Mathematics in Motion

As an arrow glides toward a target, could you describe its exact location at any time during the flight? In this module, you examine methods for modeling the paths of moving objects.

Ruth Brocklebank • Phil Lieske • Mike Trudnowski

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Introduction

The forest is on fire. Crews on the ground are battling the blaze, but they need more equipment. The dispatcher orders a plane to deliver a crate of supplies.

The crate is designed to be dropped without a parachute. As the plane flies toward the target zone, its crew must decide when to drop the crate. To do this, however, they must be able to predict the path of the falling crate. In this module, you use parametric equations to explore this and other types of motion.

Discussion 1

a. **1.** What factors influence the path of a crate during its fall?

2. Describe the effect of each of these factors on the crate's path.

b. What do you think the path of a falling crate will look like?

c. Should the crew drop the crates when the plane is directly over the target area? Explain your response.

Exploration

In the following exploration, you investigate the motion of two falling objects: one dropped straight down, and one projected horizontally. Both objects begin their fall from the same height.

a. **1.** Fold an index card over a flexible meterstick or ruler. Secure the card to the meterstick with a binder clip.

2. Fold the index card to form a platform on each side of the meterstick, parallel to the ground, as shown in Figure **1**.

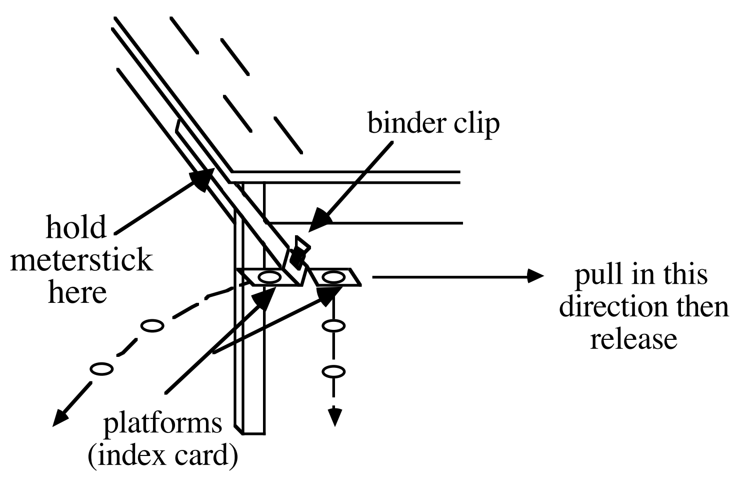

Figure 1: Two falling objects

b.
1. Hold the opposite end of the meterstick against the side of a table.
2. Place a dense object, such as a coin, on each platform. (Using dense objects lessens the effects of air resistance.)
3. Measure and record the height of the objects from the floor.
4. Pull the free end of the meterstick in the direction indicated in Figure **1**, then release it.
5. Observe the path of each object, and note when each hits the floor. Record your observations, including a sketch of each path.

c. Repeat Part **b** two or three times, varying the amount of tension on the meterstick.

d. Repeat Parts **b** and **c** with two dense objects that are not alike.

Discussion 2

a. Compare the paths of the two like objects in Part **b** of the exploration.

b. Did both objects fall from the same height?

c. Compare the time required for the two objects to reach the floor.

d. Does the time required for an object to reach the ground appear to be affected by its path?

e. How did your observations change when using two unlike objects?

f. If a feather and a coin are dropped from a height of 10 m, would you expect them to reach the floor at the same time? Explain your response.

Science Note

One of Galileo Galilei's (1564–1642) more famous accomplishments is his description of the motion of falling objects. While first investigating free fall, he is said to have simultaneously dropped a 10-kg cannonball and a 1-kg stone off the Leaning Tower of Pisa. He discovered that the objects hit the ground at approximately the same time.

About 75 years later, Isaac Newton (1642–1727) developed three laws of motion. Using his own second law of motion and the laws of planetary motion developed by Johannes Kepler (1571–1630), Newton proved that, in the absence of air resistance, any two objects dropped from the same height hit the ground at exactly the same time.

Activity 1

In the introduction, you investigated the paths of freely falling objects. In this activity, you model these paths with parametric equations.

Exploration

Consider two freely falling objects. One is dropped straight down from a height of 10 m. At the same instant, the other is projected horizontally from the same initial height.

a. The graph in Figure **2** shows the position of each object at intervals of 0.2 sec. Use the graph to approximate ordered pairs (x,y) for these positions, where x represents the horizontal distance and y represents the vertical distance.

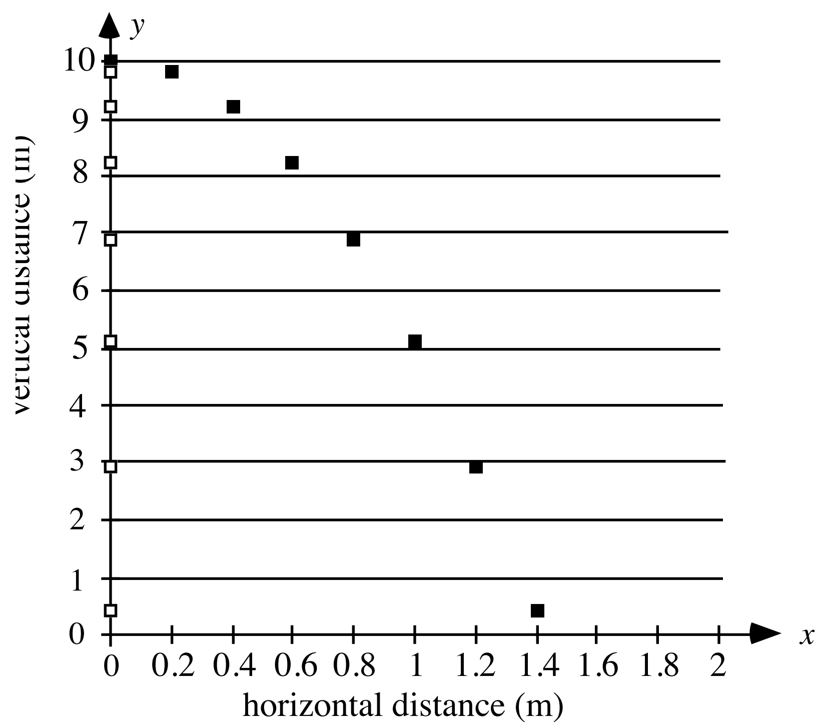

Figure 2: Positions of two freely falling objects

b. Record the values from Part **a**, along with the corresponding times, in a spreadsheet with headings like those in Table **1**.

