# The Sequence Makes the Difference

Some number patterns are easy to recognize, while others are more difficult. In this module, you'll explore one strategy for recognizing number patterns described by polynomials.

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## Introduction

In this module, you continue your explorations with number patterns. Some of these patterns are easily recognizable, while others are not. If you can recognize a number pattern, then you may be able to identify functions that can generate it.

For example, consider the function $f(n) = 2n$. Over a domain of the natural numbers, this function generates the following sequence:

$$2, 4, 6, 8, \ldots$$

The function $f(n) = 2n$ is a first-degree polynomial (or linear) function. Polynomial functions of other degrees also can be used to generate sequences.

## Discussion

**a.** Recall that the general form of a polynomial function is:

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \cdots + a_1 n^1 + a_0$$

where $k$ is a natural number. What is the degree of this polynomial?

**b.** Using a function $f(n)$ and the domain of natural numbers to generate a sequence, what value of $n$ corresponds with the first term $t_1$ of the sequence? What value of $n$ corresponds with $t_k$?

**c.** Over a domain of the natural numbers, a zero-degree polynomial of the form $f(n) = a_0$ generates a **constant sequence**. Give an example of a constant sequence.

**d.** What is another name for a sequence generated by a linear function, such as $2, 4, 6, 8, \ldots$? Explain your response.

**e.** Sequences also can be generated by second-, third-, and fourth-degree polynomials. Give an example of a sequence generated by each of the following:

**1.** a quadratic function

**2.** a cubic function

**3.** a quartic function.

**f.** Compare the use of polynomial functions to the use of explicit formulas in defining the terms of sequences.

# *Activity 1*

A scatterplot of a sequence can often give you a clue about the types of functions that may have generated its terms. However, scatterplots alone may not provide enough information. In this activity, you examine another useful tool: the **finite-difference process**.

## Exploration 1

As you have seen in earlier modules, an arithmetic sequence always has a common difference. For example, the common difference for the following arithmetic sequence is 6:

$$14, 20, 26, 32, 38, 44, \ldots$$

In this exploration, you identify differences between successive terms of non-arithmetic sequences.

**a.** Table **1** lists four different degrees of polynomial functions and shows their general forms. Select values for the coefficient(s) and constant in each of these polynomials.

**Table 1: Four types of polynomials**

| Type | General Form |
|---|---|
| linear | $f(n) = a_1 n + a_0$ |
| quadratic | $f(n) = a_2 n^2 + a_1 n + a_0$ |
| cubic | $f(n) = a_3 n^3 + a_2 n^2 + a_1 n + a_0$ |
| quartic | $f(n) = a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0$ |

**b.** Finding the differences between consecutive terms in a sequence can help you identify how the sequence may have been formed. Listed in order, these differences themselves form a sequence: a **sequence of differences**.

For example, consider the finite sequence 4, 7, 24, 60, 120, 209. Its first sequence of differences is (7 – 4), (24 – 7), (60 – 24), (120 – 60), (209 – 120), or 3, 17, 36, 60, 89.

The terms of its second sequence of differences are the differences between consecutive terms in this new sequence: (17 – 3), (36 – 17), (60 – 36), (89 – 60), or 14, 19, 24, 29.

Figure **1** shows the results of continuing this process for a third sequence of differences.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sequence | 4 | | 7 | | 24 | | 60 | | 120 | | 209 |
| 1st Sequence of Differences | | 3 | | 17 | | 36 | | 60 | | 89 | |
| 2nd Sequence of Differences | | | 14 | | 19 | | 24 | | 29 | | |
| 3rd Sequence of Differences | | | | 5 | | 5 | | 5 | | | |

**Figure 1: Successive sequences of differences**

1. List at least the first six terms of the sequence generated by the cubic function you wrote in Part **a**.
2. Determine the first sequence of differences for this sequence.
3. Determine the second sequence of differences.
4. If possible, continue the process of finding successive sequences of differences until you obtain a constant sequence.
5. If you obtained a constant sequence in Step **4**, identify the sequence of differences which corresponds with the constant sequence.

