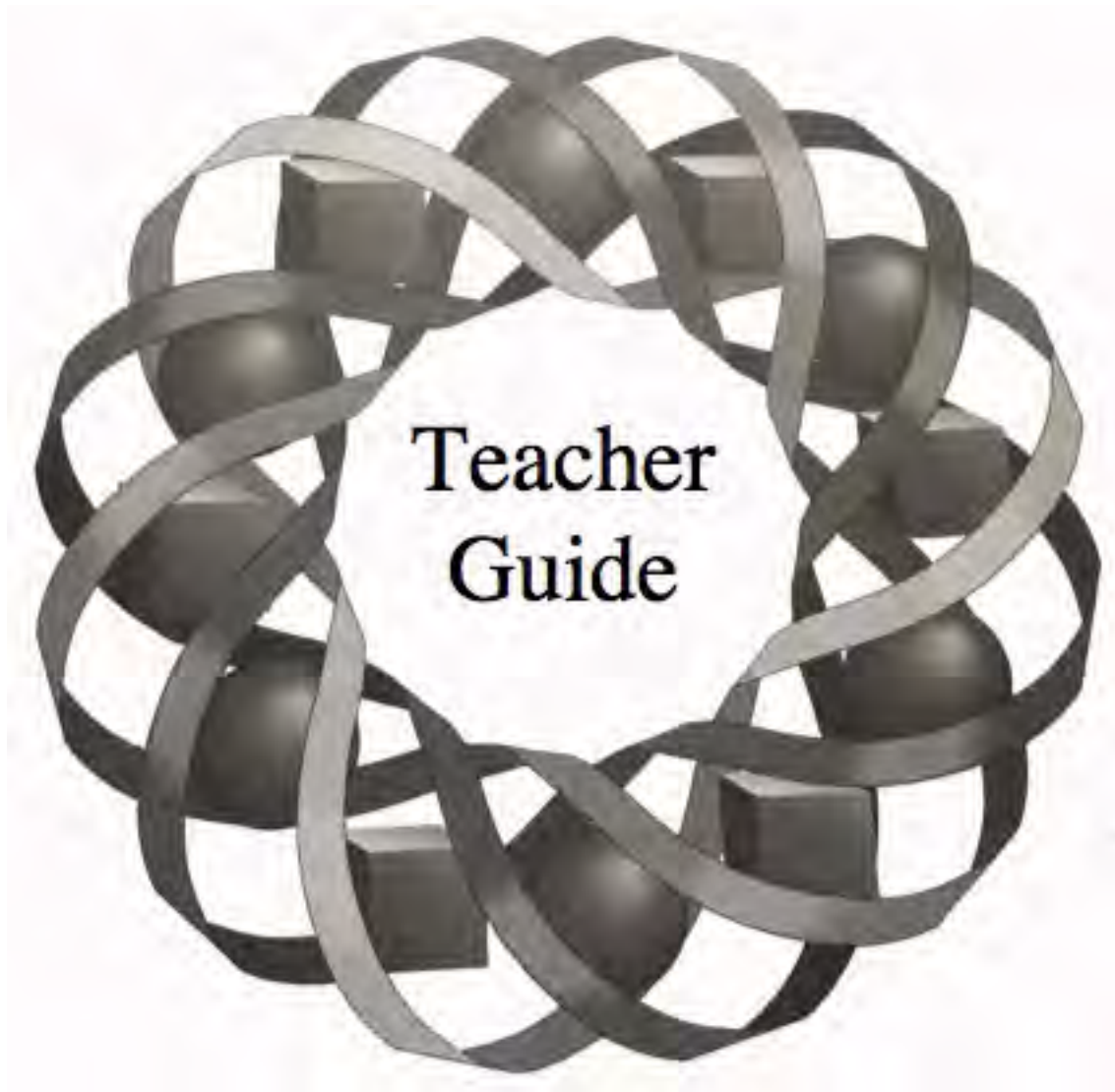


# ***SIMMS Integrated Mathematics:***

## ***A Modeling Approach Using Technology***



Level 2 Volumes 1-3



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*Teacher Guide*  
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# About *Integrated Mathematics*: A Modeling Approach Using Technology

## The Need for Change

In recent years, many voices have called for the reform of mathematics education in the United States. Teachers, scholars, and administrators alike have pointed out the symptoms of a flawed system. From the ninth grade onwards, for example, about half of the students in this country's mathematical pipeline are lost each year (National Research Council, 1990, p. 36). Attempts to identify the root causes of this decline have targeted not only the methods used to instruct and assess our students, but the nature of the mathematics they learn and the manner in which they are expected to learn. In its *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics addressed the problem in these terms:

When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics. (p. 64)

## Some Methods for Change

Among the major objectives of the *Integrated Mathematics* curriculum are:

- offering a 9–12 mathematics curriculum using an integrated inter-disciplinary approach for *all* students.
- incorporating the use of technology as a learning tool in all facets and at all levels of mathematics.
- offering a *Standards*-based curriculum for teaching, learning, and assessing mathematics.

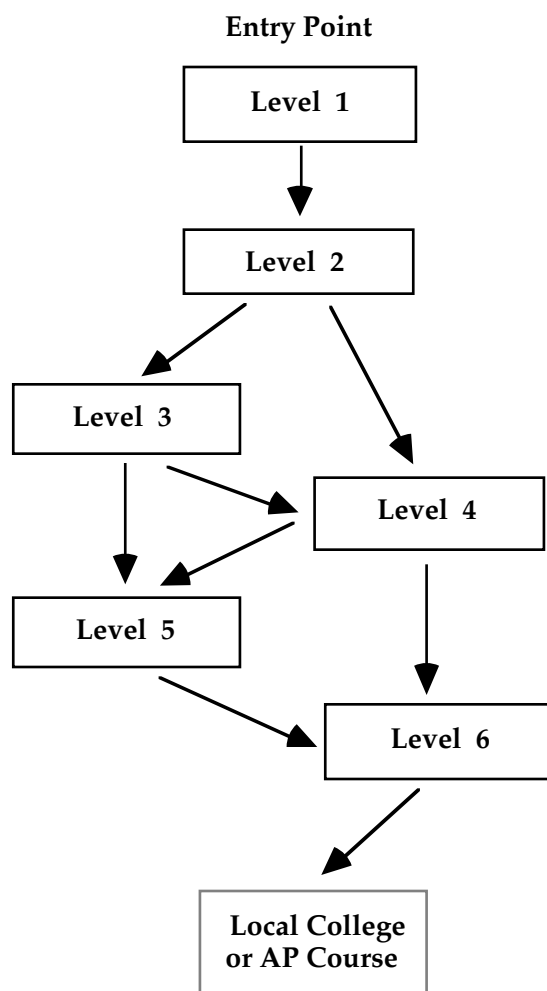
## The *Integrated Mathematics* Curriculum

An integrated mathematics program “consists of topics chosen from a wide variety of mathematical fields. . . [It] emphasizes the relationships among topics within mathematics as well as between mathematics and other disciplines” (Beal, et al., 1992; Lott, 1991). In order to create innovative, integrated, and accessible materials, *Integrated Mathematics: A Modeling Approach Using Technology* was written, revised, and reviewed by secondary teachers of mathematics and science. It is a complete, *Standards*-based mathematics program designed to replace all currently offered secondary mathematics courses, with the possible exception of advanced placement classes, and builds on middle-school reform curricula.

The *Integrated Mathematics* curriculum is grouped into six levels. All students should take at least the first two levels. In the third and fourth years, *Integrated Mathematics* offers a choice of courses to students and their parents, depending on interests and goals. A flow chart of the curriculum appears in Figure 1.

Each year-long level contains 14–16 modules. Some must be presented in

sequence, while others may be studied in any order. Modules are further divided into several activities, typically including an exploration, a discussion, a set of homework assignments, and a research project.



**Figure 1: Integrated Mathematics course sequence**

Assessment materials—including alternative assessments that emphasize writing and logical argument—are an integral part of the curriculum. Suggested assessment items for use with a standard rubric are identified in all teacher editions.

***Level 1: a first-year course for ninth graders (or possibly eighth graders)***

Level 1 concentrates on the knowledge and understanding that students need to become

mathematically literate citizens, while providing the necessary foundation for those who wish to pursue careers involving mathematics and science. Each module in Level 1, as in all levels of the curriculum, presents the relevant mathematics in an applied context. These contexts include the properties of reflected light, population growth, and the manufacture of cardboard containers. Mathematical content includes data collection, presentation, and interpretation; linear, exponential, and step functions; and three-dimensional geometry, including surface area and volume.

***Level 2: a second-year course for either ninth or tenth graders***

Level 2 continues to build on the mathematics that students need to become mathematically literate citizens. While retaining an emphasis on the presentation and interpretation of data, Level 2 introduces trigonometric ratios and matrices, while also encouraging the development of algebraic skills. Contexts include pyramid construction, small business inventory, genetics, and the allotment of seats in the U.S. House of Representatives.

***Levels 3 and 4: options for students in the third year***

Both levels build on the mathematics content in Level 2 and provide opportunities for students to expand their mathematical understanding. Most students planning careers in math and science will choose Level 4. While Level 3 also may be suitable for some of these students, it offers a slightly different mixture of context and content.

Contexts in Level 4 include launching a new business, historic rainfall patterns, the pH scale, topology, and scheduling. The mathematical content includes rational, logarithmic, and circular functions, proof, and combinatorics.

In Level 3, contexts include nutrition, surveying, and quality control. Mathematical topics include linear programming, curve-fitting, polynomial functions, and sampling.

### ***Levels 5 and 6: options for students in the fourth year***

Level 6 materials continue the presentation of mathematics through applied contexts while embracing a broader mathematical perspective. For example, Level 6 modules explore operations on functions, instantaneous rates of change, complex numbers, and parametric equations.

Level 5 focuses more specifically on applications from business and the social sciences, including hypothesis testing, Markov chains, and game theory.

## **More About Level 2**

“Marvelous Matrices” introduces matrix operations in the context of business inventories. “When to Deviate from a Mean Task” and “And the Survey Says...” focus on statistics and sampling, respectively. “A New Angle on an Old Pyramid,” “There’s No Place Like Home,” and “Crazy Cartoons” have primarily geometric themes. “Traditional Design” also explores some geometric topics—this time from perspective of American Indian star quilts. “Who Gets What and Why?” examines the mathematics of apportionment to the U.S. House of Representatives. Other modules explore linear programming, multistage probability, models of exponential decay, curve fitting, and the limits of sequences and series. Some teachers may wish to schedule the genetics module—“What Are My Child’s Chances?”—in coordination with a biology class.

## **The Teacher Edition**

To facilitate use of the curriculum, the teacher edition contains these features:

### ***Overview /Objectives/Prerequisites***

Each module begins with a brief overview of its contents. This overview is followed by a list of teaching objectives and a list of prerequisite skills and knowledge.

### ***Time Line/Materials & Technology Required***

A time line provides a rough estimate of the classroom periods required to complete each module. The materials required for the entire module are listed by activity. The technology required to complete the module appears in a similar list.

### ***Assignments/Assessment Items/Flashbacks***

Assignment problems appear at the end of each activity. These problems are separated into two sections by a series of asterisks. The problems in the first section cover all the essential elements in the activity. The second section provides optional problems for extra practice or additional homework.

Specific assignment problems recommended for assessment are preceded by a single asterisk in the teacher edition. Each module also contains a Summary Assessment in the student edition and a Module Assessment in the teacher edition, for use at the teacher’s discretion. In general, Summary Assessments offer more open-ended questions, while Module Assessments take a more traditional approach. To review prerequisite skills, each module includes brief problem sets called “Flashbacks.” Like the Module Assessment, they are designed for use at the teacher’s discretion.

## **Technology in the Classroom**

The *Integrated Mathematics* curriculum takes full advantage of the appropriate use of technology. In fact, the goals of the curriculum are impossible to achieve without it. Students must have ready access to the functionality of a graphing utility, a spreadsheet, a geometry utility, a statistics

program, a symbolic manipulator, and a word processor. In addition, students should have access to a science interface device that allows for electronic data collection from classroom experiments, as well as a telephone modem.

In the student edition, references to technology provide as much flexibility as possible to the teacher. In the teacher edition, sample responses refer to specific pieces of technology, where applicable.

### **Professional Development**

A program of professional development is recommended for all teachers planning to use the curriculum. The *Integrated Mathematics* curriculum encourages the use of cooperative learning, considers mathematical topics in a different order than in a traditional curriculum, and teaches some mathematical topics not previously encountered at the high-school level.

In addition to incorporating a wide range of context areas, *Integrated Mathematics* invites the use of a variety of instructional formats involving heterogeneous classes. Teachers should learn to use alternative assessments, to integrate writing and communication into the mathematics curriculum, and to help students incorporate technology in their own investigations of mathematical ideas.

Approximately 30 classroom teachers and 5 university professors are available to present inservice workshops for interested school districts. Please contact Kendall Hunt Publishing Company for more information.