Table 1: Positions of objects over time

	Object Dropped from Rest (□)		**Object Projected Horizontally** (■)	
Time (sec)	Horizontal Distance (m)	Vertical Distance (m)	Horizontal Distance (m)	Vertical Distance (m)
0.0				
0.2				
⋮				
1.4				

c.

1. Calculate the change in horizontal position between consecutive points for each falling object.
2. The average velocity of an object can be calculated as follows:

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Determine the average horizontal velocity (v_x) between consecutive points for each object.

3. Write a function $x(t)$ that describes each object's horizontal position with respect to time t.

d.

1. Calculate the change in vertical position between consecutive points for each falling object.
2. Determine the average vertical velocity between consecutive points for each object. Record these values in a spreadsheet with headings like those in Table **2** below.

Table 2: Vertical velocity of objects over time

Time Interval (sec)	**Object Dropped from Rest (m/sec)**	**Object Projected Horizontally (m/sec)**
$[0, 0.2)$		
$[0.2, 0.4)$		
⋮		
$[1.2, 1.4)$		

e. **Acceleration** describes an object's change in velocity per unit time. The average acceleration of an object can be calculated as follows:

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

Use the spreadsheet to calculate the average vertical acceleration between consecutive points for each object.

f. The acceleration due to gravity near Earth's surface is approximately $9.8\ \text{m}/\text{sec}^2$ in a direction toward Earth's center. Compare the average acceleration you determined in Part **e** to this value.

g. When a freely falling object has no initial velocity in the vertical direction, its height after t sec can be described by the following function, where g is the acceleration due to gravity and h_0 is the initial height:

$$y(t) = -\frac{1}{2}gt^2 + h_0$$

1. Write a function $y(t)$ that describes the vertical position of each object in Figure **2** with respect to time t. Recall that the initial height for both objects was 10 m.
2. Check your equations by substituting 0.2, 0.8, and 1.2 for t and comparing the resulting values of $y(t)$ to those in Table **1**.

Mathematics Note

Parametric equations allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. In an xy-plane, for example, both x and y can be expressed as functions of a third variable, t:

$$x = f(t)$$
$$y = g(t)$$

In these parametric equations, the independent variable is the parameter t. The dependent variables are x and y. In other words, each value of t in the domain corresponds with an ordered pair (x,y).

For example, consider an object projected horizontally at a velocity of 15 m/sec off a cliff 20 m high. This object's position after t sec can be described by the following parametric equations, where $x(t)$ represents the horizontal distance traveled and $y(t)$ represents the height above the ground:

$$x(t) = v_x t = 15t$$

$$y(t) = -\frac{1}{2}gt^2 + h_0 = -\frac{1}{2}(9.8)t^2 + 20 = -4.9t^2 + 20$$

At $t = 2$ sec, the ordered pair generated by these equations is (30,0.4). This indicates that 2 sec after leaving the cliff, the object has traveled 30 m horizontally and is 0.4 m off the ground.

h. Write parametric equations to describe the position of each object in Figure **2** with respect to time.

i. Set your graphing utility to graph parametric equations simultaneously.

1. Using appropriate intervals for x, y, and the parameter t, graph both pairs of equations from Part **h**.
2. Experiment with different increments for t. Record your observations.
3. Use the trace feature to observe and record the values of x, y, and t at various locations on each graph.

Discussion

a. Describe the graphs you created in the exploration.

b. Does the speed with which the graphs are drawn appear to be related to the actual speed of the objects? Explain your response.

c.
1. How could you determine the time required for each object to reach the ground?
2. Describe how you could find the location of each object after half this time has passed.

d. Describe how you could determine the maximum horizontal distance traveled by the object that was projected horizontally.

e. Considering an object whose height above the ground can be described by the function $y(t)$, is it reasonable to consider negative values for $y(t)$? Explain your response.

Assignment

1.1 While practicing at a target range, an archer shoots an arrow parallel to the ground at a velocity of 42 m/sec. At the moment the arrow is released, the strap on the archer's wristwatch breaks and the watch falls toward the ground. The initial height of both the arrow and the watch is 1.6 m.

a. Write a pair of parametric equations, $x(t)$ and $y(t)$, to describe each of the following:

1. the position of the watch after t sec
2. the position of the arrow after t sec.

b. Graph the equations from Part **a**.

c. Determine the height of each object after 0.25 sec.

d. Determine how long it will take for each object to hit the ground.

e. Determine the horizontal distance traveled by the arrow at the time it hits the ground.

1.2 In the introduction to this module, you discussed the airlift of a crate of supplies to some firefighters. Suppose that the plane is traveling at a horizontal velocity of 250 km/hr and the crate is dropped from a height of 100 m.

a. Write a set of parametric equations, $x(t)$ and $y(t)$, to model the path of the crate, where t represents time in seconds. Hint: The units for distance should be the same in each equation.

b. Determine how long it will take for the crate to hit the ground.

c. Determine the horizontal distance traveled by the crate during its time in the air.

d. If the plane continues to travel at the same velocity, where will it be located in relation to the crate when the crate hits the ground?

1.3 Two mountain climbers are stranded by a blizzard at an elevation of 1690 m. A search-and-rescue plane locates the climbers but cannot land to pick them up. Flying due east at a velocity of 90 m/sec and an elevation of 1960 m, the crew drops a package of food and supplies.

a. How long (to the nearest 0.1 sec) will it take for the package to reach the ground if it lands at the same elevation as the climbers?

b. How far should the plane be from the target site when the rescue team releases the package?