**c.** Repeat Part **b** for the other polynomials you created in Part **a**.

**d.**
1. Recall that a geometric sequence can be defined using a recursive formula of the form $t_n = rt_{n-1}$, where $n > 1$ and $r$ is the common ratio. Select values for $t_1$ and $r$, then repeat Part **b** for at least six terms of the resulting sequence.
2. Recall that a Fibonacci-type sequence can be defined recursively as $t_n = t_{n-1} + t_{n-2}$ where $n > 2$. Select values for $t_1$ and $t_2$, then repeat Part **b** for at least six terms of the resulting sequence.

## Discussion 1

**a.** In Exploration **1**, what type(s) of sequences eventually resulted in a constant sequence of differences?

**b.** What appears to be the relationship between the degree of a polynomial function that generates a sequence and the corresponding sequences of differences?

**Mathematics Note**

A sequence generates a constant sequence of differences if and only if that sequence can be generated by a polynomial function.

If the first constant sequence of differences is the $n$th sequence of differences, there exists a unique polynomial function of degree $n$ that generates the original sequence.

For example, consider the sequence 4, 7, 24, 60, 120, 209 and its successive sequences of differences (shown in Figure **1**). The second sequence of differences is 14, 19, 24, 29. The third sequences of differences is 5, 5, 5. Since the first constant sequence of differences occurs at the third sequence of differences, there is a unique third-degree polynomial that generates the original sequence. In this case, that polynomial function is:

$$f(n) = \frac{5}{6}n^3 + 2n^2 - \frac{53}{6}n + 10$$

where the domain is 1, 2, 3, 4, 5, 6.

**c.** Do you think that it would be possible to generate an infinite geometric sequence, where $r \neq 1$, using a polynomial function?

**d.** Given any finite sequence, can you determine with certainty the degree of the polynomial function that generates it? Explain your response.

**e.** Why would you expect it to be possible to generate every finite sequence using polynomial functions?

## Exploration 2

Each sequence of differences that results from a sequence generated by a polynomial function also can be generated by a polynomial function. In this exploration, you explore the relationship among the degree(s) of polynomial functions that generate successive sequence(s) of differences.

**a.** Determine the first 10 terms of the sequence generated by the quartic function $f(n) = n^4 - 2n^2 + n$.

**b.** **1.** The terms of this sequence's first sequence of differences can be generated by the function $f_1(n) = f(n+1) - f(n)$. Use this function to calculate the first nine terms of the first sequence of differences.

2. The function $f_1(n)$ can be expanded, as shown below:

$$f_1(n) = f(n+1) - f(n)$$
$$= \left((n+1)^4 - 2(n+1)^2 + (n+1)\right) - \left(n^4 - 2n^2 + n\right)$$

Use the distributive and associative properties to simplify this expression. Record the degree of the function in a table with headings like those in Table **2**.

**Table 2: Degrees of functions that generate sequences of differences**

| Function | Degree |
|---|---|
| $f_1(n)$ | |
| $f_2(n)$ | |
| $f_3(n)$ | |
| $f_4(n)$ | |

**c.** 1. Use the function $f_2(n) = f_1(n+1) - f_1(n)$ to calculate the first eight terms of the second sequence of differences.

2. Using the procedure described in Part **b**, determine the degree of $f_2(n)$ and record it in Table **2**.

**d.** 1. Use the function $f_3(n) = f_2(n+1) - f_2(n)$ to calculate the first seven terms of the third sequence of differences.

2. Determine the degree of $f_3(n)$ and record it in Table **2**.

**e.** 1. Use the function $f_4(n) = f_3(n+1) - f_3(n)$ to calculate the first six terms of the fourth sequence of differences. (This should be a constant sequence.)

2. Determine the degree of $f_4(n)$ and record it in Table **2**.

**f.** Describe a relationship between the degree of a polynomial that generates a sequence and the degrees of polynomials that generate successive sequences of differences.

## Discussion 2

**a.** Given a finite sequence generated by a polynomial function of degree 7, what is the least degree of a polynomial function that generates each of the following:

1. the first sequence of differences?
2. the third sequence of differences?
3. the fifth sequence of differences?
4. the sixth sequence of differences?

**b.** What is the relationship between the degree of a polynomial function that generates a finite sequence and the degrees of the polynomial functions that generate its successive sequences of differences?

**c.** Describe how you could use your knowledge of the finite-difference process to determine the terms of a sequence given the following information.