### **Student Performance**

During the development of *Integrated Mathematics*, researchers conducted an annual assessment of student performances in pilot schools. Each year, two basic measures—the PSAT and a selection of open-ended tasks—were administered to

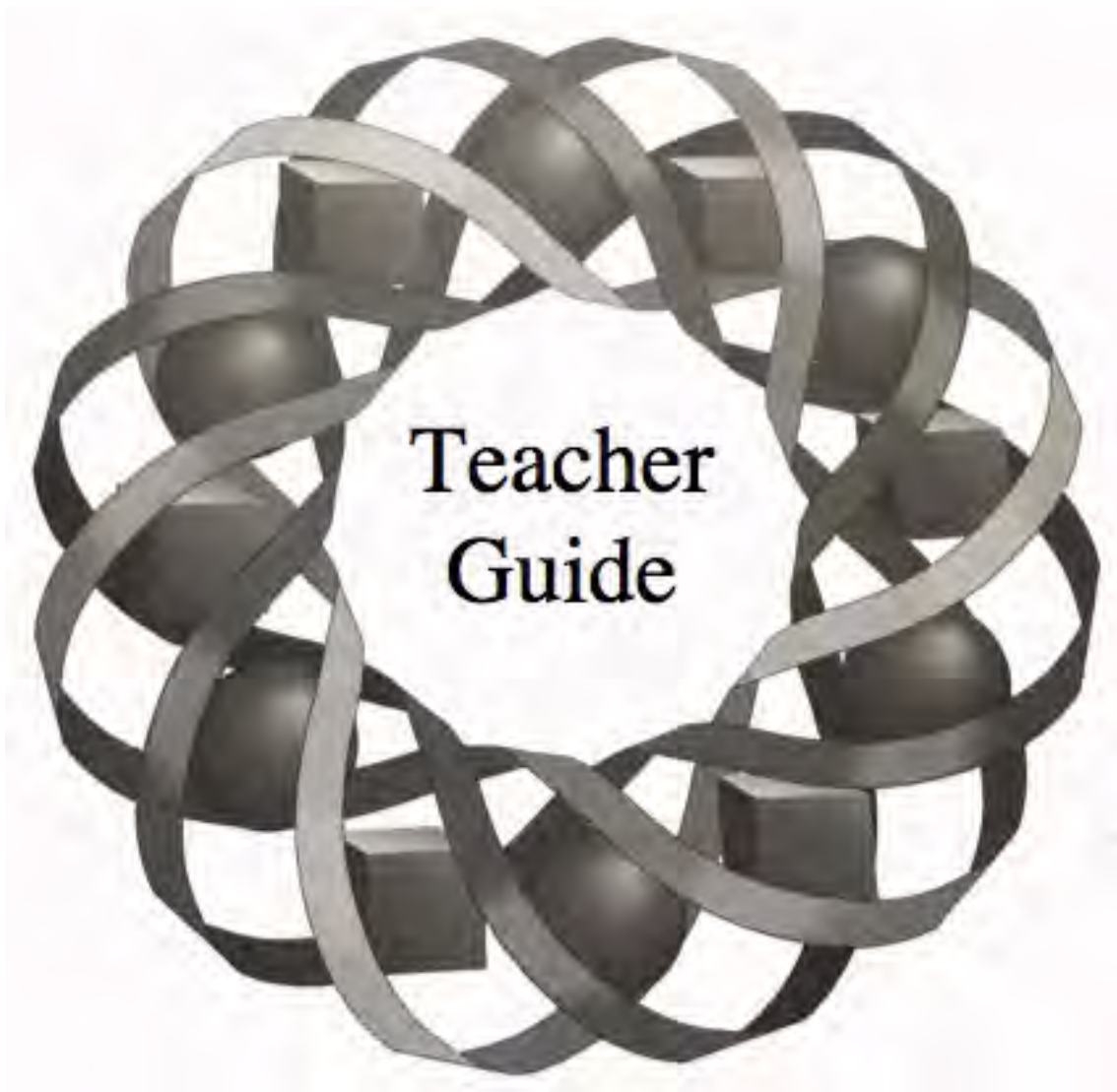
two groups: students in classes using *Integrated Mathematics* and students in classes using other materials. Students using *Integrated Mathematics* materials typically had access to technology for all class work. During administration of the PSAT, however, no technology was made available to either group. Student scores on the mathematics portion of this test indicated no significant difference in performance.

During the open-ended, end-of-year test, technology was made available to both groups. Analysis of student solutions to these tasks showed that students using *Integrated Mathematics* were more likely to provide justification for their solutions and made more and better use of graphs, charts, and diagrams. They also demonstrated a greater variety of problem-solving strategies and were more willing to attempt difficult problems.

### **References**

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# Marvelous Matrices



How do businesses—large and small—keep track of their sales, inventory, and profits? In this module, you explore some of the ways that matrices are used to store and analyze information.

*Kyle Boyce • Pete Stabio*



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# Teacher Edition

## Marvelous Matrices

### Overview

In this module, students explore matrices as tools for storing, organizing, and analyzing data. The operations of matrix addition, scalar multiplication, and matrix multiplication are introduced within a business context.

### Objectives

In this module, students will:

- organize and interpret data using matrices
- use matrices in business applications
- add and subtract two matrices
- multiply a matrix by a scalar
- multiply two matrices
- interpret the meaning of the elements within a product matrix.

### Prerequisites

For this module, students should know:

- the definition of the mean of two numbers
- how to interpret subscripted variables.

### Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	2	3	2	9

### Materials Required

- none

### Technology

Software	Activity			
	1	2	3	Summary Assessment
matrix manipulator	X	X	X	X



# Marvelous Matrices

## Introduction

(page 3)

The use of matrices for inventory control is based on ideas from management science, specifically Leontief input-output models.

(page 3)

### *Activity 1*

This activity defines matrices, simple components, composite products, requirement graphs, and total requirement matrices. Students explore matrices as a tool for storing and interpreting data.

## Materials List

- none

## Technology

- matrix manipulator (optional)

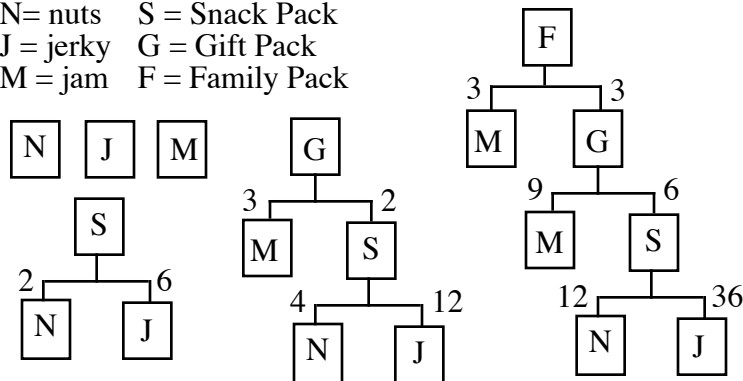
## Exploration

(page 5)

This exploration demonstrates the usefulness of requirement graphs and requirement matrices for data storage.

- a. Sample requirement graph:

N = nuts    S = Snack Pack  
J = jerky   G = Gift Pack  
M = jam    F = Family Pack



- b.**
1. A Snack Pack consists of 2 packages of nuts and 6 pieces of jerky.
  2. In terms of its simple components and composite products, a Gift Pack consists of 2 Snack Packs and 3 jars of jam.
  3. In terms of its simple components only, a Gift Pack consists of 4 packages of nuts, 12 pieces of jerky, and 3 jars of jam.
  4. In terms of its simple components and composite products, a Family Pack consists of 3 Gift Packs and 3 jars of jam.
  5. In terms of its simple components only, a Family Pack consists of 12 packages of nuts, 36 pieces of jerky, and 12 jars of jam.
- c.** A total requirement matrix for this product line is shown below.

$$\begin{array}{r}
 \mathbf{R} = \begin{array}{c}
 \begin{array}{cccccc}
 & \text{N} & \text{J} & \text{M} & \text{S} & \text{G} & \text{F} \\
 \text{N} & [1 & 0 & 0 & 2 & 4 & 12] \\
 \text{J} & |0 & 1 & 0 & 6 & 12 & 36| \\
 \text{M} & |0 & 0 & 1 & 0 & 3 & 12| \\
 \text{S} & |0 & 0 & 0 & 1 & 2 & 6| \\
 \text{G} & |0 & 0 & 0 & 0 & 1 & 3| \\
 \text{F} & |0 & 0 & 0 & 0 & 0 & 1|
 \end{array}
 \end{array}
 \end{array}$$

## Discussion

(page 6)

- a.** In this case, it means that there are 5 compasses in a Euclid Set.
- b.**
1. The elements along the diagonal represent the number of a given product contained in the product itself, which is always 1.
  2. An element of 0 indicates that a given product contains none of a specific simple component or composite product.
- c.** Each element in the requirement matrix can be calculated by adding the numbers in the appropriate product tree which correspond with a particular item. In the product tree for a Family Pack, for example, the two boxes that represent jam are accompanied by the numbers 9 and 3. The sum of these numbers (12) is the element in row M of column F of matrix **R**. This indicates that there are 12 jars of jam in a Family Pack.

- d. In the requirement matrix **R**, the elements in the rows that correspond with nuts, jerky, and jam represent the numbers of these simple components contained in each item. The desired quantity of each simple component can be found by multiplying each of these elements (shown in bold in the following equations) by the number of the corresponding item in the order, then finding the resulting sum.

$$N = 8 \cdot \mathbf{1} + 20 \cdot \mathbf{0} + 25 \cdot \mathbf{0} + 10 \cdot \mathbf{2} + 5 \cdot \mathbf{4} + 40 \cdot \mathbf{12} = 528$$

$$J = 8 \cdot \mathbf{0} + 20 \cdot \mathbf{1} + 25 \cdot \mathbf{0} + 10 \cdot \mathbf{6} + 5 \cdot \mathbf{12} + 40 \cdot \mathbf{36} = 1580$$

$$M = 8 \cdot \mathbf{0} + 20 \cdot \mathbf{0} + 25 \cdot \mathbf{1} + 10 \cdot \mathbf{0} + 5 \cdot \mathbf{3} + 40 \cdot \mathbf{12} = 520$$

Family Snack needs 528 packages of nuts, 1580 pieces of jerky, and 520 jars of jam to fill the order.

## Assignment

(page 7)

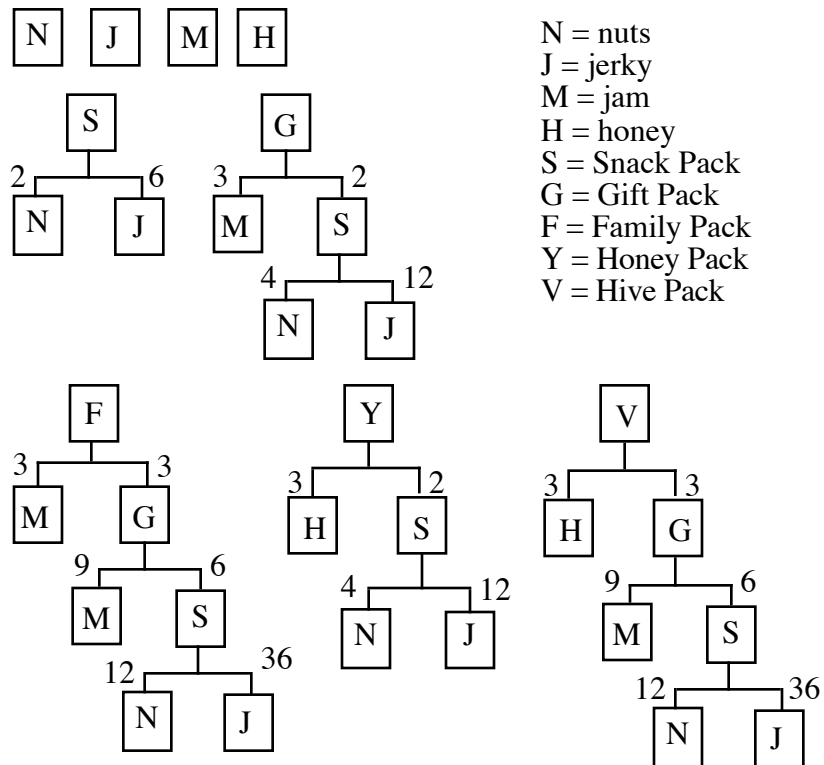
- 1.1 Using the method described in Part d of the discussion, Family Snack needs 140 packages of nuts, 420 pieces of jerky, and 108 jars of jam to fill the entire order. In the calculations below, the matrix elements are shown in bold.

$$N = 20 \cdot \mathbf{1} + 60 \cdot \mathbf{0} + 48 \cdot \mathbf{0} + 24 \cdot \mathbf{2} + 12 \cdot \mathbf{4} + 2 \cdot \mathbf{12} = 140$$

$$J = 20 \cdot \mathbf{0} + 60 \cdot \mathbf{1} + 48 \cdot \mathbf{0} + 24 \cdot \mathbf{6} + 12 \cdot \mathbf{12} + 2 \cdot \mathbf{36} = 420$$

$$M = 20 \cdot \mathbf{0} + 60 \cdot \mathbf{0} + 48 \cdot \mathbf{1} + 24 \cdot \mathbf{0} + 12 \cdot \mathbf{3} + 2 \cdot \mathbf{12} = 108$$

- \*1.2 a. Sample requirement graph:



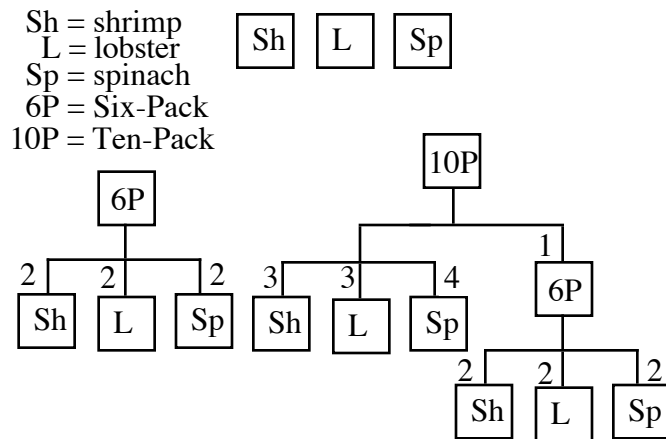
b. A total requirement matrix for this product line is shown below.

	N	J	M	H	S	G	F	Y	V
N	1	0	0	0	2	4	12	4	12
J	0	1	0	0	6	12	36	12	36
M	0	0	1	0	0	3	12	0	9
H	0	0	0	1	0	0	0	3	3
S	0	0	0	0	1	2	6	2	6
G	0	0	0	0	0	1	3	0	3
F	0	0	0	0	0	0	1	0	0
Y	0	0	0	0	0	0	0	1	0
V	0	0	0	0	0	0	0	0	1

c. There are 9 jars of jam in one Hive Pack.

