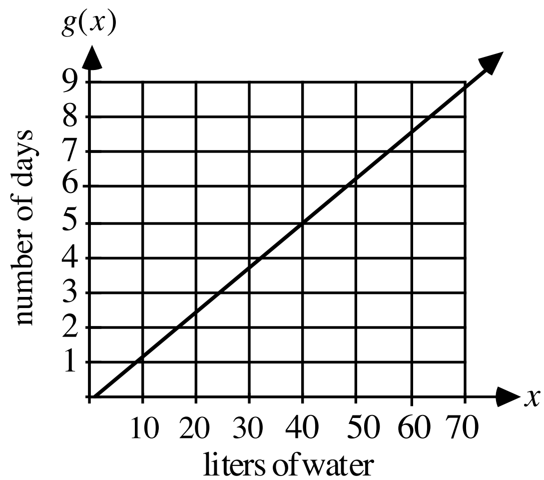
4.1 To help themselves analyze the effects of a potential drought on the ranch, Sal and Guinn decide to create two different sets of graphs.

a. If they want to maintain their current flock of birds, Sal and Guinn must examine the relationship between water consumption and time. Recall that each ostrich uses about 8 L of water per day. The following graph shows the average amount of water needed by one ostrich versus time in days.



1. Use the graph in Part **a** to estimate the amount of water an ostrich will use over 5 days.
2. Estimate the time it will take for one ostrich to use 20 L of water.
3. Write an equation that describes the liters of water used per ostrich as a function of time in days.
4. Suppose that Sal and Guinn want to maintain a flock of 40 birds. If the drought lasts for 60 days, how much water will they need?

b. If Sal and Guinn want to use only the water they currently have available on the ranch, they must determine the number of birds their present resources will maintain during a drought. The following graph shows time in days versus liters of water needed per ostrich.



1. Use the graph above to estimate how many days it will take one ostrich to consume 70 L of water.
2. Estimate the liters of water one ostrich will consume in 2 days.
3. Write an equation that describes time in days as a function of liters of water available per ostrich.
4. The water tank at the ranch holds 16,000 L. If the drought lasts for 60 days, determine the size of the flock that Sal and Guinn can maintain.

c. What relationship exists between the two equations you found in Parts **a** and **b**?

d. Graph the equations from Parts **a** and **b** on the same axes. How does this graph support your response to Part **c**?

4.2 **a.** Which of the following functions are one-to-one functions?

1. $f(x) = 2x + 3$
2. $f(x) = x^3$
3. $f(x) = (x - 4)/5$
4. $f(x) = x^4$

b. For each one-to-one function identified in Part **a**, write the equation of its inverse using $f^{-1}(x)$ notation.

c. For each one-to-one function identified in Part **a**, list the domain and range of $f(x)$ and $f^{-1}(x)$.

Mathematics Note

The inverse of the exponential function $f(x) = a^x$ where $a > 0$ is $f^{-1}(x) = \log_a x$, where $x > 0$ and read "log of x base a" or "log base a of x."

For example, if $f(x) = 12^x$, then $f^{-1}(x) = \log_{12} x$.

4.3 **a.** Describe the domain and range of $f(x) = 10^x$.

b. Write the inverse of $f(x) = 10^x$ using logarithmic notation.

c. Complete the table below without using technology.

$f(x) = 10^x$		inverse of $f(x)$	
x	10^x	x	$\log x$
–4		0.0001	
–2		0.01	
–1		0.1	
0		1	
1		10	
2		100	
4		10,000	

d. Compare the ordered pairs of $f(x) = 10^x$ to those of its inverse.

e. Graph $f(x)$ and the inverse of $f(x)$ on the same coordinate system.

f. Describe the domain and range of the inverse of $f(x) = 10^x$.

g. Is the inverse of $f(x)$ a function? Explain your response.

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4.4 **Parametric equations** allow rectangular coordinates to be expressed in terms of another variable (the parameter). On an *xy*-plane, for example, both x and y can be expressed as functions of a third variable t.

a. Consider the relation $y = 5x^2$. Graph this relation parametrically using the equations $x = t$ and $y = 5t^2$. Graph its inverse using the equations $x = 5t^2$ and $y = t$.

b. How can you verify that the graphs in Part **a** represent inverses?

c. Why does the method outlined in Part **a** result in inverse relations?