- The first term of the sequence is 17.
- The first term of the first sequence of differences is 9.
- The first term of the second sequence of differences is 6.
- The terms of the third sequence of differences are 5, 5, 5.

## Assignment

**1.1** **a.** Complete a copy of the following table for the finite sequence generated by $t_n = 3n^3 + 4n - 3$, for $n = 1, 2, 3, \ldots, 10$.

| $n$ | **Sequence** | **Sequences of Differences** | | |
|---|---|---|---|---|
| 1 | | **First** | | |
| 2 | | | **Second** | |
| 3 | | | | **Third** |
| 4 | | | | |
| $\vdots$ | | | | |
| 10 | | | | |

**b.** Identify the least degree of polynomial function that could generate the sequence in each column of the table.

**1.2** Create a polynomial function that generates a finite sequence in which the fourth sequence of differences is the first constant sequence of differences.

**1.3** **a.** Identify the missing term in the following finite sequence and describe how you determined your answer.

$$2, 9, ?, 65, 126, 217, 344, 513$$

**b.** Identify the least degree of polynomial function that could generate the sequence in Part **a**.

**1.4** Explain why a sequence generated by a fifth-degree polynomial eventually results in a constant sequence of differences.

**1.5** Each of the following sequences was generated by a cubic function. Identify the missing term in each one.

**a.** 4, 15, 38, 79, 144, 239, ?

**b.** –144, –286, –420, –540, –640, –714, –756, –760, ?

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**1.6** Determine the least degree of polynomial function that could have generated each sequence below.

**a.** 3, 11, 31, 69, 131, 223, 351, 521, 739

**b.** 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

**1.7** Use the infinite sequences defined by the following recursive rules to complete Parts **a–c**.

$$\begin{cases} p_1 = 1 \\ p_n = p_{n-1} + n, \ n > 1 \end{cases} \qquad \begin{cases} q_1 = 1 \\ q_n = q_{n-1} + n^2, \ n > 1 \end{cases} \qquad \begin{cases} r_1 = 1 \\ r_n = r_{n-1} + n^3, \ n > 1 \end{cases}$$

**a.** Determine the first eight terms of each sequence.

**b.** Determine whether or not each sequence could be generated by a polynomial function.

**c.** If the sequence could be generated by a polynomial function, determine the least degree of this polynomial.

**1.8** **a.** Identify the missing term in the following sequence, given that it was generated by a polynomial:

5, –22, –147, ?, –1219, –2550, –4747, –8122, –13035, –19894

**b.** Determine the least degree of polynomial function that could have generated the sequence in Part **a**.

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## *Activity 2*

Retailers and builders often stack objects in pyramids. For example, a grocer might display a pyramid of fruit in the produce department, while a builder might store a pyramid of concrete blocks on a job site. In this activity, you examine how sequences generated by polynomial functions can be used to model these situations.

### Exploration

**a.** Create a stack of blocks shaped like a pyramid with a square base, using 25 blocks on the bottom level and 1 block on the top level. When completed, your stack should resemble the diagram in Figure **2**.

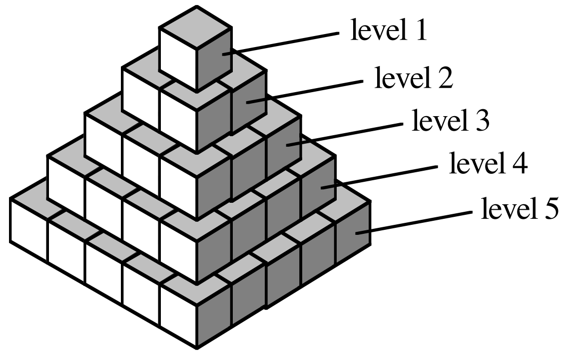



**Figure 2: A stack of blocks**

**b.** **1.** Write a sequence to model the total number of blocks in the pyramid after each level, starting with the top. Let the term number of the sequence equal the level of the pyramid, and let the value of the term equal the total number of blocks in the pyramid after each level.

**2.** Determine a recursive formula for the sequence.

**3.** Find the next three terms of the sequence.

**c.** One method for finding an explicit formula for a polynomial sequence involves using finite differences. Use the finite-difference process to determine the least degree of the polynomial that might generate the sequence found in Part **b**.