**\*1.3**

a. Sample requirement graph:

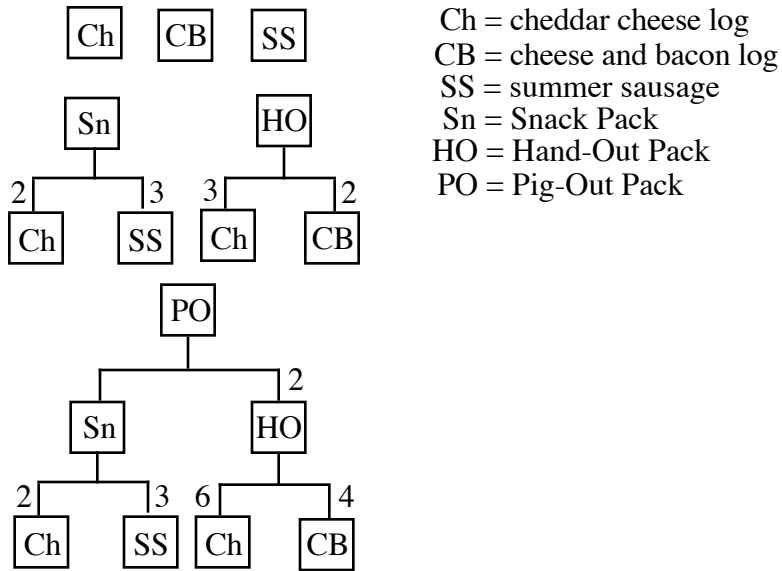


b. In a column for a simple component, the element 1 occurs once; the rest of the elements in the column are 0.

c. Sample response: Read the number of simple components in the Ten-Pack column. This tells how many of each are required for each Ten-Pack. Multiply these numbers by 8. You would need 24 shrimp spreads, 24 lobster spreads, and 32 spinach spreads.

\*\*\*\*\*

\*1.4 a. Sample requirement graph:



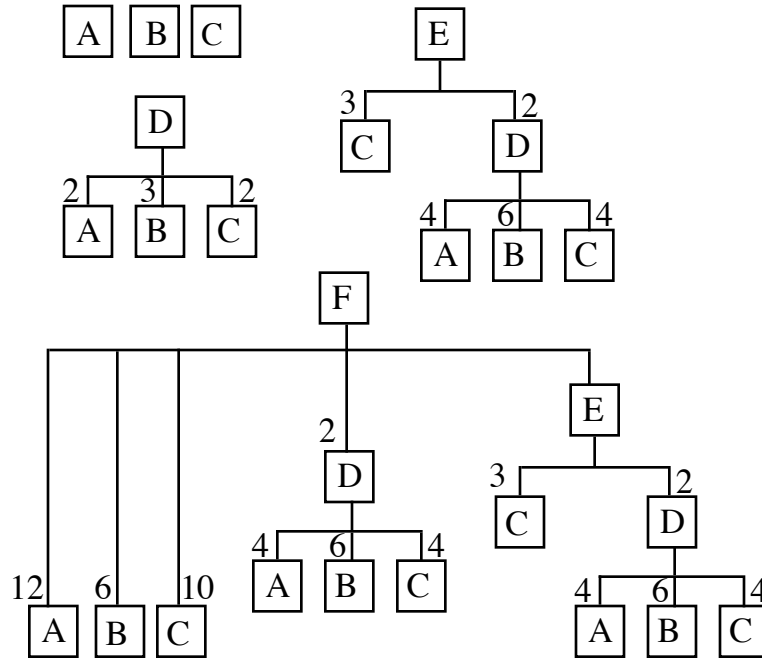
b. The corresponding total requirement matrix is shown below.

	Ch	CB	SS	Sn	HO	PO
Ch	1	0	0	2	3	8
CB	0	1	0	0	2	4
SS	0	0	1	3	0	3
Sn	0	0	0	1	0	1
HO	0	0	0	0	1	2
PO	0	0	0	0	0	1

c. There are 3 summer sausage logs in one Pig-Out Pack.

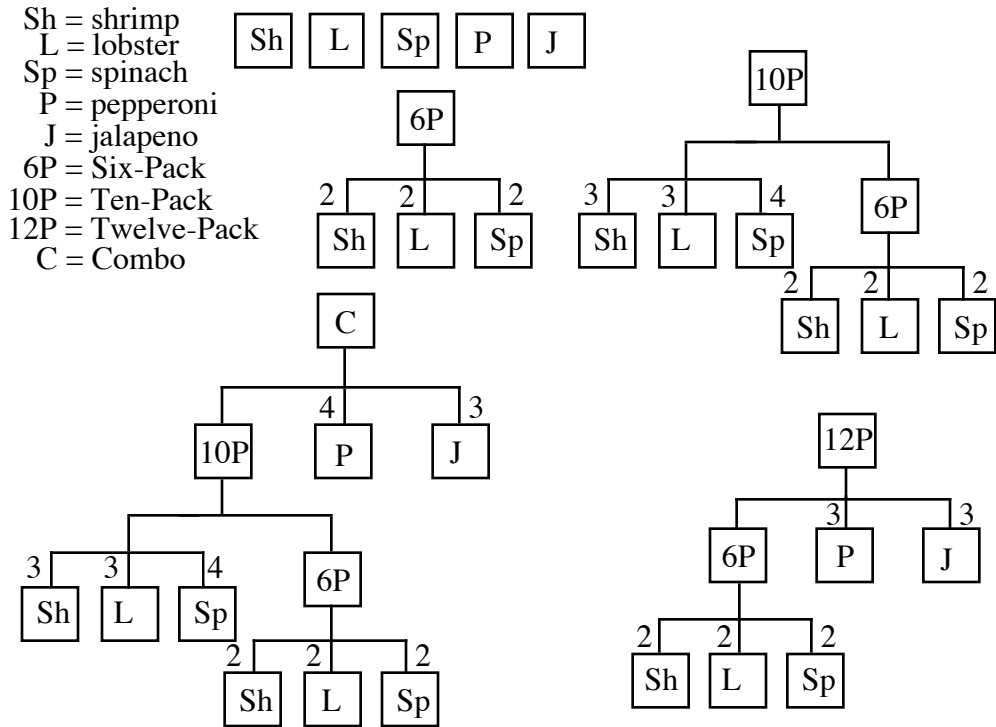
d. There are 8 cheddar cheese logs in one Pig-Out Pack.

1.5 a. Sample requirement graph:



- b. Sample response: The simple components in matrix **T** are A, B, and C. In a column for a simple component, the element 1 occurs once; the rest of the elements in the column are 0.
- c. Sample response: Read the number of simple components (As, Bs, and Cs) in the F column. This tells how many of each are required for each F. Multiply these numbers by 10. You would need 200 As, 180 Bs, and 210 Cs.

**\*1.6 a. Sample requirement graph:**



b. The corresponding total requirement matrix is shown below.

	Sh	L	Sp	P	J	6P	10P	12P	C
Sh	1	0	0	0	0	2	3	2	5
L	0	1	0	0	0	2	3	2	5
Sp	0	0	1	0	0	2	4	2	6
P	0	0	0	1	0	0	0	3	4
J	0	0	0	0	1	0	0	3	3
6P	0	0	0	0	0	1	1	1	1
10P	0	0	0	0	0	0	1	0	1
12P	0	0	0	0	0	0	0	1	0
C	0	0	0	0	0	0	0	0	1

c. There are 6 spinach spreads in one Combo Pack.

\*\*\*\*\*

(page 9)

**Activity 2**

In this activity, students are introduced to matrix addition and scalar multiplication in the context of inventory control.

## Materials List

- none

## Technology

- matrix manipulator (optional)

## Exploration

(page 9)

- a. 1. The dimensions of this matrix are  $3 \times 6$ .

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 8 \quad 12 \quad 0 \quad 16 \quad 28 \quad 8 ] \\ \mathbf{S} = \text{Z} [ 12 \quad 4 \quad 24 \quad 8 \quad 24 \quad 12 ] \\ \text{T} [ 12 \quad 0 \quad 12 \quad 12 \quad 36 \quad 4 ] \end{array}$$

2. The dimensions of this matrix are  $6 \times 3$ .

$$\begin{array}{r} \text{K} \quad \text{Z} \quad \text{T} \\ \text{N} [ 8 \quad 12 \quad 12 ] \\ \text{J} [ 12 \quad 4 \quad 0 ] \\ \text{M} [ 0 \quad 24 \quad 12 ] \\ \mathbf{R} = \text{S} [ 16 \quad 8 \quad 12 ] \\ \text{G} [ 28 \quad 24 \quad 36 ] \\ \text{F} [ 8 \quad 12 \quad 4 ] \end{array}$$

- b. Although they illustrate the same information, the matrices created in Part a are not equal because they have different dimensions.

- c. Sample  $3 \times 6$  matrix:

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 8 \quad 4 \quad 24 \quad 12 \quad 8 \quad 36 ] \\ \mathbf{O} = \text{Z} [ 16 \quad 8 \quad 28 \quad 16 \quad 20 \quad 32 ] \\ \text{T} [ 8 \quad 12 \quad 4 \quad 16 \quad 8 \quad 0 ] \end{array}$$

- d. 1. The matrix **T** below shows the total of the corresponding elements in matrices **S** and **O**.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 16 \quad 16 \quad 24 \quad 28 \quad 36 \quad 44 ] \\ \mathbf{T} = \text{Z} [ 28 \quad 12 \quad 52 \quad 24 \quad 44 \quad 44 ] \\ \text{T} [ 20 \quad 12 \quad 16 \quad 28 \quad 44 \quad 4 ] \end{array}$$



2. The matrix **D** below shows the differences between the corresponding elements in matrices **O** and **S**.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 0 \quad -8 \quad 24 \quad -4 \quad -20 \quad 28 ] \\ \mathbf{D} = \mathbf{Z} [ 4 \quad 4 \quad 4 \quad 8 \quad -4 \quad 20 ] \\ \text{T} [ -4 \quad 12 \quad -8 \quad 4 \quad -28 \quad -4 ] \end{array}$$

- e. The matrix **N** shows the product of each element in matrix **O** multiplied by 2.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 16 \quad 8 \quad 48 \quad 24 \quad 16 \quad 72 ] \\ \mathbf{N} = \mathbf{Z} [ 32 \quad 16 \quad 56 \quad 32 \quad 40 \quad 64 ] \\ \text{T} [ 16 \quad 24 \quad 8 \quad 32 \quad 16 \quad 0 ] \end{array}$$

- f. 1. This matrix can be determined by matrix addition.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 24 \quad 12 \quad 72 \quad 36 \quad 24 \quad 108 ] \\ \mathbf{N} + \mathbf{O} = \mathbf{Z} [ 48 \quad 24 \quad 84 \quad 48 \quad 60 \quad 96 ] \\ \text{T} [ 24 \quad 36 \quad 12 \quad 48 \quad 24 \quad 0 ] \end{array}$$

2. This matrix can be determined by matrix subtraction.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 8 \quad -4 \quad 48 \quad 8 \quad -12 \quad 64 ] \\ \mathbf{N} - \mathbf{S} = \mathbf{Z} [ 20 \quad 12 \quad 32 \quad 24 \quad 16 \quad 52 ] \\ \text{T} [ 4 \quad 24 \quad -4 \quad 20 \quad -20 \quad -4 ] \end{array}$$

3. This matrix can be determined by matrix addition.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 32 \quad 24 \quad 72 \quad 52 \quad 52 \quad 116 ] \\ \mathbf{S} + \mathbf{O} + \mathbf{N} = \mathbf{Z} [ 60 \quad 28 \quad 108 \quad 56 \quad 84 \quad 108 ] \\ \text{T} [ 36 \quad 36 \quad 24 \quad 60 \quad 60 \quad 4 ] \end{array}$$

- g. 1. This matrix can be determined as follows.

$$\begin{array}{r} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} [ 8 \quad 8 \quad 12 \quad 14 \quad 18 \quad 22 ] \\ \frac{1}{2}(\mathbf{S} + \mathbf{O}) = \mathbf{Z} [ 14 \quad 6 \quad 26 \quad 12 \quad 22 \quad 22 ] \\ \text{T} [ 10 \quad 6 \quad 8 \quad 14 \quad 22 \quad 2 ] \end{array}$$

2. This matrix can be determined as shown below.

$$\frac{1}{3}(S + O + N) = \begin{matrix} & \begin{matrix} \text{N} & \text{J} & \text{M} & \text{S} & \text{G} & \text{F} \end{matrix} \\ \begin{matrix} \text{K} \\ \text{Z} \\ \text{T} \end{matrix} & \begin{bmatrix} 10\frac{2}{3} & 8 & 24 & 17\frac{1}{3} & 17\frac{1}{3} & 38\frac{2}{3} \\ 20 & 9\frac{1}{3} & 36 & 18\frac{2}{3} & 28 & 36 \\ 12 & 12 & 8 & 20 & 20 & 1\frac{1}{3} \end{bmatrix} \end{matrix}$$

## Discussion

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- Sample response: The mean sales matrix was found by adding the matrices for September, October, and November, then multiplying the result by the scalar  $1/3$ .
- Only matrices with the same dimensions can be added together.
- The dimensions of the resulting matrix are the same as the dimensions of the original matrix (or matrices).
- Matrix addition involves adding corresponding elements of matrices with the same dimensions. Because order does not matter when adding two real numbers, matrix addition is commutative.

## Assignment

(page 12)

\*2.1 a.  $\begin{bmatrix} 5 & 9 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 0 & 9 \end{bmatrix}$

- b. Addition is not possible since the matrices do not have the same dimensions.

c.  $\pi \cdot \begin{bmatrix} 5 & 7 \\ 11 & -4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 5\pi & 7\pi \\ 11\pi & -4\pi \\ 3\pi & 0\pi \end{bmatrix} \approx \begin{bmatrix} 15.7 & 22.0 \\ 34.6 & -12.6 \\ 9.4 & 0 \end{bmatrix}$

d.  $4 \cdot \begin{bmatrix} 2 & 6 & -9 \\ 3 & 7 & 11 \end{bmatrix} - \begin{bmatrix} 8 & 15 & 2 \\ -5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 9 & -38 \\ 17 & 22 & 44 \end{bmatrix}$

e.  $\begin{bmatrix} 3 & 4 \\ 1 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & -7 \end{bmatrix}$

- 2.2 a. The element in row 4, column 5 represents the distance from Kansas City to Seattle.
- b. In row 5, column 3, 2779 represents the distance from Seattle to Chicago. In row 3, column 5, 2779 represents the distance from Chicago to Seattle.
- c. The zeros represent the distance from a city to itself.