* * * * *

1.4 Under the watchful eye of your skydiving instructor, you step out of a plane. The plane is traveling at a constant velocity of 65 m/sec and an altitude of 1300 m. You wait 10 sec before pulling the ripcord of your parachute.

a. Ignoring air resistance, describe your path during the 10 sec of free fall.

b. Write a set of parametric equations that models your path during this interval.

c. Determine how far you have fallen vertically before pulling the ripcord.

d. Determine the horizontal distance you have traveled before pulling the ripcord.

e. At the time you pull the ripcord, where is the airplane relative to your position? Explain your response.

1.5 The object of the game "Sure-Aim" is to roll a marble off a table and into a cup. The table is 0.8 m high. The cup is 0.1 m high, with a diameter of 5 cm. The horizontal distance from the table to the cup's rim is 0.75 m.

Determine the approximate velocity at which a marble must leave the table in order to land in the cup. Defend your response.

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Activity 2

In Activity **1**, you explored the motion of objects falling from rest or projected with a horizontal velocity. In this activity, you investigate the motion of objects projected into the air at an angle.

Discussion 1

a. When a batter hits a ball, what forces are involved?

b. What factors influence the distance traveled by the ball?

Exploration

While watching a videotape of herself in the batting cage, Kami noticed that she hit the ball at many different angles of elevation, from line drives to pop-ups. After speaking with her fast-pitch softball coach, she wondered what angle of elevation would make her hits travel as far as possible.

You may recall from the Level 4 module, "Flying the Big Sky with Vectors," that it is possible to analyze this situation using vectors. In this exploration, you develop a vector model to help answer Kami's question.

Mathematics Note

A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically symbolized by a bold, lowercase letter, such as vector **u**. In handwritten work, the same vector can be symbolized by $\vec{u}$. The magnitude of a vector **u** is denoted by $|\mathbf{u}|$.

The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector **u** is denoted by $\mathbf{u}_x$ (read "**u** sub *x*"), while its vertical component is denoted by $\mathbf{u}_y$.

For example, the arrowhead on vector **a** in Figure **3** indicates its direction. The length of vector **a** indicates its magnitude. Its horizontal and vertical components are $\mathbf{a}_x$ and $\mathbf{a}_y$, respectively.

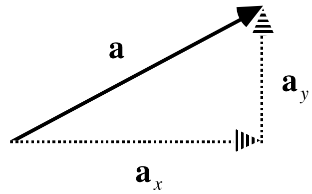


Figure 3: Vector a and its components

a. To analyze the paths of the hit balls, Kami ignores air resistance and assumes that each ball leaves the bat at the same speed of 40 m/sec.

When the initial velocity of a hit ball is represented by a vector **v**, the vector's direction is determined by the angle θ at which the ball leaves the bat. Its magnitude is the velocity at which the ball is hit. Figure **4** shows vector **v** and its components.

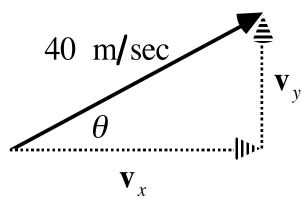


Figure 4: Vector v and its components

1. Write an expression for the horizontal velocity $\mathbf{v}_x$ in terms of the initial velocity of 40 m/sec and the angle θ.
2. Write an expression for the vertical velocity $\mathbf{v}_y$ in terms of the initial velocity and θ.

b. Complete Table **3** for softballs hit at angles of elevation between 0° and 90°, in increments of 5°.

Table 3: Component velocities of a softball

Initial Velocity (m/sec)	Angle of Elevation (degrees)	Horizontal Component $\mathbf{v}_x$	Vertical Component $\mathbf{v}_y$
40	0	40	0
40	5	39.85	3.49
⋮	⋮	⋮	⋮
40	90		

c. In general, the height of a projectile above the ground at any time t can be modeled by the following function:

$$h(t) = -\frac{1}{2}gt^2 + \mathbf{v}_y t + h_0$$

where g is the acceleration due to gravity, v_y is the vertical component of the initial velocity, and h_0 is the initial height.

1. Consider a softball hit with an initial velocity of 40 m/sec at a 30° angle of elevation from an initial height of 1 m. Write a function that models the height of this softball with respect to time.
2. Determine the height of the softball 4 sec after it is hit.

d. The softball's horizontal motion can be analyzed independently of its vertical motion. In general, the horizontal distance traveled at any time t can be modeled by the following function:

$$x(t) = \mathbf{v}_x t$$

where $\mathbf{v}_x$ is the horizontal component of the initial velocity.

1. Write a function that models the horizontal distance traveled by the softball described in Part **c**.
2. Find the horizontal distance traveled by the softball 4 sec after it is hit.

e. Graph the parametric equations from Parts **c** and **d**. Use the graph to determine the horizontal distance traveled by the softball before it hits the ground.

f. Repeat Parts **c–e** using several different values for θ, the angle of elevation. Estimate the measure of the angle that will allow a hit ball to travel the farthest distance.

Discussion 2

a. Describe the paths of the softball in Part **e** of the exploration.

b. Given the initial velocity and angle of elevation for a hit softball, how could you determine each of the following?

1. the maximum horizontal distance traveled by the softball
2. the time required for the softball to reach its maximum height
3. the maximum height reached by the softball.

c. What angle of elevation appears to result in the maximum horizontal distance for a hit ball?

d. Suppose that the wind is blowing when Kami hits the ball.

1. Does wind affect the horizontal or vertical component of a ball's velocity? Explain your response.
2. How would you adjust your parametric equations if the wind was blowing toward Kami?
3. How would you adjust your parametric equations if the wind was blowing away from Kami?