**d.** Once the least degree of the polynomial has been identified using finite differences, the polynomial function itself can be found by solving a system of equations.

For example, Table **3** shows the results of using the finite-difference process for the sequence 49, 72, 99, 130, 165.

**Table 3: Differences for a sequence created by a quadratic**

| *n* | **Sequence** | **Sequences of Differences** | |
|---|---|---|---|
| 1 | 49 | **First** | |
| 2 | 72 | 23 | **Second** |
| 3 | 99 | 27 | 4 |
| 4 | 130 | 31 | 4 |
| 5 | 165 | 35 | 4 |

Since the second sequence of differences is the first constant sequence, the least degree of the polynomial that generates this sequence is 2.

From Table **3**, when $n = 1$, $f(n) = 49$; when $n = 2$, $f(n) = 72$; and when $n = 3$, $f(n) = 99$.

These values, along with the general equation for a quadratic polynomial, $f(n) = a_2n^2 + a_1n + a_0$, can be used to create the following system of equations:

$$\begin{cases} 49 = a_2(1)^2 + a_1(1) + a_0 \\ 72 = a_2(2)^2 + a_1(2) + a_0 \\ 99 = a_2(3)^2 + a_1(3) + a_0 \end{cases}$$

Solving this system for $a_2$, $a_1$, and $a_0$, results in the polynomial function that generates the sequence: $f(n) = 2n^2 + n - 6$.

Use this process to determine a polynomial that generates the sequence in Part **b**.

**e.** Another method for identifying a polynomial that generates a sequence involves curve-fitting techniques.

1. Create a scatterplot of term value versus term number for the sequence in Part **b**.
2. Considering your response to Part **c**, use a polynomial regression to find an explicit formula for the sequence.

## Discussion

**a.** Compare the two polynomial functions you identified in Parts **d** and **e** of the exploration.

**b.**
1. How can a recursive formula be used to find the 100th term of a sequence?
2. How can an explicit formula be used to find the 100th term of a sequence?

**c.** Given only the first few terms, what assumption must be made in order to predict the 100th term of a sequence?

**d.** Consider the sequence $1, 2, 4, 8, 16\ldots$ .

1. Predict the sixth term of this sequence and describe how you made your prediction.
2. The sequence given above could be generated by the following polynomial function:

   $$f(n) = \frac{41}{120}n^5 - \frac{61}{12}n^4 + \frac{691}{24}n^3 - \frac{911}{12}n^2 + \frac{1393}{15}n - 40$$

   In this case, what would be the sixth term of the sequence?

**e.** What do your responses to Part **d** above imply about the reliability of making predictions about subsequent terms in a given sequence?

**f.** Suppose that you know the recursive formula for an infinite sequence. The pattern revealed by the first several terms indicates that it can be generated by a polynomial. Would you expect there to be more than one polynomial that could generate this sequence?

## Assignment

**2.1** Suggest a polynomial function for generating each sequence below and use it to predict the next three terms.

**a.** **1.** 1, 6, 18, 40, 75, 126, 196, ...

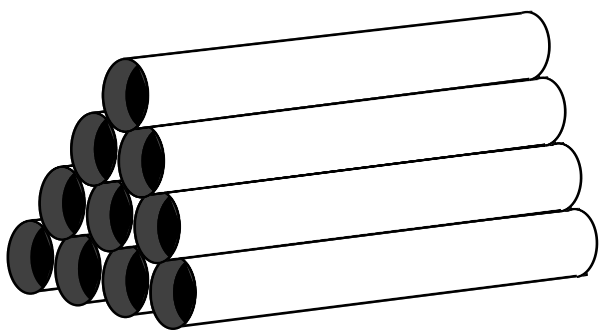
**2.** 6.5, 17, 36.5, 65, 102.5, …

**3.** 8.9, –43.6, –215.1, –631.6, –1469.5, –2955.6, …

**4.** –1, 2, 3, 2, –1, –6, –13, –22, …

**b.** Are the functions you suggested in Part **a** the only polynomials that could have generated these sequences? Explain your response.

**2.2** A pipe manufacturer stores its products in piles like the one shown in the diagram below.



**a.** The total number of pipes after any row, beginning with the top row, defines a sequence. For example, after 1 row, there is 1 pipe; after 2 rows, there are 3 pipes; and so on.