- 2.3 a. Sample matrix:

$$\mathbf{T} = \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \end{array} \begin{bmatrix} 8 & 7 & 4 & 10 \\ 10 & 4 & 5 & 8.5 \\ 6 & 3.5 & 4 & 9 \\ 8 & 6.5 & 8 & 6 \end{bmatrix}$$

- b. The following assignment requires a minimum of 6 hr to complete all four tasks: Assign task 3 to employee 1, task 2 to employee 2, task 1 to employee 3, and task 4 to employee 4.
- c. There is no other set of assignments that completes the day's tasks in 6 hr. To complete all the tasks in 6 hr, employee 4 must do task 4 (because all the other tasks take this employee more than 6 hr). Likewise, employee 1 must do task 3. Employee 2 must do task 2 or task 1, since tasks 2 and 4 are already taken. Since task 1 requires more than 6 hr for employee 2 to complete, employee 2 must do Task 2. This leaves employee 3 with task 1.

- \*2.4 a. The dimensions of matrix  $\mathbf{I}$  are  $3 \times 4$ .

- b. This element (22) represents the inventory of large, red sweat pants.
- c. The new inventory is shown in matrix  $\mathbf{A}$  below.

$$\mathbf{A} = \begin{array}{c} B \\ R \\ Y \end{array} \begin{array}{c} S \\ M \\ L \\ XL \end{array} \begin{bmatrix} 31 & 45 & 44 & 28 \\ 32 & 47 & 42 & 30 \\ 29 & 35 & 38 & 32 \end{bmatrix}$$

- d. Using scalar multiplication:

$$3 \cdot \mathbf{I} = \begin{array}{c} B \\ R \\ Y \end{array} \begin{array}{c} S \\ M \\ L \\ XL \end{array} \begin{bmatrix} 33 & 75 & 72 & 24 \\ 36 & 81 & 66 & 30 \\ 27 & 45 & 54 & 36 \end{bmatrix}$$

- e. The sum of the following matrix and matrix  $\mathbf{I}$  is matrix  $\mathbf{A}$ .

$$\begin{array}{c} B \\ R \\ Y \end{array} \begin{array}{c} S \\ M \\ L \\ XL \end{array} \begin{bmatrix} 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \end{bmatrix}$$

\* \* \* \* \*

**\*2.5** a. Sample matrix:

$$\mathbf{U} = \begin{array}{c} \text{MT} \quad \text{TR} \quad \text{CC} \\ 43 \begin{bmatrix} 112 & 117 & 111 \end{bmatrix} \\ 48 \begin{bmatrix} 190 & 122 & 92 \end{bmatrix} \\ 54 \begin{bmatrix} 101 & 216 & 132 \end{bmatrix} \end{array}$$

- b. 1. Answers may vary. Sample response: Since sales are predicted to increase by 20%, the numbers of bicycles needed will be approximately equal to the entries in  $1.2 \cdot \mathbf{U}$ . After performing this scalar multiplication, you can use nearby integers to estimate next month's sales.
2. The elements in the following matrix correspond to the numbers of bicycles that are needed to meet next month's sales.

$$1.2 \cdot \mathbf{U} \approx \begin{array}{c} \text{MT} \quad \text{TR} \quad \text{CC} \\ 43 \begin{bmatrix} 135 & 141 & 134 \end{bmatrix} \\ 48 \begin{bmatrix} 228 & 147 & 111 \end{bmatrix} \\ 54 \begin{bmatrix} 122 & 260 & 159 \end{bmatrix} \end{array}$$

**\*2.6** a. The number of mountain bikes in stock on December 1 was  $5 + 17 + 13 = 35$ .

b. The number of 48-cm bikes in stock was  $17 + 9 = 26$ .

c. The total number of bicycles in stock was  $5 + 17 + 13 + 11 + 9 + 18 = 73$ .

d. Through matrix subtraction, the number of bicycles sold during December can be represented as follows:

$$\mathbf{I}_1 - \mathbf{I}_{31} = \begin{array}{c} \text{mountain} \quad \begin{array}{cc} 43 & 48 & 54 \\ \begin{bmatrix} 2 & 6 & 8 \end{bmatrix} \\ \text{touring} \quad \begin{bmatrix} 5 & 9 & 8 \end{bmatrix} \end{array} \end{array}$$

e. The 48-cm touring bike was the best-selling bike in December.

f. Eight of the 54-cm mountain bikes sold during the month.

g. The 48-cm touring bike is the only model that needs to be reordered.

- 2.7 a. 1. Adding matrices **C** and **N** results in a matrix **T** that represents the total inventory after next week.
2. This matrix addition is shown below.

$$\mathbf{T} = \mathbf{C} + \mathbf{N} = \begin{array}{r} \text{MT} \quad \text{TR} \quad \text{CC} \\ \text{white} \left[ \begin{array}{ccc} 106 & 112 & 73 \end{array} \right] \\ \text{red} \left[ \begin{array}{ccc} 133 & 135 & 93 \end{array} \right] \\ \text{blue} \left[ \begin{array}{ccc} 82 & 137 & 40 \end{array} \right] \\ \text{green} \left[ \begin{array}{ccc} 116 & 101 & 149 \end{array} \right] \end{array}$$

- b. To determine the numbers of bicycles required, students should perform the following matrix subtraction:

$$\mathbf{S} - \mathbf{T} = \begin{array}{r} \text{MT} \quad \text{TR} \quad \text{CC} \\ \text{white} \left[ \begin{array}{ccc} 10 & 20 & 14 \end{array} \right] \\ \text{red} \left[ \begin{array}{ccc} 12 & 12 & 10 \end{array} \right] \\ \text{blue} \left[ \begin{array}{ccc} 12 & 15 & 14 \end{array} \right] \\ \text{green} \left[ \begin{array}{ccc} 14 & 16 & 9 \end{array} \right] \end{array}$$

- 2.8 a. Sample matrix:

$$\begin{array}{r} \text{A} \quad \text{B} \quad \text{C} \quad \text{D} \\ \text{W}_1 \left[ \begin{array}{cccc} 3 & 6 & 7 & 4 \end{array} \right] \\ \text{W}_2 \left[ \begin{array}{cccc} 4 & 5 & 5 & 6 \end{array} \right] \\ \text{W}_3 \left[ \begin{array}{cccc} 6 & 3 & 4 & 4 \end{array} \right] \\ \text{W}_4 \left[ \begin{array}{cccc} 5 & 5 & 3 & 6 \end{array} \right] \end{array}$$

- b. Answers may vary. Sample response: Worker 1 should knit style C because this is the maximum number of sweaters that can be knitted by any worker. Similarly, worker 2 should knit style D; worker 3 should knit style A; and worker 4 should knit style B. This arrangement allows the production of 24 sweaters per day.
- c. The maximum number of sweaters possible is 24.
- d. There is more than one possible response. Assigning worker 1 to style C; worker 2 to style B; worker 3 to style A; and worker 4 to style D also results in the production of 24 sweaters per day.

\* \* \* \* \*

### Activity 3

In this activity, students are introduced to matrix multiplication.

#### Materials List

- none

#### Technology

- matrix manipulator

#### Exploration 1

(page 16)

- a. 1. Keyes made \$860 in sales, while Zhang made \$795 in sales.
2. Sample response: To determine the value of the sales made by each person, you must multiply the number of cases of each item sold by the selling price of that item, then add these products.  
**Note:** Some students may recognize that each row in the sales matrix  $\mathbf{W}$  can be multiplied by the column matrix  $\mathbf{M}_p$ , as shown below.

$$\mathbf{W} \cdot \mathbf{M}_p = \begin{array}{c} \text{Sales} \\ \text{K} [860.00] \\ \text{Z} [795.00] \end{array}$$

- b. 1. Sample matrix:

$$\mathbf{M}_c = \begin{array}{c} \text{Com.} \\ \text{S} [2.25] \\ \text{G} [4.20] \\ \text{F} [7.20] \end{array}$$

2. The matrix below illustrates the product  $\mathbf{W} \cdot \mathbf{M}_c$ .

$$\mathbf{W} \cdot \mathbf{M}_c = \begin{array}{c} \text{Com.} \\ \text{K} [52.80] \\ \text{Z} [51.30] \end{array}$$

- c. 1. The product matrix is shown below.

$$\mathbf{W} \cdot \mathbf{M} = \begin{array}{cc} & \text{Sales} & \text{Com.} \\ \text{K} & [860.00 & 52.80] \\ \text{Z} & [795.00 & 51.30] \end{array}$$

2. The dimensions of the matrix are  $2 \times 2$ .

3. The information in the matrix represents the total value of the sales and commissions for each salesperson.
- d. 1. The product matrix is shown below.
- $$\mathbf{M} \cdot \mathbf{W} = \begin{array}{c} \text{S} \quad \text{G} \quad \text{F} \\ \text{S} [184.5 \quad 328.5 \quad 96.75] \\ \text{G} [288.4 \quad 515.2 \quad 152.6] \\ \text{F} [394.4 \quad 708.2 \quad 211.6] \end{array}$$
2. The dimensions of the matrix are  $3 \times 3$ .
3. Sample response: The information in the matrix has no meaning in this context. For example, the entry 184.5 represents the sum of the sales made by Keyes and the commission earned by Zhang on Snack Packs. This information is not of any use to the company.

### Discussion 1

(page 18)

- a. Answers will vary, depending on the methods students used in Part a. Sample response: Matrix multiplication is the better method because it is more orderly.
- b. The number of columns in matrix **A** must equal the number of rows in matrix **B**.
- c. Sample response: The operation  $\mathbf{W} \cdot \mathbf{M}$  is more efficient because it is a one-step process that includes all of the information in one place.
- d. 1. Yes. The entries in the matrix indicate the amount of commission earned by each salesperson.
2. No. The elements in the matrix are not meaningful in this context. (See response to Part d3 of Exploration 1.) **Note:** The fact that matrix multiplication is not commutative will be examined in Exploration 2. You may wish to emphasize that multiplication of matrices does not guarantee a product matrix which contains useful information. Each product must be evaluated in context.
- e. Keyes earned \$1.50 more in commissions than Zhang as shown in Part b.2 of Exploration 1.

## Exploration 2

(page 19)

- a. Students multiply matrices  $\mathbf{M}$  and  $\mathbf{W}$  using technology. They should observe that the product is the same as that obtained in Exploration 1.
- b. Sample matrix:

$$\mathbf{W}_2 = \begin{array}{c} \text{S} \quad \text{G} \quad \text{F} \\ \text{K} \begin{bmatrix} 4 & 7 & 2 \end{bmatrix} \\ \text{Z} \begin{bmatrix} 2 & 6 & 3 \end{bmatrix} \\ \text{T} \begin{bmatrix} 3 & 9 & 1 \end{bmatrix} \\ \text{L} \begin{bmatrix} 6 & 0 & 0 \end{bmatrix} \end{array}$$

- c. Students should use technology to obtain the  $4 \times 2$  product matrix below. This matrix shows the value of the sales and commissions for each salesperson.

$$\mathbf{W}_2 \cdot \mathbf{M} = \begin{array}{c} \text{Sales} \quad \text{Com.} \\ \text{K} \begin{bmatrix} 860.00 & 52.80 \end{bmatrix} \\ \text{Z} \begin{bmatrix} 795.00 & 51.30 \end{bmatrix} \\ \text{T} \begin{bmatrix} 855.00 & 51.75 \end{bmatrix} \\ \text{L} \begin{bmatrix} 270.00 & 13.50 \end{bmatrix} \end{array}$$

- d. The product does not exist because the number of columns in  $\mathbf{M}$  does not equal the number of rows in  $\mathbf{W}_2$ . When attempting to multiply  $\mathbf{M} \cdot \mathbf{W}_2$  using technology, students will receive an error message.
- e. Sample matrix:

$$\mathbf{W}_3 = \begin{array}{c} \text{N} \quad \text{J} \quad \text{M} \quad \text{S} \quad \text{G} \quad \text{F} \\ \text{K} \begin{bmatrix} 2 & 3 & 0 & 4 & 7 & 2 \end{bmatrix} \\ \text{Z} \begin{bmatrix} 3 & 1 & 6 & 2 & 6 & 3 \end{bmatrix} \\ \text{T} \begin{bmatrix} 3 & 0 & 3 & 3 & 9 & 1 \end{bmatrix} \\ \text{L} \begin{bmatrix} 2 & 4 & 1 & 6 & 0 & 0 \end{bmatrix} \end{array}$$

- f. Sample matrix:

$$\mathbf{S} = \begin{array}{c} \text{Price} \quad \text{Com.} \\ \text{N} \begin{bmatrix} 6.00 & 0.18 \end{bmatrix} \\ \text{J} \begin{bmatrix} 10.00 & 0.30 \end{bmatrix} \\ \text{M} \begin{bmatrix} 8.00 & 0.32 \end{bmatrix} \\ \text{S} \begin{bmatrix} 45.00 & 2.25 \end{bmatrix} \\ \text{G} \begin{bmatrix} 70.00 & 4.20 \end{bmatrix} \\ \text{F} \begin{bmatrix} 95.00 & 7.20 \end{bmatrix} \end{array}$$



- g. Students should use technology to obtain the product below. This matrix shows the value of the sales and commissions for each salesperson on Family Snack's entire product line.

$$\mathbf{W}_3 \cdot \mathbf{S} = \begin{array}{r} \text{Sales} \quad \text{Com.} \\ \text{K} \begin{bmatrix} 902.00 & 54.06 \end{bmatrix} \\ \text{Z} \begin{bmatrix} 871.00 & 54.06 \end{bmatrix} \\ \text{T} \begin{bmatrix} 902.00 & 53.25 \end{bmatrix} \\ \text{L} \begin{bmatrix} 330.00 & 15.38 \end{bmatrix} \end{array}$$

- h. The product does not exist because the number of columns in  $\mathbf{S}$  does not equal the number of rows in  $\mathbf{W}_3$ .

## Discussion 2

(page 20)

- a. Sample response: To multiply an  $m \times n$  matrix by a  $p \times q$  matrix, it is first necessary to make sure that  $n$  equals  $p$ . In other words, the number of columns in the first matrix must equal the number of rows in the second matrix. Once that condition has been met, multiplication of the matrices  $\mathbf{A}$  and  $\mathbf{B}$  below results in the product matrix  $\mathbf{C}$ .