4.5 Since only one-to-one functions have inverses that are also functions, it is sometimes desirable to restrict the domain of a function that is not one-to-one in order to create a one-to-one function.

a. Create a scatterplot of the graph of $f(x) = \sin x$ over the domain $[-4\pi, 4\pi]$.

b. On the same coordinate system, create a scatterplot of the inverse of the function defined in Part **a**.

c. Explain why the inverse of $f(x) = \sin x$ over the domain $[-4\pi, 4\pi]$ is not a function.

d. Find a restricted domain of $f(x) = \sin x$ so that its inverse is also a function, and so that all possible values of the range of $f(x) = \sin x$ are values of the domain in the inverse.

e. Graph $f(x) = \sin x$ and its inverse over this restricted domain.

4.6 **a.** Graph the function $f(x) = \tan x$.

b. Identify an interval in which $f(x) = \tan x$ is a one-to-one function.

c. What is the range of the function $f(x) = \tan x$ over the interval in Part **b**?

d. Find the domain and range of the inverse of $f(x)$ over the interval in Part **b**.

e. Graph $f^{-1}(x)$ over the restricted domain in Part **d**.

4.7 What is true about the composition of a function with its inverse function?

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Summary Assessment

Sal and Guinn have just sold one breeding pair of ostriches to their friend Terry. In approximately two years, this breeding pair will become mature adults and produce approximately 40 eggs per year. Sal and Guinn have recommended that Terry increase the size of his flock by keeping two pairs of ostriches out of each year's eggs and selling the rest. It will take approximately three more years for each new pair of ostriches to mature and produce their own eggs. Function f models the approximate number of ostrich pairs Terry will have in x years:

$$f(x) = 10^{0.1618x}$$

1. Use this function to predict the total number of ostrich pairs Terry will have in seven years.

2. Terry wants to know how many years it will take before he has a certain number of ostrich pairs. Use function f to determine a new function that will give the number of years it will take to produce x ostrich pairs.

3.
- **a.** Graph the functions in Problems **1** and **2** on the same set of axes.
- **b.** Describe the graphs in Part **a**.
- **c.** Explain whether or not the functions in Problems **1** and **2** are one-to-one functions.

4. Terry would like to determine the cost of fencing a pasture for the ostriches. The cost per meter for a chain-link fence is $30 for materials and $10 for labor. According to Sal and Guinn, each pair of ostriches should have 1400 m^2 of pasture.

Assuming that Terry fences a single square pasture, determine functions for each of the following:

- **a.** the length of fence required for x pairs of ostriches
- **b.** the cost of materials for x meters of fence
- **c.** the cost of labor for x meters of fence.

5. Identify the domain and range of each function in Problem **4**.

6. Use the functions in Problems **4b** and **4c** to determine a new function that describes the total cost for x meters of fencing.

7.
- **a.** Compose two or more of the functions in the problems above to determine a function that describes the total cost of fencing sufficient to contain Terry's growing flock of ostriches for x years.
- **b.** Identify the domain and range of the function in Part **a**.

Module Summary

- A **relation** is a set of ordered pairs in which the **domain** is the set of first elements and the **range** is the set of second elements.
- A **function** is a relation from a domain to a range in which each element of the domain occurs in exactly one ordered pair.
- One way to represent a function between two sets is to use a **set diagram** with an arrow to represent the rule. A second way to represent a function between two sets is a **mapping diagram**.
- The function $(f+g)$ is defined by $(f+g)(x) = f(x) + g(x)$. Likewise, $(f-g)(x) = f(x) - g(x)$, $(f \bullet g)(x) = f(x) \bullet g(x)$, and $(f/g)(x) = f(x)/g(x)$ where $g(x) \neq 0$.
- When adding, subtracting or multiplying two or more functions, these operations are defined only for those values common to the domains of all the functions involved.
- Given two functions *f* and *g*, the **composite function** $f \circ g$, read as "*f* composed with *g*" is defined as

  $$(f \circ g)(x) = f\big(g(x)\big)$$

  The domain of $f \circ g$ is the set of all values of *x* in the domain of *g* such that $g(x)$ is in the domain of *f*.
- The **inverse** of a relation results when the elements in each ordered pair of the relation are interchanged. The domain of the original relation becomes the range of the inverse, while the range of the original relation becomes the domain of the inverse.
- If a relation is a function, and its inverse is also a function, the inverse is an **inverse function**. The inverse function of *f* is denoted by f^{-1}.
- The graph of the inverse is a reflection of the graph of the function in the line $y = x$.
- A **one-to-one function** is a function such that each element in the range corresponds to a unique element of the domain. In other words, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. One-to-one functions are important because they are the only functions whose inverses are also functions.
- The inverse of the exponential function $f(x) = a^x$, where $a > 0$, is $f^{-1}(x) = \log_a x$, where $x > 0$ and read "log of *x* base *a*" or "log base *a* of *x*."

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