Assignment

2.1 While watching the videotape of herself in the batting cage, Kami noticed that she hit one pitch especially well. Estimating that the angle of elevation measured 20°, she wondered if that hit would have been a home run.

Assume that the softball left the bat with an initial velocity of 40 m/sec at a height of 1 m.

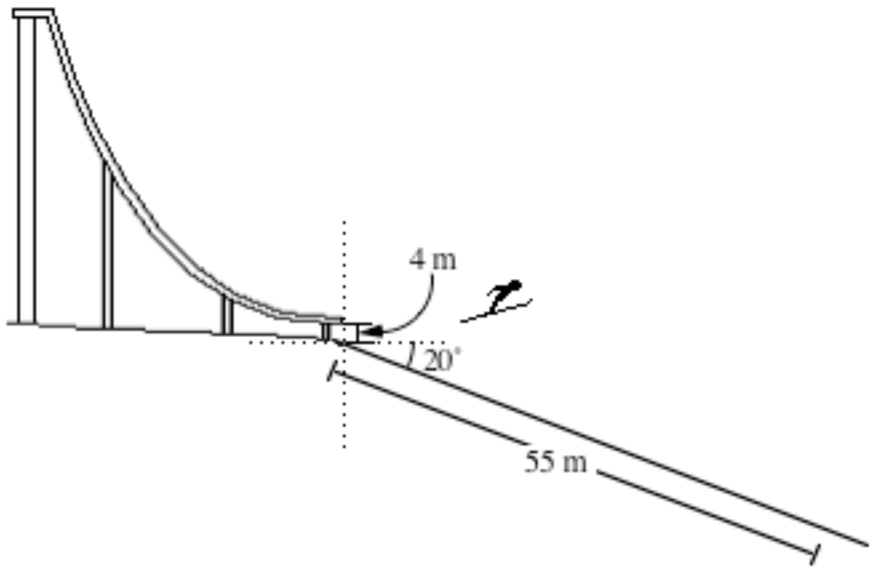
a. At what time would the softball have reached its maximum height?

b. What would have been its maximum height?

c. The outfield fence is 2 m high and 80 m from home plate. Would the ball have cleared the fence? If so, determine the distance by which the ball would have cleared the fence. If not, determine the distance by which the ball would have fallen short.

2.2 Imagine that the wind is blowing directly toward home plate at 8.5 m/sec. If Kami hits the ball as in Problem **2.1**, will the ball clear the fence? Check your response using a graph of the appropriate parametric equations.

2.3 The distance traveled by a ski jumper is measured from the base of the ramp to the landing point. As shown in the diagram below, the end of the ramp is 4 m above the snow. The angle formed by the plane of the landing area and the horizontal is 20˚. Ignoring air resistance, find the horizontal velocity that the skier would need to jump 55 m.



2.4 Imagine that you are an engineer for the Buildaroad Construction Company. In order to widen a highway, the company must blast through a mountain. You have been asked to determine a safe distance from the blast for the construction workers on the site. The charge of dynamite will propel rocks and debris at a maximum initial velocity of 55 m/sec. Write a report explaining your recommendations, including a minimum "safe" distance.

* * * * *

2.5 At the circus, Rowdy the Riot is shot out of a cannon and into a square net which measures 10 m on each side. To land safely, Rowdy must land at least 2 m from the edge of the net. The barrel of the cannon is 2 m off the ground and has a 40˚ angle of elevation. The net is 1 m off the ground. Its nearest edge is 30 m from the cannon.

Ignoring air resistance, determine an interval of initial velocities that will allow Rowdy to land safely in the net. Justify your response by showing an appropriate vector analysis of the situation.

2.6 When Lief hits a golf ball, the distance it travels depends on which golf club he uses. The following table shows the ball's angle of elevation and initial velocity when hit with four different golf clubs.

Golf Club	Angle of Elevation	Initial Velocity
six iron	32°	44.5 m/sec
seven iron	36°	41.5 m/sec
eight iron	40°	38.5 m/sec
nine iron	44°	36.5 m/sec

From his position on the fairway, Lief wants to hit a golf ball so that it lands in the middle of the green. The front of the green is 162 m away, while the back is 181 m away. Use the information in the table to determine which club Lief should select.

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Activity 3

In the previous activities, you used parametric equations to model parabolic paths. In this activity, you use parametric equations to investigate circular and elliptical paths.

Exploration 1

Recall from the Level 4 module, "Controlling the Sky with Parametrics," that a circle with center at point (h,k) and radius r can be defined by the following pair of parametric equations:

$$x(\theta) = h + r\cos\theta$$
$$y(\theta) = k + r\sin\theta$$

where θ is the measure of the central angle formed by two radii of the circle, one of which is parallel to the x-axis. As shown in Figure **5**, each value of θ corresponds with a specific point on the circle.

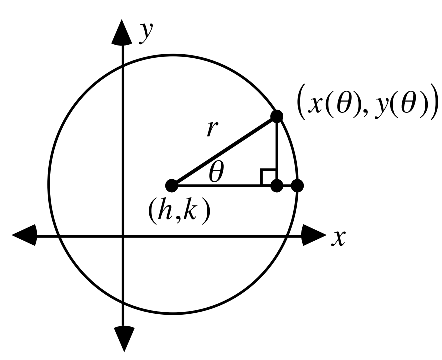


Figure 5: A circle with center at (*h*,*k*) and radius *r*

a. Figure **6** shows a toy airplane attached by a string to a weighted base. As the plane flies, it follows a circular path whose radius is the length of the string.

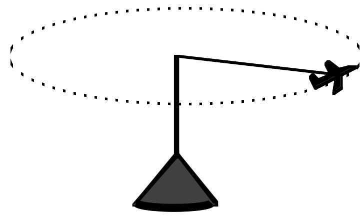


Figure 6: A toy airplane

1. Use parametric equations to model the path of the airplane, given that the length of the string is 2 m long and the end attached to the base is at the origin.
2. Graph your equations from Step **1**. **Note:** Remember to set your graphing utility to measure angles in radians.

b. The plane completes 1 revolution in 1 sec. Determine its average speed in meters per second.