Do you think that this sequence can be generated by a polynomial function? Explain your response.

**b.** The warehouse manager would like to develop a method for determining the total number of pipes in a pile by counting the number of pipes in the bottom row.

Find a formula that describes the total number of pipes in a pile given the number of pipes in the bottom row.

**c.** The warehouse has a pile with 40 pipes in the bottom row. How many pipes are in this pile?

**2.3** Consider the sequence 1, 2, 3, ... .

**a.** Predict the fourth term in the sequence.

**b.** Determine a polynomial function that could be used to generate the sequence containing your four terms.

**c.** Suggest a value for the fourth term other than the one you predicted in Part **a**.

**d.** Use the first four terms of the sequence from Part **c** and the general form of a cubic function, $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, to determine a polynomial that could be used to generate these four terms.

**e.** Compare the fifth terms of the sequences generated by the polynomials in Parts **b** and **d**.

**2.4** Consider an arrangement of the natural numbers in a triangular pattern, as shown below.

1 — row 1
2 3 — row 2
4 5 6 — row 3
… *etc.* …

**a.** Generate the next two rows of the triangular pattern.

**b.** Which row will contain the number 1000?

**c.** What is the sum of the numbers in the 100th row?

**2.5** Consider the sequence –5, 7.5, –11.25, 16.875, … .

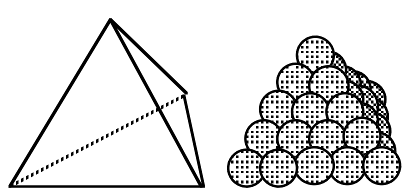
**a.** Suggest a recursive formula for this sequence.

**b.** Predict the next three terms.

**c.** Suggest an explicit formula for this sequence.

**d.** Predict the 50th term.

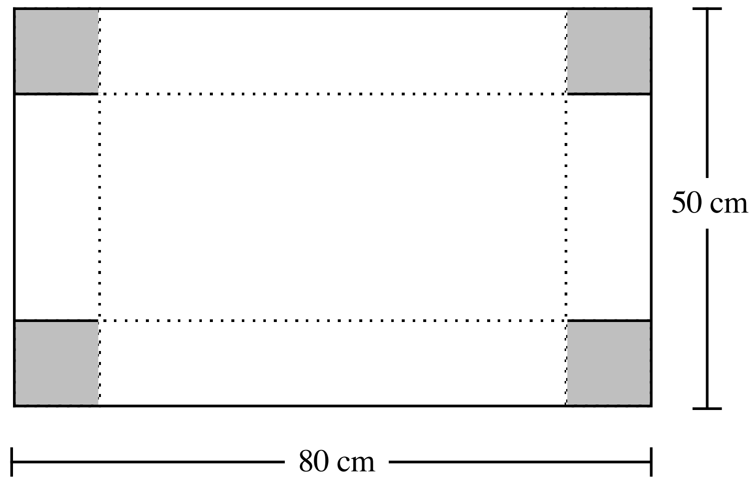
**2.6** Grocery stores often display fruit in stacks shaped like tetrahedrons. The diagram below shows such a display, with 15 oranges on the bottom level and 1 orange on the top level.



**a.** Determine the number of oranges that could be displayed in a tetrahedral stack with 10 levels.

**b.** A produce department has 7 cases of oranges. Each case contains 48 oranges. If the manager decides to display these oranges in a tetrahedral stack, how many levels will it have?

**2.7** A box manufacturer creates clothing boxes from a template like the one shown below.




Cuts are made along the solid lines and folds are made along the dotted lines. The square tabs (shaded areas) are folded and glued inside to form the box. The machinery that makes the cuts will cut only square tabs with dimensions measured in whole centimeters.

**a.** Create a sequence in which the term number represents the length of the side of the square tab and the terms describe the surface area of the resulting box.

**b.** Create a sequence in which the term number represents the length of the side of the square tab and the terms describe the volume of the resulting box.