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{p1} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + a_{13}b_{3p} + \cdots + a_{1n}b_{pq} \\ \vdots & \vdots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + a_{m3}b_{31} + \cdots + a_{mn}b_{p1} & \cdots & a_{m1}b_{1p} + a_{m2}b_{2p} + a_{m3}b_{3p} + \cdots + a_{mn}b_{pq} \end{bmatrix}$$

- b. Sample response: The product  $\mathbf{S} \cdot \mathbf{W}_3$  cannot be done because the number of columns in  $\mathbf{S}$  does not equal the number of rows in  $\mathbf{W}_3$ .
- c. Sample response: Matrix multiplication is not commutative because  $\mathbf{A} \cdot \mathbf{B}$  does not always equal  $\mathbf{B} \cdot \mathbf{A}$ . For example, if the dimensions of matrix  $\mathbf{A}$  are  $3 \times 4$  and the dimensions of  $\mathbf{B}$  are  $4 \times 5$ , then the product of  $\mathbf{A} \cdot \mathbf{B}$  exists, but the product of  $\mathbf{B} \cdot \mathbf{A}$  is not defined.

If the dimensions of matrix  $\mathbf{A}$  are  $3 \times 4$  and the dimensions of  $\mathbf{B}$  are  $4 \times 3$ , then both products exist, but their dimensions are different. Therefore, the two product matrices are not equal.

## Assignment

(page 20)

3.1 a.  $\mathbf{A} \cdot \mathbf{B} = [1(-1) + 2(-2) + 3(-3) + 4(-4)] = [-30]$

b.  $\mathbf{C} \cdot \mathbf{D} = \begin{bmatrix} 3 \cdot 3 + (-1) \cdot 1 + 4 \cdot 4 \\ 0 \cdot 3 + 1 \cdot 1 + (-2) \cdot 4 \end{bmatrix} = \begin{bmatrix} 24 \\ -7 \end{bmatrix}$

c. In this case, matrix multiplication is not possible. Since matrix  $\mathbf{E}$  is  $5 \times 3$  and matrix  $\mathbf{B}$  is  $4 \times 1$ , the number of columns in  $\mathbf{E}$  is not the same as the number of rows in  $\mathbf{B}$ .

3.2 a. 1. The dimensions of the product matrix are  $2 \times 2$ .

$$\mathbf{X} \cdot \mathbf{Y} = \begin{bmatrix} 25 & 23 \\ -3 & 7 \end{bmatrix}$$

2. The row dimension is the number of rows in  $\mathbf{X}$ , while the column dimension is the number of columns in  $\mathbf{Y}$ .

b. 1. The dimensions of the product matrix are  $3 \times 3$ .

$$\mathbf{Y} \cdot \mathbf{X} = \begin{bmatrix} 20 & 6 & 13 \\ 9 & 4 & 26 \\ -2 & 0 & 8 \end{bmatrix}$$

2. The row dimension is the number of rows in  $\mathbf{Y}$ , while the column dimension is the number of columns in  $\mathbf{X}$ .

c. Answers will vary. Students should mention that since the two product matrices have different dimensions and different elements,  $\mathbf{X} \cdot \mathbf{Y} \neq \mathbf{Y} \cdot \mathbf{X}$ . Therefore, matrix multiplication is not commutative.

\*3.3 a. By performing the following matrix multiplication, students should determine that 140 boxes of nuts, 420 pieces of jerky, and 108 jars of jam are needed to fill the order.

$$\mathbf{R} \cdot \mathbf{O} = \begin{bmatrix} \text{N} [140] \\ \text{J} [420] \\ \text{M} [108] \\ \text{S} [60] \\ \text{G} [18] \\ \text{F} [2] \end{bmatrix}$$

- b. By performing the matrix multiplication below, students should find that 318 boxes of nuts, 1104 pieces of jerky, and 84 jars of jam are needed to fill the order.

$$\mathbf{R} \cdot \begin{array}{l} \text{N} \begin{bmatrix} 150 \\ 600 \\ 48 \\ 60 \\ 12 \\ 0 \end{bmatrix} \\ \text{J} \begin{bmatrix} 318 \\ 1104 \\ 84 \\ 84 \\ 12 \\ 0 \end{bmatrix} \\ \text{M} \begin{bmatrix} 84 \\ 84 \\ 12 \\ 0 \end{bmatrix} \end{array} = \begin{array}{l} \text{N} \begin{bmatrix} 318 \end{bmatrix} \\ \text{J} \begin{bmatrix} 1104 \end{bmatrix} \\ \text{M} \begin{bmatrix} 84 \end{bmatrix} \\ \text{S} \begin{bmatrix} 84 \end{bmatrix} \\ \text{G} \begin{bmatrix} 12 \end{bmatrix} \\ \text{F} \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

- \*3.4 a. Sample matrix:

$$\mathbf{S} = \begin{array}{l} 1 \begin{bmatrix} 10 \end{bmatrix} \\ 2 \begin{bmatrix} 6 \end{bmatrix} \\ 3 \begin{bmatrix} 4 \end{bmatrix} \\ 4 \begin{bmatrix} 3 \end{bmatrix} \end{array}$$

- b. The columns of  $\mathbf{R}$  must match the rows of  $\mathbf{S}$  in Part a.

$$\mathbf{R} = \begin{array}{l} E_1 \begin{bmatrix} 4 & 0 & 4 & 4 \end{bmatrix} \\ E_2 \begin{bmatrix} 3 & 4 & 0 & 5 \end{bmatrix} \\ E_3 \begin{bmatrix} 2 & 5 & 5 & 0 \end{bmatrix} \\ E_4 \begin{bmatrix} 3 & 3 & 3 & 3 \end{bmatrix} \end{array}$$

- c. As shown in the product matrix below, employee 3 was first, employees 2 and 4 tied for second, and employee 1 was fourth.

$$\mathbf{R} \cdot \mathbf{S} = \begin{array}{l} \text{Pts.} \\ E_1 \begin{bmatrix} 68 \end{bmatrix} \\ E_2 \begin{bmatrix} 69 \end{bmatrix} \\ E_3 \begin{bmatrix} 70 \end{bmatrix} \\ E_4 \begin{bmatrix} 69 \end{bmatrix} \end{array}$$

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- 3.5** The matrix below is the product of the requirement matrix **R** and matrix **O**.

$$\mathbf{R} \cdot \mathbf{O} = \begin{array}{c} \phantom{\mathbf{R} \cdot \mathbf{O} =} \\ \phantom{\mathbf{R} \cdot \mathbf{O} =} \\ \phantom{\mathbf{R} \cdot \mathbf{O} =} \\ \phantom{\mathbf{R} \cdot \mathbf{O} =} \\ \phantom{\mathbf{R} \cdot \mathbf{O} =} \\ \phantom{\mathbf{R} \cdot \mathbf{O} =} \end{array} \begin{array}{ccccc} & O_1 & O_2 & O_3 & O_4 & O_5 \\ N & [ 68 & 104 & 272 & 148 & 32 ] \\ J & [ 178 & 270 & 681 & 396 & 74 ] \\ M & [ 52 & 72 & 182 & 130 & 14 ] \\ S & [ 28 & 42 & 106 & 64 & 12 ] \\ G & [ 10 & 16 & 44 & 28 & 2 ] \\ F & [ 2 & 4 & 12 & 8 & 0 ] \end{array}$$

- Family Snack needs 272 boxes of nuts to fill  $O_3$ .
  - Family Snack needs 624 boxes of nuts to fill all five orders.
  - Family Snack needs 270 pieces of jerky to fill  $O_2$ .
  - Family Snack needs 1599 pieces of jerky to fill all five orders.
  - Family Snack needs 450 jars of jam to fill all five orders.
- \*3.6**
- As shown below, the value of the sales for each model can be calculated by matrix multiplication.

$$\mathbf{J} \cdot \mathbf{P} = \begin{array}{cc} & \text{MT} & \text{TR} \\ \text{MT} & [ 3570 & 3960 ] \\ \text{TR} & [ 6000 & 6660 ] \end{array}$$

- Because it is the product of the first row in **J** and the first column in **P** (both of which represent values for mountain bikes), \$3570 represents the sales of mountain bikes. Similarly, \$6660 represents the sales from touring bikes. The other two elements are not relevant because they involve numbers representing mountain bikes multiplied by numbers representing touring bikes.
- Yes, it is possible to perform the matrix operation in Part **a** if the order of the matrices is reversed. In the product matrix below, \$1900 is the income from 43-cm bikes, \$4320 is the income from the 48-cm bikes, and \$4010 is the income from 54-cm bikes. (As in Part **b**, the only relevant elements are those along the diagonal.)

$$\mathbf{P} \cdot \mathbf{J} = \begin{array}{ccc} & 43 & 48 & 54 \\ 43 & [ 1900 & 4020 & 3490 ] \\ 48 & [ 2040 & 4320 & 3750 ] \\ 54 & [ 2180 & 4620 & 4010 ] \end{array}$$

- The total July sales is less than \$15,000. To find the total sales, students should add only the relevant elements in the product matrix:  $\$3570 + \$6660 = \$10,230$  or  $\$1900 + \$4320 + \$4010 = \$10,230$ .

- 3.7 a. The dimensions of  $\mathbf{A} \cdot \mathbf{B}$  are  $2 \times 4$ .

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 20 & 30 & -7 & 23 \\ 89 & 104 & -78 & 67 \end{bmatrix}$$

Since the column dimension of  $\mathbf{B}$  does not match the row dimension of  $\mathbf{A}$ ,  $\mathbf{B} \cdot \mathbf{A}$  is not possible.

- b. The dimensions of  $\mathbf{A} \cdot \mathbf{C}$  are  $2 \times 3$ .

$$\mathbf{A} \cdot \mathbf{C} = \begin{bmatrix} 40 & 1 & -11 \\ 131 & -14 & 20 \end{bmatrix}$$

Since the column dimension of  $\mathbf{C}$  does not match the row dimension of  $\mathbf{A}$ ,  $\mathbf{C} \cdot \mathbf{A}$  is not possible.

- c. Since the column dimension of  $\mathbf{B}$  does not match the row dimension of  $\mathbf{C}$ ,  $\mathbf{B} \cdot \mathbf{C}$  is not possible. The dimensions of  $\mathbf{C} \cdot \mathbf{B}$  are  $3 \times 4$ .

$$\mathbf{C} \cdot \mathbf{B} = \begin{bmatrix} -55 & 4 & 20 & 41 \\ 23 & -32 & 61 & -54 \\ 41 & 30 & 28 & 9 \end{bmatrix}$$

- d. The dimensions of both product matrices are  $3 \times 3$ .

$$\mathbf{C} \cdot \mathbf{D} = \begin{bmatrix} 2 & 5 & -6 \\ 9 & -4 & 0 \\ 9 & 3 & 1 \end{bmatrix} \quad \mathbf{D} \cdot \mathbf{C} = \begin{bmatrix} 2 & 5 & -6 \\ 9 & -4 & 0 \\ 9 & 3 & 1 \end{bmatrix}$$

- e. The dimensions of  $\mathbf{B} \cdot \mathbf{E}$  are  $3 \times 4$ .

$$\mathbf{B} \cdot \mathbf{E} = \begin{bmatrix} 21 & -37 & -5 & -48 \\ 14 & 112 & -1 & 114 \\ -13 & 111 & -75 & 132 \end{bmatrix}$$

Since the column dimension of  $\mathbf{E}$  does not match the row dimension of  $\mathbf{B}$ ,  $\mathbf{E} \cdot \mathbf{B}$  is not possible.

\*\*\*\*\*

- a. Multiplication of  $2 \times 2$  matrices is associative, as shown below.

$$(\mathbf{A} \cdot \mathbf{B}) = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$\begin{aligned} (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} &= \begin{bmatrix} (ae + bg)i + (af + bh)k & (ae + bg)j + (af + bh)l \\ (ce + dg)i + (cf + dh)k & (ce + dg)j + (cf + dh)l \end{bmatrix} \\ &= \begin{bmatrix} a(ei + fk) + b(gi + hk) & a(ej + fl) + b(gj + hl) \\ c(ei + fk) + d(gi + hk) & c(ej + fl) + d(gj + hl) \end{bmatrix} \end{aligned}$$

$$(\mathbf{B} \cdot \mathbf{C}) = \begin{bmatrix} ei + fk & ej + fl \\ gi + hk & gj + hl \end{bmatrix}$$

$$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = \begin{bmatrix} a(ei + fk) + b(gi + hk) & a(ej + fl) + b(gj + hl) \\ c(ei + fk) + d(gi + hk) & c(ej + fl) + d(gj + hl) \end{bmatrix}$$

Therefore,  $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$ .

- b. The distributive property of multiplication over addition is valid for  $2 \times 2$  matrices, as shown below.

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e + i & f + j \\ g + k & h + l \end{bmatrix} \\ &= \begin{bmatrix} a(e + i) + b(g + k) & a(f + j) + b(h + l) \\ c(e + i) + d(g + k) & c(f + j) + d(h + l) \end{bmatrix} \\ &= \begin{bmatrix} ae + ai + bg + bk & af + aj + bh + bl \\ ce + ci + dg + dk & cf + cj + dh + dl \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C}) &= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} + \begin{bmatrix} ai + bk & aj + bl \\ ci + dk & cj + dl \end{bmatrix} \\ &= \begin{bmatrix} ae + bg + ai + bk & af + bh + aj + bl \\ ce + dg + ci + dk & cf + dh + cj + dl \end{bmatrix} \end{aligned}$$

Therefore,  $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$ .

- c. The identity for multiplication of  $2 \times 2$  matrices is shown below.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- d.** The Zero Product Matrix for  $2 \times 2$  matrices is shown below. This matrix is unique.