Mathematics Note

The average **angular speed** of a moving point P, relative to a fixed point O, is the measure of the angle θ through which the line containing O and P passes per unit time. For example, consider the wheel in Figure **7** below.

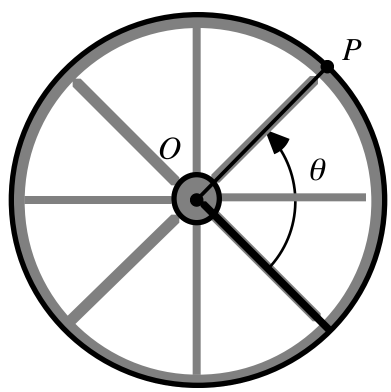


Figure 7: Point *P* on a wheel

Suppose P moves $1/4$ the circumference of the wheel in 2 sec. In this case, the line containing O and P has passed through an angle measure of $2\pi/4$ or $\pi/2$ radians in 2 sec. Therefore, the average angular speed of P is:

$$\frac{\pi/2 \text{ radians}}{2 \text{ sec}} = \frac{\pi}{4} \text{ radians/sec}$$

c. Determine the plane's average angular speed in radians per second.

Mathematics Note

The position of an object traveling counterclockwise at a constant angular speed c, on a circle with center at point (h,k) and radius r, can be modeled by the following parametric equations:

$$x(t) = h + r\cos(ct)$$
$$y(t) = k + r\sin(ct)$$

where t represents time.

For example, consider a chair on a Ferris wheel with a radius of 10 m, where the center of the wheel is 12 m off the ground. The Ferris wheel completes 1 revolution every 20 sec. In this case, the angular speed c is $2\pi/20$, or $\pi/10$ radians/sec. If the origin is located on the ground directly below the wheel's center, the chair's position with respect to time can be modeled by the following parametric equations:

$$x(t) = 10\cos\left(\frac{\pi}{10}t\right)$$
$$y(t) = 12 + 10\sin\left(\frac{\pi}{10}t\right)$$

d. Each of the following pairs of parametric equations models the movement of a toy airplane at the end of a 2-m string, where t represents time in seconds. Determine how long it takes each plane to complete 1 revolution.

1. $x(t) = 2\cos(\pi t)$
   $y(t) = 2\sin(\pi t)$
2. $x(t) = 2\cos(2\pi t)$
   $y(t) = 2\sin(2\pi t)$
3. $x(t) = 2\cos\left(\frac{2\pi}{3}t\right)$
   $y(t) = 2\sin\left(\frac{2\pi}{3}t\right)$

e. Determine a pair of parametric equations that models the motion of a toy airplane that completes 1 revolution in each of the following intervals:

1. 2.5 sec
2. 0.8 sec
3. a sec.

f. For each pair of parametric equations in Part **e**, determine the average speed of the plane.

Discussion 1

a. Is the speed at which the graph is plotted related to the actual speed of the object moving around the circle? Explain your response.

b. Using parametric equations of the form given in the previous mathematics note, what is the position of the object when $t = 0$? Justify your response.

c. Figure **8** below shows three different points on a circle: A, B, and C. How would you model the movement of an object whose initial position is at one of these points?

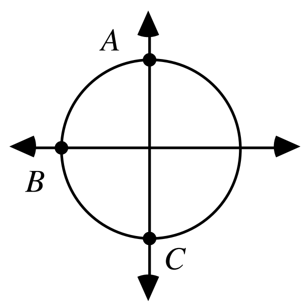


Figure 8: A circle with center at the origin

d. Consider an object moving on a circle with center at (–4,3) and radius 7 units. If the object completes 1 revolution every 5 sec, describe how to use parametric equations to model its position over time.

e. Describe how to determine the speed of an object whose position with respect to time can be modeled by the parametric equations below, where t represents time in hours:

$$x(t) = 9\cos(6t)$$
$$y(t) = 9\sin(6t)$$

f. Figure **9** below shows two concentric circles and a segment OP containing a point Q.

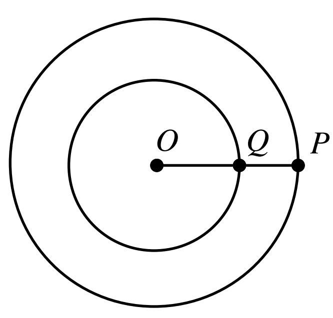


Figure 9: Two concentric circles

1. Compare the speeds of P and Q as the segment rotates about O.
2. Compare the angular speeds of P and Q as the segment rotates about O.

Exploration 2

Parametric equations also can be used to model elliptical paths. In this exploration, you discover how to use parametric equations to define an ellipse.

a. Use a geometry utility to complete the following steps.

1. Construct two circles with center at the origin *O* and different radii. Create a moveable point on the outer circle. Label this point *A*.
2. Draw a ray from *O* through *A*. Locate the point of intersection of the ray and the inner circle. Label this point *B*.
3. From *A*, construct a segment perpendicular to the *x*-axis. Locate the intersection of the perpendicular and the *x*-axis. Label this point *C*.
4. From *B*, construct a segment perpendicular to the *x*-axis and a line perpendicular to the *y*-axis. Label the line perpendicular to the *y*-axis *m*. Locate the intersection of the perpendicular segment and the *x*-axis. Label this point *E*.
5. Locate the point of intersection of $\overline{AC}$ and line *m*. Label this intersection *D*. This point represents one point on your graph of an ellipse. Your construction should now resemble the diagram in Figure **10**.