**c.** Determine the dimensions of the square tabs that result in a box with the largest possible surface area.

**d.** Determine the dimensions of the square tabs that result in a box with the largest possible volume.

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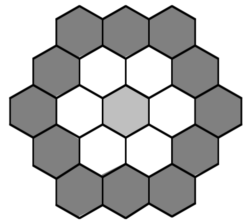
**2.8** According to a popular holiday song, "The Twelve Days of Christmas," a "true love" gives an increasing number of gifts on 12 consecutive days. On the first day, the true love gives a partridge in a pear tree. On the second day, the true love gives two turtle doves and a partridge in a pear tree. On the third day, the true love gives three French hens, two turtle doves, and a partridge in a pear tree. This pattern continues.

**a.** On the 12th day, how many gifts does the true love give? Justify your response.

**b.** During the entire 12 days, how many gifts does the loved one receive? Explain your response.

**2.9** Quilts are sometimes created by starting with a hexagonal piece of cloth, then adding rings of hexagons, as shown below.



**a.** Write a sequence for the number of hexagons in each of the first 10 rings of this quilt pattern.

**b.** How many hexagons would there be in the 21st ring of such a quilt?

**2.10** A person starts a savings plan by saving \$10 in the first month, \$12 in the second month, \$14 in the third month, and so on.

**a.** How much should be saved in the 12th month of the fifth year?

**b.** What is the total amount of money saved after 5 years?

**2.11** **a.** Create a sequence by following the steps below.

1. Construct a circle. Construct one point on the circle. Record the number of regions in the interior of the circle.
2. Construct a circle. Construct a diameter of the circle. Record the number of regions in the interior of the circle.
3. Construct a circle. Inscribe an equilateral triangle in the circle and connect each vertex to every other vertex with a line segment. Record the number of regions in the interior of the circle.
4. Construct a circle. Inscribe a square in the circle and connect each vertex to every other vertex with a line segment. Record the number of regions in the interior of the circle.
5. Construct a circle. Inscribe a regular pentagon in the circle and connect each vertex to every other vertex with a line segment. Record the number of regions in the interior of the circle.

**b.** Determine a pattern in the numbers of interior regions formed.

**c.** Predict the numbers of regions formed by each of the following:

1. inscribing a regular hexagon in a circle and connecting each vertex to every other vertex with a line segment
2. inscribing a regular heptagon in a circle and connecting each vertex to every other vertex with a line segment.

**d.** Verify your predictions from Part **c**.

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## Research Project

The Tower of Hanoi is a classic game involving sequences. The goal of the game is to move all the rings from one peg to another peg, retaining the same order, in a minimum number of moves. The game is subject to the following restrictions:

- Only one ring may be moved at a time.
- Once a ring is removed from one peg, it must be placed on one of the other two pegs.
- A ring may not be placed on top of a smaller ring.

Figure **3** shows the starting positions for 8 rings, an intermediate step, and the final positions of the rings.