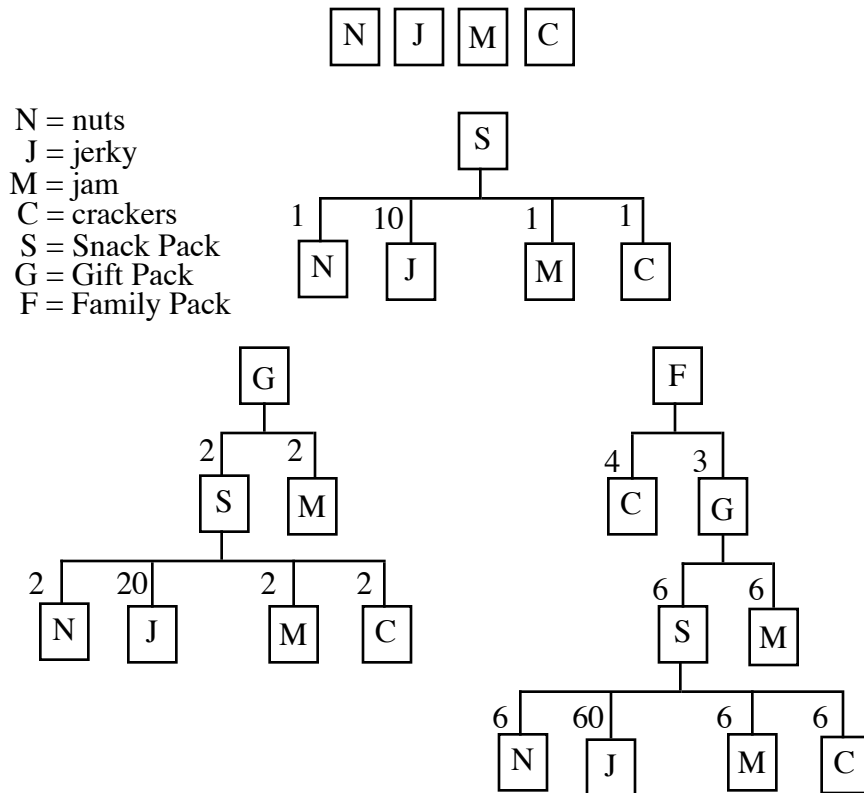
$$\mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

---

**Answers to Summary Assessment**

(page 24)

1. a. Sample requirement graph:



- b. The corresponding total requirement matrix is shown below.

$$\mathbf{R} = \begin{array}{c} \begin{array}{cccc} \text{N} & \text{J} & \text{M} & \text{C} \end{array} \\ \begin{array}{l} \text{N} \\ \text{J} \\ \text{M} \\ \text{C} \\ \text{S} \\ \text{G} \\ \text{F} \end{array} \end{array} \begin{array}{l} \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 0 & 0 & 10 & 20 & 60 \\ 0 & 0 & 1 & 0 & 1 & 4 & 12 \\ 0 & 0 & 0 & 1 & 1 & 2 & 10 \\ 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \end{array}$$

2. Answers may vary, depending on how students organize their matrices.

- a. Sample matrix:

$$\mathbf{I} = \begin{array}{c} \begin{array}{ccc} \text{one} & \text{two} & \text{three} \end{array} \\ \begin{array}{l} \text{f} \\ \text{u} \end{array} \end{array} \begin{array}{l} \left[ \begin{array}{ccc} 10 & 5 & 5 \\ 10 & 10 & 5 \end{array} \right] \end{array}$$



b. Sample matrix:

$$\mathbf{R} = \begin{array}{c} \text{one} \\ \text{two} \\ \text{three} \end{array} \begin{array}{cc} \text{f} & \text{u} \\ [375 & 345] \\ [400 & 370] \\ [450 & 420] \end{array}$$

c. The product of the matrices given in Parts **a** and **b** can be done in two ways.

$$\mathbf{R} \cdot \mathbf{I} = \begin{array}{c} \text{one} \\ \text{two} \\ \text{three} \end{array} \begin{array}{ccc} \text{one} & \text{two} & \text{three} \\ [7200 & 5325 & 3600] \\ [7700 & 5700 & 3850] \\ [8700 & 6450 & 4350] \end{array}$$

d. The elements along the main diagonal in the product matrix  $\mathbf{I} \cdot \mathbf{R}$  represent the total rents for the furnished (\$8000) and unfurnished (\$9250) apartments.

The elements along the main diagonal in the product matrix  $\mathbf{R} \cdot \mathbf{I}$  represent the total rents for the one-bedroom (\$7200), two-bedroom (\$5700) and three-bedroom apartments (\$4350).

The other elements in both matrices are irrelevant. The irrelevant elements are the result of combining values from different types of apartments. For example, the rent for a one-bedroom apartment multiplied by the number of two-bedroom apartments is meaningless.

e. Students may use scalar multiplication to determine this matrix, as shown below.

$$1.1 \cdot \mathbf{R} = \begin{array}{c} \text{one} \\ \text{two} \\ \text{three} \end{array} \begin{array}{cc} \text{f} & \text{u} \\ [412.50 & 379.50] \\ [440.00 & 407.00] \\ [495.00 & 462.00] \end{array}$$

f. Students may have to adjust their matrices so that multiplication is defined (and the product is meaningful). Sample response:

$$\mathbf{I} \cdot \begin{array}{c} \text{one} \\ \text{two} \\ \text{three} \end{array} \begin{array}{c} [80] \\ [95] \\ [105] \end{array} = \begin{array}{c} \text{f} \\ \text{u} \end{array} \begin{array}{c} [1800] \\ [2275] \end{array}$$

g. Sample response:

$$\mathbf{I} \cdot \begin{array}{c} \text{one} \\ \text{two} \\ \text{three} \end{array} \begin{array}{ccc} \text{e} & \text{s} & \text{w} \\ [80 & 15 & 18] \\ [95 & 20 & 26] \\ [105 & 30 & 42] \end{array} = \begin{array}{c} \text{f} \\ \text{u} \end{array} \begin{array}{ccc} \text{e} & \text{s} & \text{w} \\ [1800 & 400 & 520] \\ [2275 & 500 & 650] \end{array}$$

## Module Assessment

1. Find the values of  $r$  and  $s$  in each of the following equations.

a. 
$$\begin{bmatrix} 5 & r & 3 \\ -2 & 8 & s \end{bmatrix} + \begin{bmatrix} 2 & 11 & 4 \\ 6 & 7 & s \end{bmatrix} = \begin{bmatrix} 7 & 5 & 7 \\ 4 & 15 & 4 \end{bmatrix}$$

b. 
$$5 \cdot \begin{bmatrix} r & 0 \\ s & -2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -11 & -10 \end{bmatrix}$$

c. 
$$\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 23 \end{bmatrix}$$

2. Find the values of the variables in each of the following equations.

a. 
$$\begin{bmatrix} 3 & -4 & 12 \\ -4 & 6 & 7 \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -5 & 23 & 8 \\ -7 & 6 & 15 \end{bmatrix}$$

b. 
$$8 \cdot \begin{bmatrix} 9 & -8 \\ 0 & g \\ -5 & h \end{bmatrix} = \begin{bmatrix} 72 & -64 \\ 0 & 96 \\ -40 & -72 \end{bmatrix}$$

3. Matrix  $\mathbf{D}$  below shows the distances in kilometers between selected cities.

	Kansas City	Seattle	Los Angeles
$\mathbf{D} =$	Miami	1997	4400
	New York	1765	3875
	Chicago	666	2795
		2808	

- a. What does the element in row 2, column 3 represent?
- b. If you flew from New York to Los Angeles with stops in Kansas City and Chicago, how far would you travel?
- c. Create a matrix that shows the flight times between the cities in matrix  $\mathbf{D}$  if an airplane averages 570 km per hour.
- d. What would the values in the matrix  $2 \cdot \mathbf{D}$  represent?

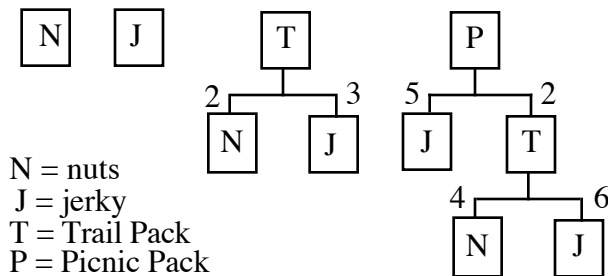
4. Describe the steps used to obtain the element in row 1, column 2 of the product matrix below.

$$\begin{bmatrix} 3 & -4 & 12 \\ -4 & 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} 9 & -8 \\ 0 & -3 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -33 & 36 \\ -71 & 42 \end{bmatrix}$$

5. A ski shop sells three different types of skis. The three types—listed in order of quality—are laminate construction, torsion-box construction, and monoblock construction. The shop’s inventory of each type is displayed in the matrix below. The column headings designate the lengths of the skis in centimeters.

$$\mathbf{I} = \begin{matrix} & 160 & 170 & 180 & 190 \\ \text{L} & [5 & 11 & 17 & 4] \\ \text{TB} & [3 & 16 & 18 & 2] \\ \text{M} & [4 & 13 & 14 & 5] \end{matrix}$$

- a. The shop pays \$1.25 per centimeter for laminate construction, \$1.60 per centimeter for torsion-box construction, and \$2.15 per centimeter for monoblock construction.
- Create a matrix that can be multiplied by matrix **I** to determine how much the store paid for its inventory.
- b. Determine the total cost of all the skis in the store.
6. Family Snack has added two new items to its original product line—the Trail Pack and the Picnic Pack. A Trail Pack consists of 2 boxes of nuts and 3 sticks of jerky, while a Picnic Pack contains 2 Trail Packs plus 5 more sticks of jerky. A total requirement graph for these four products is shown below.



- a. Create the corresponding total requirement matrix.
- b. Family Snack has received an order for 111 boxes of nuts, 55 pieces of jerky, 46 Trail Packs, and 87 Picnic Packs. Use matrix multiplication to determine the number of each simple component required to fill this order.

## Answers to Module Assessment

1.
  - a.  $r = -6; s = 2$
  - b.  $r = 8/5 = 1.6; s = -11/5 = -2.2$
  - c.  $r = 3; s = 4$
2.
  - a.  $a = -8, b = 27, c = -4, d = -3, e = 0,$  and  $f = 8$
  - b.  $g = 12; h = -9$
3.
  - a. This element represents the distance in kilometers between New York and Los Angeles.
  - b. The distance from New York to Kansas City is 1765 km; the distance from Kansas City to Chicago is 666 km; and the distance from Chicago to Los Angeles is 2808 km. The total distance traveled is 5239 km.
  - c. The elements in matrix **T** below are rounded to the nearest 0.01 hr.

$$\mathbf{T} = \frac{1}{570} \cdot \mathbf{D} = \begin{matrix} & \begin{matrix} \text{Kansas} \\ \text{City} \end{matrix} & \text{Seattle} & \begin{matrix} \text{Los} \\ \text{Angeles} \end{matrix} \\ \begin{matrix} \text{Miami} \\ \text{New York} \\ \text{Chicago} \end{matrix} & \begin{bmatrix} 3.50 & 7.72 & 6.60 \\ 3.10 & 6.80 & 6.92 \\ 1.17 & 4.90 & 4.93 \end{bmatrix} \end{matrix}$$

- d. Answers may vary. Sample response: The values in the matrix would represent round-trip distances between the cities.
4. Answers will vary. Sample response: First, multiply the number in the first row, first column of the  $2 \times 3$  matrix with the number in the first row, second column of the  $3 \times 2$  matrix as follows:  $3 \cdot -8 = -24$ .

Then multiply the number in the first row, second column of the  $2 \times 3$  matrix with the number in the second row, second column of the  $3 \times 2$  matrix:  $-4 \cdot -3 = 12$ .

Then multiply the number in the first row, third column of the  $2 \times 3$  matrix with the number in the third row, second column of the  $3 \times 2$  matrix:  $12 \cdot 4 = 48$ .

Finally, find the sum of these three products:  $-24 + 12 + 48 = 36$ .

5. a. Matrix **C** shown below is designed for multiplication on the right side of **I**. (Multiplication on the left would require the transpose of this matrix.)

$$\mathbf{C} = \begin{array}{r} \begin{array}{ccc} & \text{L} & \text{TB} & \text{M} \\ 160 & [200.00 & 256.00 & 344.00] \\ 170 & |212.50 & 272.00 & 365.50| \\ 180 & |225.00 & 288.00 & 387.00| \\ 190 & [237.50 & 304.00 & 408.50] \end{array} \end{array}$$

- b. The total cost is the sum of the elements on the main diagonal in the product matrix below, or \$32,612.50.

$$\mathbf{I} \cdot \mathbf{C} = \begin{array}{r} \begin{array}{ccc} & \text{L} & \text{TB} & \text{M} \\ \text{L} & [8112.50 & 10,384.00 & 13,953.50] \\ \text{TB} & |8525.00 & 10,912.00 & 14,663.00| \\ \text{M} & [7900.00 & 10,112.00 & 13,588.00] \end{array} \end{array}$$

6. a. Sample matrix:

$$\mathbf{R} = \begin{array}{r} \begin{array}{cccc} & \text{N} & \text{J} & \text{T} & \text{P} \\ \text{N} & [1 & 0 & 2 & 4] \\ \text{J} & |0 & 1 & 3 & 11| \\ \text{T} & |0 & 0 & 1 & 2| \\ \text{P} & [0 & 0 & 0 & 1] \end{array} \end{array}$$

- b. As shown in the product matrix below, 551 boxes of nuts and 1150 pieces of jerky are required to fill the order.

$$\mathbf{R} \cdot \begin{array}{r} \begin{array}{cc} \text{N} [111] & \text{N} [551] \\ \text{J} |55| & \text{J} |1150| \\ \text{T} |46| & \text{T} |220| \\ \text{P} [87] & \text{P} [87] \end{array} \end{array} =$$

### **Selected References**

DeLange, J. *Matrices*. Utrecht, The Netherlands: Research Group on Mathematics Education, 1990.

Harshbarger, R. J., and J. J. Reynolds. *Mathematical Applications for Management, Life, and Social Sciences*. Lexington, MA: D. C. Heath, 1992.

Vazsonyi, A. *Finite Mathematics: Quantitative Analysis for Management*. Santa Barbara, CA: Wiley/Hamilton, 1977.