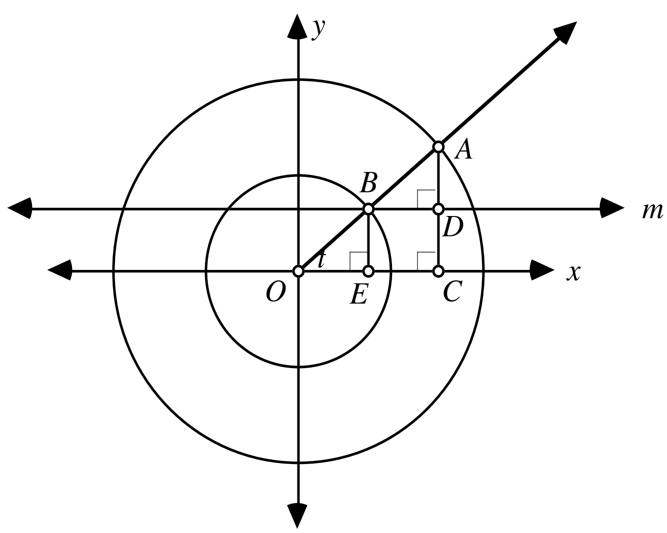


Figure 10: Construction for modeling an ellipse

b. Trace the locus of point *D* as point *A* moves about the outer circle.

c. Using your construction, let *t* represent $m\angle BOE = m\angle AOC$. Let (x,y) represent the coordinates of point *D*.

1. Express *x* in terms of *t* and OA
2. Express *y* in terms of *t* and OB.

d. Suppose that the radius of the larger circle in Figure **9** is 4 units, while the radius of the smaller circle is 2 units. Write parametric equations that model the paths of points A, B, and D as A moves about the larger circle.

Discussion 2

a. Compare your construction with those of your classmates. What differences do you observe?

b.
1. Compare the parametric equations you found in Part **d** of Exploration **2** with those of your classmates.
2. Describe a method you could use to determine these equations.

Mathematics Note

An ellipse with center at the origin can be defined parametrically by the equations $x(t) = a\cos t$ and $y(t) = b\sin t$, where a is the positive x-intercept of the ellipse and b is the positive y-intercept.

For example, consider an ellipse with center at the origin that intersects the x-axis at (12,0) and the y-axis at (0,7). In this case, $a = 12$ and $b = 7$. Therefore, the parametric equations of the ellipse are $x(t) = 12\cos t$ and $y(t) = 7\sin t$.

c. Consider the locus of points traced by D in Part **b** of Exploration **2**. Does a graph of these points appear to be a function? Explain your response.

d. When the locus of points traced by point D is expressed using parametric equations, its graph is a function of t.

1. Describe the domain and range of this function.
2. Explain why it is a function.

e. How are the values of a and b in the equations $x(t) = a\cos t$ and $y(t) = b\sin t$ related to the lengths of the axes of an ellipse?

f. Describe the type of ellipse formed when $a = b$.

g. What advantages are there in using parametric equations to sketch ellipses?

h. The parametric equations $x = a\cos t$ and $y = b\sin t$ define the coordinates of the points of an ellipse. Solving these equations for $\cos t$ and $\sin t$, respectively, results in the following: $\cos t = x/a$ and $\sin t = y/b$.

If these equations are squared and added together, how is the resulting equation related to an ellipse? Hint: Recall from the Level 6 module, "Ostriches are Composed," that $\sin^2 x + \cos^2 x = 1$ is true for all values of x.

Assignment

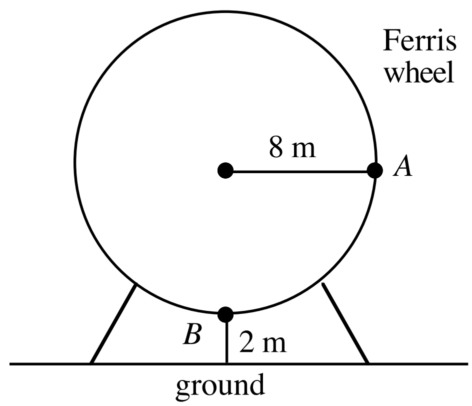
3.1 Assume that the radius of the larger circle in Exploration **2** is 5 units, while the radius of the smaller circle is 3 units.

a. Write parametric equations that describe the locus of points traced by D in terms of the sine and cosine of the angle t. (See Figure **10** for reference.)

b. Graph the equations from Part **a** on a graphing utility. Describe the resulting figure, including the locations of its foci.

c. Graph the parametric equations $x(t) = 3\cos t$ and $y(t) = 5\sin t$. Describe the resulting figure, including the locations of its foci.

3.2 The diagram below shows a Ferris wheel with a radius of 8 m. The bottom of the Ferris wheel is 2 m above the ground.



a. The wheel completes 1 revolution every 20 sec. Determine the speed and angular speed of a chair on this Ferris wheel.

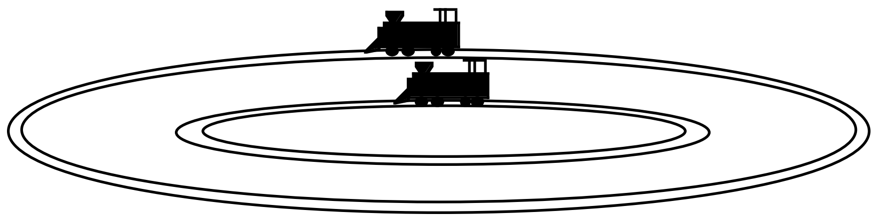
b. **1.** Consider a chair whose initial position is at point A. Use parametric equations to model the position of this chair over time.

2. How high above the ground will this chair be if the wheel stops 10 sec after the chair passes point A? Explain your response.

3. How long will it take for this chair to reach a height of 16 m?

c. Describe how you could model the movement of a chair whose initial position is at point B.

3.3 The following diagram shows two toy trains traveling on concentric sets of circular tracks.



The train on the outer track is 1 m from the center, while the one on the inner track is 0.5 m from the center.

a. Suppose that each train completes 1 lap around its respective track in 15 sec.

1. Determine the angular speed of each train.

2. Determine the speed of each train.

3. Model each train's position over time with parametric equations, given that at $t = 0$, both trains are located on the x-axis of a two-dimensional coordinate system.

b. Suppose that each train travels at a constant speed of 0.25 m/sec.