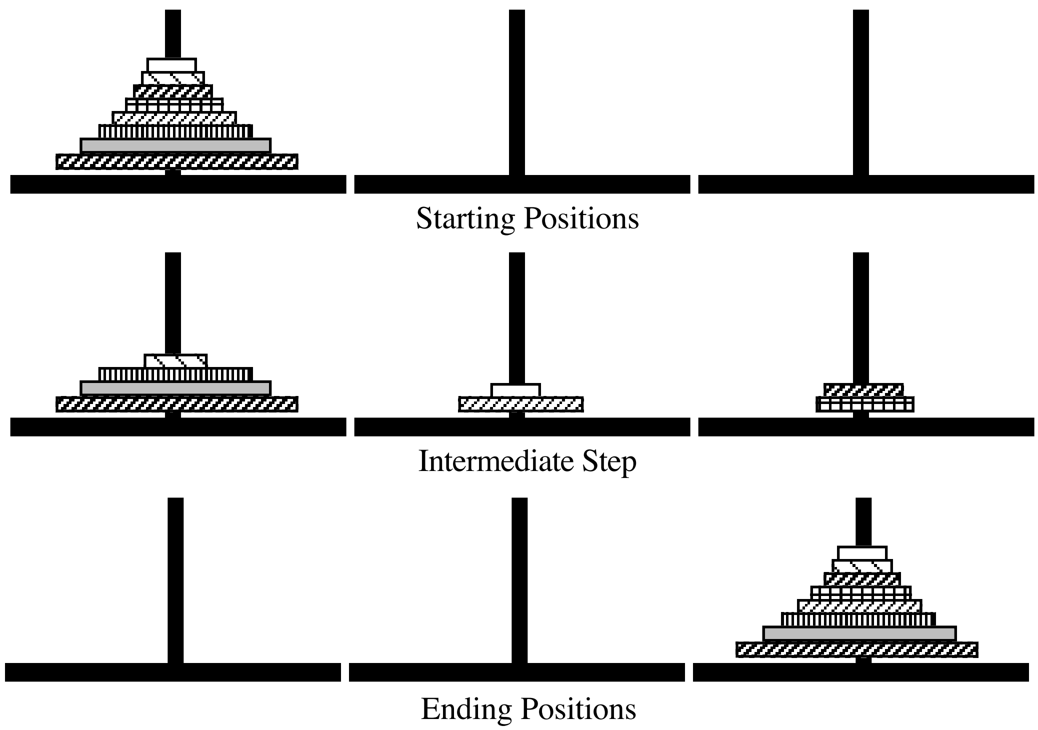



**Figure 3: Playing the Tower of Hanoi with eight rings**

**a.** Determine the minimum number of moves needed to move a stack of *n* rings from one peg to another while retaining the same order.

**b.** Determine the relationship between the number of rings and the minimum number of moves necessary to complete the task. Express this relationship as:

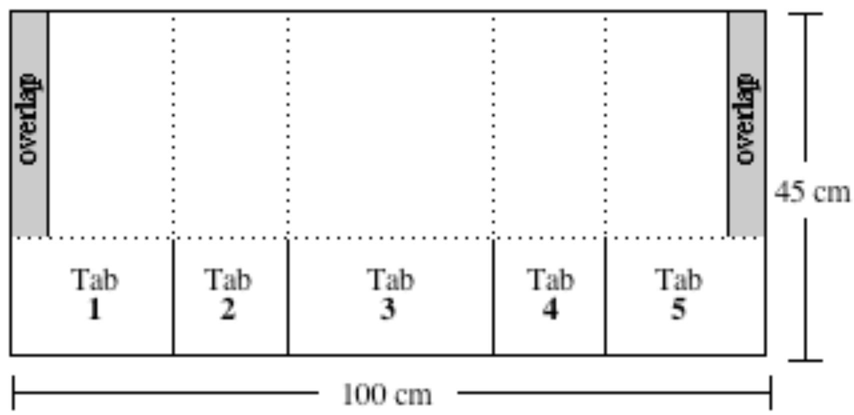
  **1.** a recursive formula

  **2.** an explicit formula.

## *Summary Assessment*

Imagine that you are an engineer for a company that manufactures paper bags. Your department has been asked to design a bag with a capacity of at least 12,500 $cm^3$. For advertising purposes, the bag also must have the largest possible outside surface area.

The bag must be cut and folded from a rectangular sheet of paper measuring 100 cm by 45 cm, as shown in the following diagram.




The solid lines on the template indicate cuts; the dotted lines indicate folds. Tabs **2** and **4** are squares of equal measure. Tabs **1** and **5** are $3/4$ the length of Tab **3**. (This creates an overlap for gluing and strength.) The dimensions of the base of the bag are the length of Tab **2** by the length of Tab **3**. Due to the nature of the machinery that makes the cuts, the dimensions of the tabs must be in whole centimeters.

Once you have designed an acceptable bag, you must make a presentation to the company's board of directors. Your presentation should include the dimensions of the bag, a description of the method you used to determine these dimensions, and a paragraph that explains why you believe your bag is the best one possible. As part of your presentation, include a scale model made from a sheet of notebook paper.

## *Module Summary*

- A **sequence of differences** can be generated from a finite sequence $t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ \ldots,\ t_n$ by taking the differences of consecutive terms. The first sequence of differences is $t_2 - t_1,\ t_3 - t_2,\ t_4 - t_3,\ t_5 - t_4,\ \ldots,\ t_n - t_{n-1}$.
- The process of finding successive sequences of differences is called the **finite-difference process**, which continues until the first constant sequence of differences is found.
- A sequence generates a constant sequence of differences if and only if that sequence can be generated by a polynomial function.
- If the first constant sequence of differences is the *n*th sequence of differences, there exists a unique polynomial function of degree *n* that generates the original sequence.

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