# Flashbacks

## Activity 1

1.1 Use the following table for this problem.

**Attraction Costs (dollars)**

<b>City</b>	<b>Amusement Park</b>	<b>Water Park</b>	<b>Helicopter Tours</b>
San Antonio	\$50	\$15	\$125
Orlando	\$65	\$10	\$150
San Diego	\$45	\$18	\$95
Honolulu	\$70	\$20	\$100

- Which city has the least expensive helicopter tours?
- Which city would be the most expensive if you wanted to include all three activities?
- Use the row number and column number to identify the location in the table of the cost of the water park in San Antonio?
- What row number and column number identifies the location of the entry "70"?

1.2 Evaluate each of the following expressions.

- $-23 + 10$
- $-18 - 12$
- $15 - 29$
- $-16 + 20$
- $-24 - 19$

## Activity 2

- 2.1** Four circus clowns—Curley, Larry, Moe, and Fred—are selling red, green, blue, and mauve balloons. The number of balloons each clown has at the beginning of the day are shown in the matrix below.

$$\begin{array}{c} \text{R} \quad \text{G} \quad \text{B} \quad \text{M} \\ \text{Curley} \left[ \begin{array}{cccc} 5 & 3 & 2 & 12 \end{array} \right] \\ \text{Larry} \left[ \begin{array}{cccc} 7 & 5 & 4 & 6 \end{array} \right] \\ \text{Moe} \left[ \begin{array}{cccc} 9 & 7 & 8 & 4 \end{array} \right] \\ \text{Fred} \left[ \begin{array}{cccc} 11 & 9 & 3 & 4 \end{array} \right] \end{array}$$

- a. What are the dimensions of this matrix?
  - b.
    1. How many green balloons does Moe have at the beginning of the day?
    2. How many blue balloons does Larry have at the beginning of the day?
    3. How many mauve balloons does Fred have at the beginning of the day?
- 2.2** The matrix **S** shown below shows the amount of money in a savings account for the first four months of 1994.

$$\mathbf{S} = \begin{array}{c} 1994 \\ \text{Jan} \left[ \$500 \right] \\ \text{Feb} \left[ \$600 \right] \\ \text{Mar} \left[ \$750 \right] \\ \text{Apr} \left[ \$800 \right] \end{array}$$

- a. What are the dimensions of this matrix?
- b. What was the average monthly increase in the account during this time?



### Activity 3

3.1 Consider matrices **A** and **B** below.

$$\mathbf{A} = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -3 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & -2 & -2 \\ 0 & 2 & 9 \\ 0 & -3 & -1 \end{bmatrix}$$

Use these matrices to perform each of the following operations.

- $6\mathbf{A}$
- $-2\mathbf{B}$
- $\mathbf{A} + \mathbf{B}$
- $\mathbf{B} - \mathbf{A}$

3.2 Evaluate each of the following expressions.

- $2 \cdot 5 + 8 \cdot 4$
- $2 \cdot -8 + 4 \cdot 4 + 3 \cdot 2 - 4 \cdot 10$
- $12 \cdot 25 + 18 \cdot 24 + 6 \cdot 8$
- $-2 \cdot -5 + -8 \cdot -4 - 4 \cdot 1 + 5 \cdot 6$

3.3 Consider matrices **X**, **Y**, and **W** below.

$$\mathbf{X} = \begin{bmatrix} 2 & -1 & 8 \\ -2 & 4 & -3 \\ 10 & -4 & 6 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 5 & -1 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 3 & -7 \\ -5 & 0 \end{bmatrix}$$

- List the dimensions of each matrix.
- Perform each of the following calculations, if possible. If the operation is not possible, explain why not.
  - $\mathbf{X} + \mathbf{Y}$
  - $-4\mathbf{X}$
  - $5\mathbf{W}$

# Answers to Flashbacks

## Activity 1

- 1.1**
- a. San Diego has the least expensive helicopter tours.
  - b. Orlando is the most expensive city if you include all three activities.
  - c. Row 1, Column 2 is the location of the cost of the water park in San Antonio.
  - d. Row 4, Column 1 identifies the location of the entry "70".
- 1.2**
- a.  $-23 + 10 = -13$
  - b.  $-18 - 12 = -30$
  - c.  $15 - 29 = -14$
  - d.  $-16 + 20 = 4$
  - e.  $-24 - 19 = -43$

## Activity 2

- 2.1**
- a.  $4 \times 4$
  - b.
    - 1. Moe starts with 7 green balloons.
    - 2. Larry starts with 4 blue balloons.
    - 3. Fred starts with 4 mauve balloons.
- 2.2**
- a.  $4 \times 1$
  - b. The average monthly increase for the account was \$100.

### Activity 3

$$3.1 \quad \text{a. } 6\mathbf{A} = \begin{bmatrix} -12 & -12 & 36 \\ 24 & -18 & 6 \\ 0 & 24 & -6 \end{bmatrix}$$

$$\text{b. } -2\mathbf{B} = \begin{bmatrix} -8 & 4 & 4 \\ 0 & -4 & -18 \\ 0 & 6 & 2 \end{bmatrix}$$

$$\text{c. } \mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & -4 & 4 \\ 4 & -1 & 10 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\text{d. } \mathbf{B} - \mathbf{A} = \begin{bmatrix} 6 & 0 & -8 \\ -4 & 5 & 8 \\ 0 & -7 & 0 \end{bmatrix}$$

$$3.2 \quad \text{a. } 10 + 32 = 42$$

$$\text{b. } -16 + 16 + 6 - 40 = -34$$

$$\text{c. } 300 + 432 + 48 = 780$$

$$\text{d. } 10 + 32 - 4 + 30 = 68$$

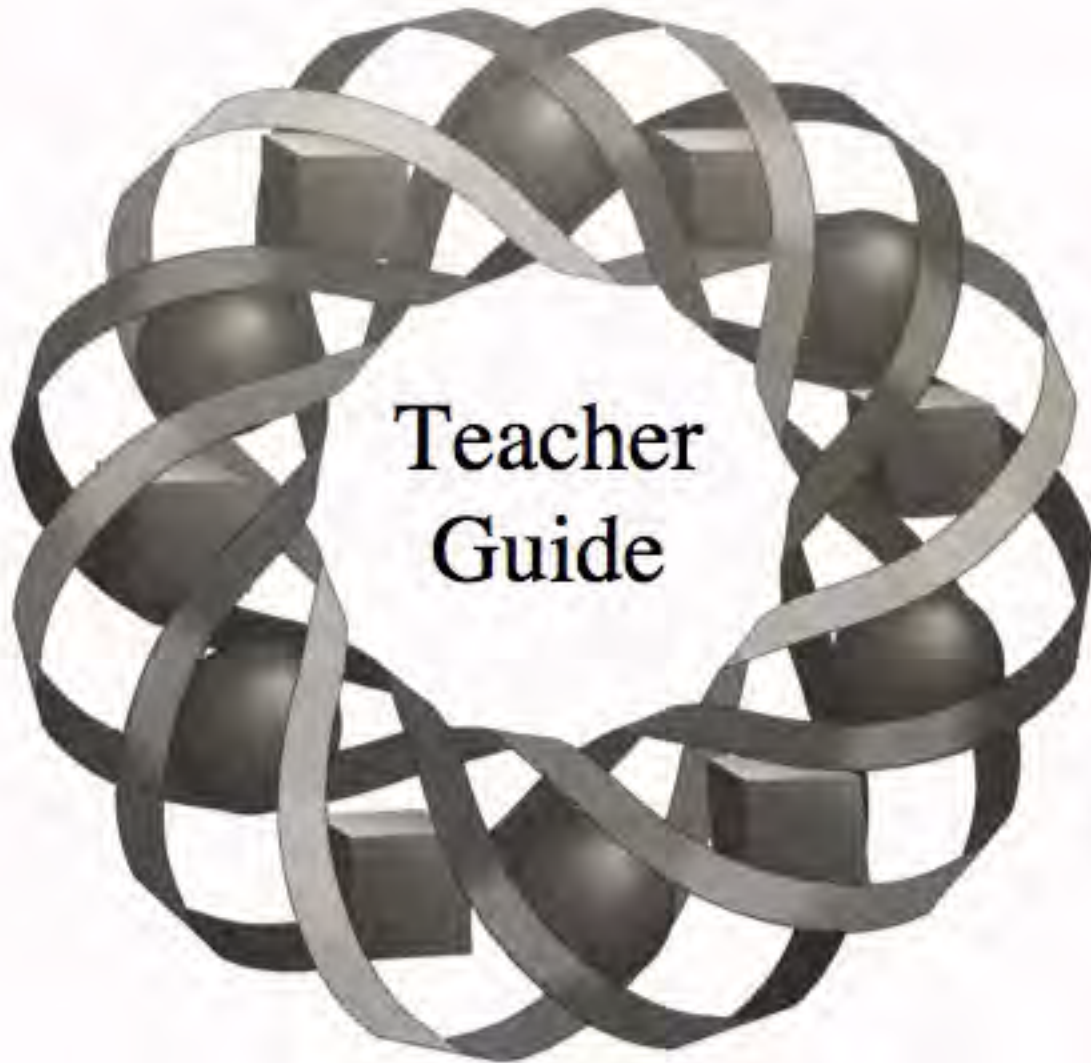
3.3 a. The dimensions of matrix  $\mathbf{X}$  are  $3 \times 3$ , of matrix  $\mathbf{Y}$  are  $3 \times 2$ , and of matrix  $\mathbf{W}$  are  $2 \times 2$ .

b. 1. This operation is not possible because the two matrices have different dimensions.

$$2. -4\mathbf{X} = \begin{bmatrix} -8 & 4 & -32 \\ 8 & -16 & 12 \\ -40 & 16 & -24 \end{bmatrix}$$

$$3. 5\mathbf{W} = \begin{bmatrix} 15 & -35 \\ -25 & 0 \end{bmatrix}$$

# A New Angle on an Old Pyramid



## Teacher Guide

What mathematical skills did the ancient Egyptians use to build the pyramids? In this module, you explore some of the theories regarding the construction of these architectural wonders—and see how modern mathematical knowledge could have made part of the job much simpler.

*Clay Burkett • Randy Carspecken*



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## Teacher Edition

# A New Angle on an Old Pyramid

### Overview

This module reviews similar triangles, the Pythagorean theorem, and triangle notation in the context of Egypt's Great Pyramid. Students explore the tangent, sine, and cosine ratios and their inverses.

### Objectives

In this module, students will:

- use similarity to determine unknown measures in triangles
- use the Pythagorean theorem and its converse to solve right-triangle problems
- use technology to develop a table of trigonometric values
- develop and apply the sine, cosine, and tangent ratios
- develop and apply the inverses of sine, cosine, and tangent.

### Prerequisites

For this module, students should know:

- how to use a protractor
- how to solve proportions
- the sum of the angles of a triangle
- the definition of similarity.

## Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	2	2	1	9

## Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
butcher paper	X				
metersticks	X				
centimeter rulers	X				
protractors	X		X	X	
triangle template	X				
puzzle template		X			
scissors		X			
string		X	X		
graph paper		X			
drinking straws			X		
paper clips			X		

## Teacher Note

Blackline masters of the templates appear at the end of the teacher edition FOR THIS MODULE.

## Technology

Software	Activity				
	1	2	3	4	Summary Assessment
geometry utility		X	X	X	
spreadsheet		X	X	X	

## Teacher Note

In Activities 3 and 4, students should set the technology to use degree measure (not radian measure).

# A New Angle on an Old Pyramid

## Introduction

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The construction of Egypt's Great Pyramid provides the context for most of the activities in this module.

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## *Activity 1*

In this activity, students review similar triangles and proportional reasoning.

## Materials List

- butcher paper (one sheet per group; approximately 1 m by 2 m)
- 30-cm rulers (one per group)
- protractors (one per group)
- metersticks (one per group)
- triangle template (one per student; a blackline master appears at the end of the teacher edition for this module)

## Exploration

(page 32)

- Students use protractors and rulers to construct a  $90^\circ$  corner.
- To illustrate the potential range of error, students mark their locations for point  $P$  on a common sheet of paper, as shown in Figure 3.
- The distance between  $X$  and  $Y$  may be as large as several centimeters.
- Students should predict that the distance between  $R$  and  $S$  will be 100 times the distance between  $X$  and  $Y$ .

## Discussion

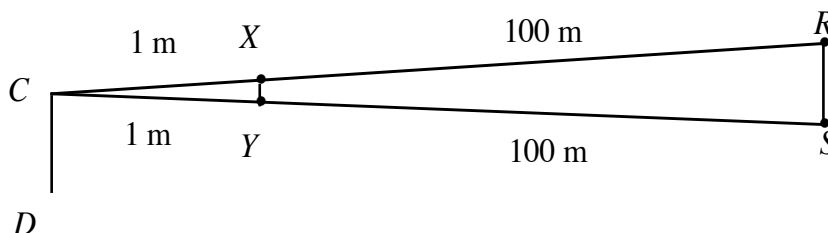
(page 33)

- Students should discuss the accuracy of their measurements, tools, and methods.
- Since  $\overline{CY}$  and  $\overline{CX}$  are congruent,  $\triangle CXY$  is an isosceles triangle. Because  $\triangle CXY$  is isosceles,  $\angle CXY$  and  $\angle CYX$  are congruent.

- c. It is important to emphasize that this notation indicates corresponding sides and angles. For example, the statement  $\triangle ABC \sim \triangle EDF$  indicates that  $\angle B$  and  $\angle D$  are corresponding angles and must be congruent, among other things. For the similar triangles in Figure 4, this is incorrect.

**Note:** For the triangles in Figure 4, there is only one correspondence between the angles and sides that results in a similarity relation. For some triangles, however, there can be several different similarity relations. If  $\triangle ABC$  and  $\triangle DEF$  are equilateral triangles, for example, then there are six different ways that angles and sides correspond:  $\triangle ABC \sim \triangle DEF$ ,  $\triangle ABC \sim \triangle EDF$ ,  $\triangle ABC \sim \triangle FED$ ,  $\triangle ABC \sim \triangle DFE$ ,  $\triangle ABC \sim \triangle EFD$ , and  $\triangle ABC \sim \triangle FDE$ .

- d. Students should recognize that  $\triangle CXY$  and  $\triangle CRS$  are isosceles triangles with the same vertex angle, as shown below. Therefore,  $\triangle CXY \sim \triangle CRS$ .