1. Model each train's position over time with parametric equations.

2. How long will it take the train on the inner track to gain a one-lap lead over the train on the outer track?

3.4 Use your response to Part **h** of Discussion **2** to complete the following.

a. An ellipse can be represented parametrically by the equations $x = 12\cos t$ and $y = 7\sin t$. Write the equation of this ellipse in standard form.

b. Write a set of parametric equations for the ellipse defined by the following equation:

$$\frac{x^2}{36} + \frac{y^2}{4} = 1$$

3.5 **a.** Write a set of parametric equations that define an ellipse with center at (2,3), a major axis with a length of 7 units, and a minor axis with a length of 3 units.

b. At what points does a graph of this ellipse intersect the lines $x = 2$ and $y = 3$?

c. Write a set of parametric equations for an ellipse with center at (h,k), a major axis with length $2a$, and a minor axis with length $2b$.

Mathematics Note

The **area** of an ellipse can be calculated using the formula $A = \pi ab$, where $2a$, and $2b$ are the lengths of the axes.

For example, consider the ellipse defined parametrically by $x(t) = 6\cos t$ and $y(t) = 11\sin t$. In this case, the length of the major axis is 22 units, while the length of the minor axis is 12 units. The area of this ellipse is $6(11)\pi = 66\pi$ units^2.

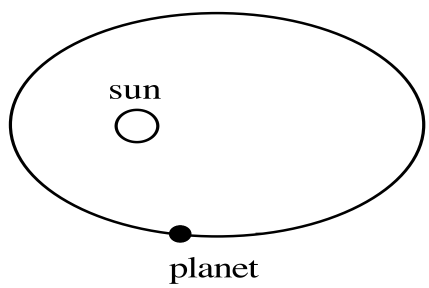
3.6 **a.** Graph an ellipse defined by parametric equations of the form $x(t) = a\cos t$ and $y(t) = b\sin t$, where $a \neq b$.

b. Use the formula $A = \pi ab$ to calculate the area of this ellipse.

c. How is the formula for the area of an ellipse related to the formula for the area of a circle?

* * * * *

3.7 Johannes Kepler's (1571–1630) first law of planetary motion states that the planets move in elliptical orbits in which the sun is located at one focus of the ellipse, as shown in the following diagram.



The shape of Earth's elliptical orbit can be modeled parametrically by the equations $x(t) = \left(1.4958 \bullet 10^8\right)\cos t$ and $y(t) = \left(1.4955 \bullet 10^8\right)\sin t$, where x and y represent distances in kilometers and t represents angle measures.

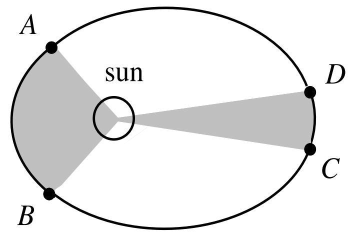
Johannes Kepler approximated the circumference of an ellipse using the equation $\pi(a + b)$.

a. Explain why this formula provides a reasonable approximation for Earth's orbit by comparing it to the formula for the circumference of a circle.

b. Using Kepler's approximation, how far does Earth travel in its yearly orbit?

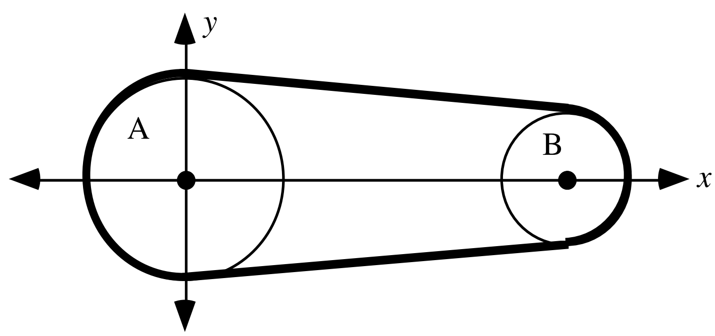
3.8 Kepler's second law of planetary motion states that a ray drawn from the sun to a planet will sweep out equal areas in equal times.

In the diagram below, for example, the time required for a planet to travel from *A* to *B* equals the time it takes for the planet to move from *C* to *D*. Therefore, according to Kepler's second law, the two shaded areas are equal.



Use Kepler's second law to approximate the area that a ray drawn from the sun to Earth would sweep in 30 days.

3.9 The following diagram shows a belt and two circular pulleys. The radius of pulley A is 10 cm, while the radius of pulley B is 6 cm.



The center of pulley A is located at the origin of a two-dimensional coordinate system. The center of pulley B is 40 cm to the right of the origin on the *x*-axis.

a. Suppose that pulley A completes 1 revolution every 0.1 sec. Determine the speed of a point on the circumference of pulley A.

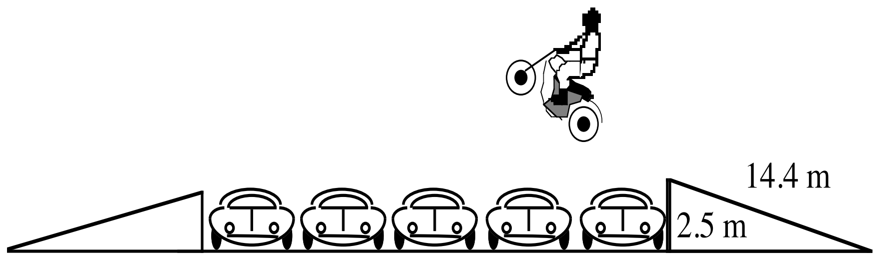
b. Write parametric equations to model the movement of a point on pulley A.

c. When either pulley turns, the belt causes the other pulley to turn also. Given this fact, which quantities would you expect to be equal: the pulleys' speeds, or their angular speeds? Explain your response.

d. Write parametric equations to model the movement of a point on pulley B.

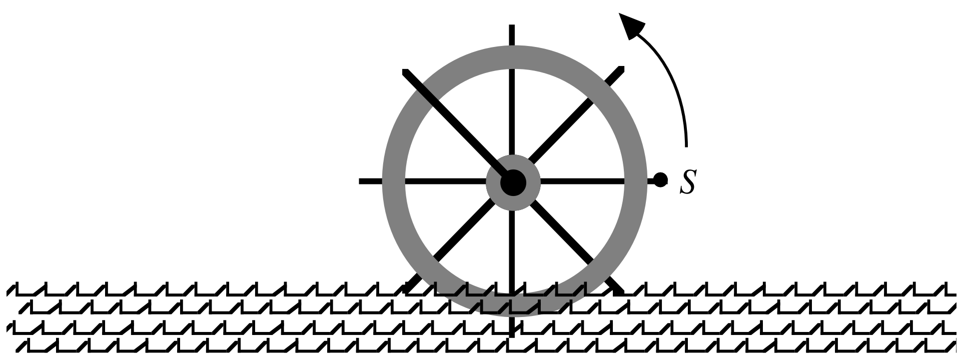
Summary Assessment

1. A motorcycle stunt rider is planning to jump a line of cars arranged side by side, as shown in the diagram below. The approach ramp is 14.4 m long and 2.5 m high, and the motorcycle will have a velocity of 130 km/hr when it leaves the ramp.



The average width of each car is 1.7 m, and the last car in line is 1.5 m high. Determine the maximum number of cars that the stunt rider could clear (ignoring air resistance). Justify your response.

2. The following diagram shows a water wheel with eight paddles.



The center of the wheel is 1.2 m above the water's surface. The distance from the wheel's center to the end of each paddle is 1.8 m. The current flows at a speed of 4.5 km/hr .

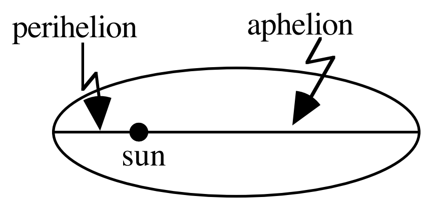
a. Assuming that the speed of point *S* equals the speed of the current, use parametric equations to model the position of *S* with respect to time.

b. Determine how long point *S* is under water during each revolution of the wheel.

c. Given that the eight paddles are evenly spaced, how long are two consecutive paddles under water during a single turn of the wheel? Explain your response.

3. The orbits of planets can be modeled by ellipses with one focus at the sun. Orbits are often described by their *aphelion* (farthest point from the sun), *perihelion* (closest point to the sun), and orbital *eccentricity*.

Eccentricity is a measure of the orbit's elongation, and is equal to the ratio of the distance c between the center and one focus to half the length of the major axis. In other words, $e = c/a$.



In our solar system, Pluto has the most elongated orbit. Its orbital eccentricity is 0.2482. Pluto's aphelion and perihelion are $7.3812 \bullet 10^9$ km and $4.4458 \bullet 10^9$ km, respectively.

Determine parametric equations to model Pluto's orbit. Graph these equations and describe the shape of the orbit.

Module Summary

- **Acceleration** describes an object's change in velocity per unit time. The average acceleration of an object can be calculated as shown below:

  $$\text{average acceleration} = \text{change in velocity}/\text{change in time}$$

- The height h of a falling object after t sec can be described by the function:

  $$h(t) = -\frac{1}{2}gt^2 + h_0$$

  where g is the acceleration due to gravity and h_0 is the object's initial height. The acceleration due to gravity on earth is about $9.8\text{ m}/\text{sec}^2$ in a direction toward Earth's center.

- **Parametric equations** allow rectangular coordinates to be expressed in terms of another variable, the **parameter**. In an *xy*-plane, for example, both x and y can be expressed as functions of a third variable, t:

  $$x = f(t)$$
  $$y = g(t)$$

  In these parametric equations, the independent variable is the parameter t. The dependent variables are x and y. In other words, each value of t in the domain corresponds with an ordered pair (x,y).

- A **vector** is a quantity that has both magnitude (size) and direction. In printed work, a vector is typically symbolized by a bold, lowercase letter, such as vector $\mathbf{u}$. In handwritten work, the same vector can be symbolized by $\vec{u}$. The magnitude of a vector $\mathbf{u}$ is denoted by $|\mathbf{u}|$.

- The pair of horizontal and vertical vectors that when added result in a given vector are the **components** of that vector. The horizontal component of a vector $\mathbf{u}$ is denoted by $\mathbf{u}_x$ (read "**u** sub *x*"), while its vertical component is denoted by $\mathbf{u}_y$.

- In general, the height of a projectile above the ground at any time t is described by the function:

  $$h(t) = -\frac{1}{2}gt^2 + \mathbf{v}_y t + h_0$$

  where g is the acceleration due to gravity, $\mathbf{v}_y$ is the magnitude of the vertical component of the velocity, and h_0 is the initial height.

- A circle with center at point (h,k) and radius r can be defined by the following pair of parametric equations:

$$x(\theta) = h + r\cos\theta$$
$$y(\theta) = k + r\sin\theta$$

  where θ is the measure of the central angle formed by two radii of the circle, one of which is parallel to the x-axis.

- The average **angular speed** of a moving point P, relative to a fixed point O, is the measure of the angle θ through which the line containing O and P passes per unit time.

- The position of an object traveling counterclockwise at a constant angular speed c, on a circle with center at point (h,k) and radius r, can be modeled by the following parametric equations:

$$x(t) = h + r\cos(ct)$$
$$y(t) = k + r\sin(ct)$$

  where t represents time.

- An ellipse with center at the origin can be defined parametrically by the equations $x(t) = a\cos t$ and $y(t) = b\sin t$, where a is the positive x-intercept of the ellipse and b is the positive y-intercept.

- The **area** of an ellipse can be calculated using the formula $A = \pi ab$, where $2a$, and $2b$ are the lengths of the axes.

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