- e. Since  $\triangle CXY \sim \triangle CRS$ , the corresponding sides are proportional. The distance  $RS$  can be found using the proportion below.

$$\frac{XY}{RS} = \frac{CX}{CR} = \frac{1}{100}$$

### Teacher Note

Each student will require a copy of the triangle template to complete Problem 1.2.

### Assignment

(page 34)

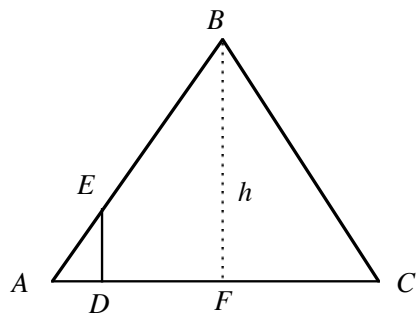
- 1.1 Answers will vary, depending on the class value for  $XY$ . For example, if  $XY = 5$  cm, then by similarity:

$$\frac{1}{0.05} = \frac{230}{x}$$

$$x = 11.5 \text{ m}$$



- 1.2** In this problem, students investigate the properties of isosceles triangles.
- a–b.** The line of symmetry is perpendicular to the base and bisects the base.
  - c.** The line of symmetry contains the altitude of the triangle.
  - d.** When the triangle is folded along the line of symmetry, a right triangle is formed.
  - e.** The line of symmetry bisects the vertex angle.
- 1.3** Sample response: Since the base of each pyramid is a square, the bases are similar. Each face is an isosceles triangle. Since the sum of the angles in a triangle is  $180^\circ$ , then  $m\angle ACB = 180 - (m\angle CAB + m\angle CBA)$  and  $m\angle DFE = 180 - (m\angle FDE + m\angle FED)$ . Therefore,  $\angle ACB \cong \angle DFE$ . By the AAA Property,  $\triangle ACB \sim \triangle DFE$ . Because both the bases and the faces are similar, the pyramids are similar.
- 1.4** Sample response: The least amount of information needed is that two pairs of corresponding angles are congruent. This is because the sum of the measures of the angles in a triangle is  $180^\circ$ . Therefore, if two pairs of angles are congruent, then the third pair must also be congruent and the AAA Property would apply.
- 1.5**
- a.** Since the two right triangles with heights  $h$  and  $k$  also share another angle, then they have two pairs of congruent, corresponding angles. Therefore, they are similar triangles. Because they are similar, the ratios of corresponding sides are equal.
  - b.** If an acute angle in one right triangle is congruent to an acute angle in another right triangle, then the two triangles are similar by the AAA Property.
- \*1.6** Sample response: In the following diagram,  $\triangle ADE \sim \triangle AFB$  by the AAA Property.



Because the triangles are similar, the ratios of corresponding sides are proportional:

$$\frac{AD}{AF} = \frac{ED}{h}$$

$$\frac{5/2}{230} = \frac{3}{h}$$

$$h = 276 \text{ cubits}$$

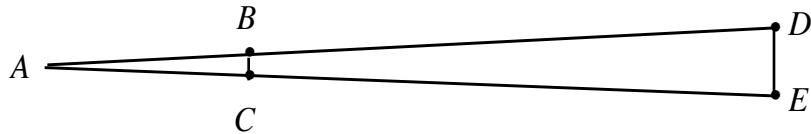
- 1.7** Since  $\triangle ABC \sim \triangle IJH$ , the ratios of corresponding sides are proportional:

$$\frac{AC}{IH} = \frac{AB}{IJ} = \frac{BC}{JH}$$

$$\frac{20}{10} = \frac{16}{x} = \frac{y}{7}$$

Solving these proportions gives  $x = 8$  and  $y = 14$ .

- 1.8** In the diagram below (not drawn to scale),  $AB = AC = 15$  cm and  $BC = 1$  mm.



After 3 m of tile are installed,  $AD = AE = 3$  m and  $\triangle ADE$  is similar to  $\triangle ABC$ . Therefore,

$$\frac{DE}{BC} = \frac{AD}{AB}$$

$$\frac{x}{0.001} = \frac{3}{0.15}$$

$$x = 0.02 \text{ m or } 2 \text{ cm}$$

\*\*\*\*\*

- 1.9**
- Sample response: The two triangles are similar because they share a common angle,  $\angle CAB$ , and they both have a right angle.
  - Since the lengths of the corresponding sides of the similar triangles are proportional,

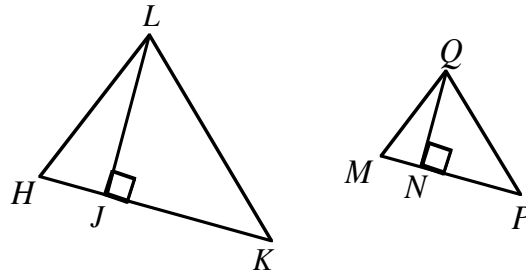
$$\frac{CB}{AB} = \frac{DE}{AE}$$

Solving the proportion for  $DE$  gives the height of the flagpole.

- 1.10** Sample response: The object should be 168 cm away from the front of the camera. Since the pyramids are similar, corresponding lengths are proportional. Solving for  $d$  (distance the camera is from the object) in the following proportion results in the distance being 168 cm.

$$\frac{20}{5} = \frac{d}{42}$$

- 1.11** Yes, if two triangles are similar, then their altitudes are proportional. In the diagram below, for example,  $\triangle LHK \sim \triangle QMP$ . Since  $\angle LKJ \cong \angle QPN$  and  $\angle LJK \cong \angle QNP$ , then  $\triangle LJK \sim \triangle QNP$ . Since these triangles are similar, their corresponding sides  $\overline{LJ}$  and  $\overline{QN}$  are proportional. These segments are also the altitudes of  $\triangle LHK$  and  $\triangle QMP$ .



\*\*\*\*\*

(page 39)

## Activity 2

In this activity, students review the Pythagorean theorem, then develop its converse using a geometry utility.

### Materials List

- scissors (one pair per group)
- 30–40 cm lengths of string (one per student)
- puzzle template (one per student; a blackline master appears at the end of the teacher edition for this module)
- graph paper

### Technology

- geometry utility
- spreadsheet

### Exploration

(page 40)

- a.
  1. The converse of the Pythagorean theorem may be stated as follows: “If the square of the length of the longest side equals the sum of the squares of the lengths of the other sides, then the triangle is a right triangle.”
  2. Students may or may not believe that the converse is true.

- b. By keeping point  $C$  within the given constraints, all obtuse or right triangles generated will have  $\overline{AB}$  as the longest side. Sample data:

$m\angle C$	$AB$	$AC$	$BC$	$(AB)^2 - [(AC)^2 + (BC)^2]$
$62.2^\circ$	1.59	1.38	1.66	-2.14
$135^\circ$	1.59	0.55	1.16	0.89
$90^\circ$	1.59	0.95	1.27	0

- c.
1. obtuse
  2. acute
  3. right

## Discussion

(page 41)

- a. The expression  $(AB)^2 - [(AC)^2 + (BC)^2]$  represents the difference between the square of the side opposite  $\angle C$  and the sum of the squares of the other two sides. This expression equals 0 when  $m\angle C = 90^\circ$ . At this point,  $\triangle ABC$  is a right triangle, where  $\overline{AB}$  is the hypotenuse and  $\overline{AC}$  and  $\overline{BC}$  are the legs. In this case, the Pythagorean theorem is restated as  $c^2 - (a^2 + b^2) = 0$ , where  $c$  is the length of the hypotenuse and  $b$  and  $a$  are the lengths of the legs.
- b. Sample response: In obtuse triangles, the square of the length of the side opposite the obtuse angle is greater than the sum of the squares of the lengths of the other two sides. This is shown by a positive value in the right-hand column of Table 1.
- c. Sample response: In acute triangles, the square of the length of any side is less than the sum of the squares of the lengths of the other two sides. This is shown by a negative value in the right-hand column of Table 1.
- d. From the data in Table 1, students should observe that the converse of the Pythagorean theorem appears to be true. **Note:** It should be pointed out that the exploration does not constitute a proof.
- e. Sample response: Measure the length  $a$  and the width  $b$  of the corner and consider these the legs of a potential right triangle. Then measure the potential hypotenuse  $c$  of the triangle. If these lengths satisfy the equation  $a^2 + b^2 = c^2$ , then the triangle is a right triangle and the angle is a right angle.

## Teacher Note

Students require string to complete Problem 2.2. The puzzle template is needed for Problem 2.3. Two different puzzles are included at the end of this teacher edition. In Problem 2.4, students explore the triangle inequality property.

## Assignment

(page 42)

- 2.1 a. Students should verify that, when used as the lengths of the sides of a triangle, each triple in the table makes the Pythagorean equation true. Since the converse of the Pythagorean theorem is true, these triangles are all right triangles. For example, using the triple in the first row:

$$\begin{aligned}120^2 + 119^2 & \stackrel{?}{=} 169^2 \\ 28,561 & = 28,561\end{aligned}$$

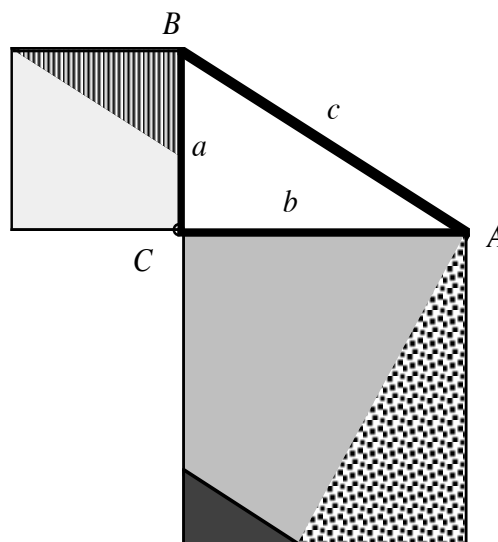
- b. Any multiple of a Pythagorean triple is also a Pythagorean triple. For example, suppose  $a$ ,  $b$ , and  $c$  are whole numbers and  $a^2 + b^2 = c^2$ . Then  $2a$ ,  $2b$ , and  $2c$  are whole numbers and:

$$\begin{aligned}(2a)^2 + (2b)^2 & = 4(a^2 + b^2) \\ & = 4(c^2) \\ & = (2c)^2\end{aligned}$$

In general, if  $a^2 + b^2 = c^2$ , then  $(ma)^2 + (mb)^2 = m^2(a^2 + b^2) = m^2(c^2)$ .

- 2.2 By using the string to form a 3-4-5 triangle, students can create one right angle for the square. The other right angles may be formed in a similar manner. Students can then verify that their figures are squares by calculating the lengths of the diagonals. The diagonals of a square are congruent, perpendicular, and bisect each other. If the side length of a square is 3 units, the diagonal length is approximately 4.2 units.

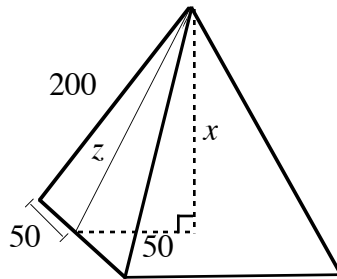
- 2.3 Sample arrangement:



**Note:** Although this puzzle provides a graphic illustration of the Pythagorean relationship as a statement about areas, any demonstration of the Pythagorean theorem by the puzzle method

should be treated with caution. Many math puzzles ask students to cut up a rectangle with an area of, for example,  $80 \text{ cm}^2$ , then rearrange the pieces to obtain a rectangle with an area of  $81 \text{ cm}^2$ . You may wish to emphasize that a visually convincing demonstration does not constitute a proof.

- 2.4**
- Answers will vary. Sample response: (2, 3, 6); (1, 4, 5); (9, 10, 25).
  - Sample response: The two shorter sides cannot meet to form the third vertex of the triangle.
  - Any three segments whose lengths satisfy the inequality  $a + b \leq c$  will not form a triangle.
- 2.5**
- Using the Pythagorean theorem,  $20^2 = x^2 + 10^2$ . Solving for  $x$ ,  $x = \sqrt{300} \approx 17.3$ .
  - Using the Pythagorean theorem,  $x^2 = 12^2 + 13^2$ . Solving for  $x$ ,  $x = \sqrt{313} \approx 17.7$ .
  - Since this is a square pyramid, the altitude intersects the center of the base. The length of one leg of the right triangle is therefore 50 m. Using the Pythagorean theorem,  $150^2 = x^2 + 50^2$ . Solving for  $x$ ,  $x = \sqrt{20,000} \approx 141.4$ .
  - From the diagram below, the height of the triangular face can be found as follows:  $z^2 + 50^2 = 200^2$ ;  $z = \sqrt{37,500}$ .

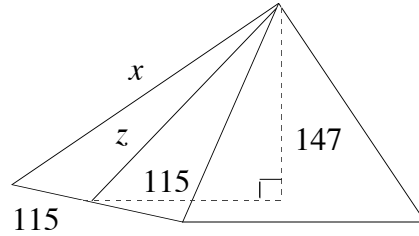


Substituting for  $z^2$  in the equation below:

$$x^2 + 50^2 = z^2$$

$$x = \sqrt{35,000} \approx 187 \text{ m}$$

- e. From the diagram below, the height of the triangular face can be found as follows:  $z^2 = 147^2 + 115^2$ ;  $z = \sqrt{34,834}$ .



Substituting for  $z^2$  in the equation below:

$$x^2 = z^2 + 115^2$$

$$x = \sqrt{48,059} \approx 219 \text{ m}$$

- 2.6**
- Since  $13^2 = 5^2 + 12^2$ , these lengths form a right triangle.
  - Since  $12^2 > 6^2 + 8^2$ , these lengths form an obtuse triangle.
  - Since  $7 + 10 < 19$ , no triangle is possible. (See Problem **2.4**.)
  - Since  $13^2 + 8^2 > 15^2$ , these lengths form an acute triangle.
- 2.7**
- $AC = 2$ ;  $BC = 3$ . Since  $2^2 + 3^2 = 13$ ,  $AB = \sqrt{13}$ .
  - Since  $5^2 + 4^2 = 41$ ,  $DE = \sqrt{41}$ .
  - Since  $(4 - 1)^2 + (-1 - 3)^2 = 25$ , the length of the segment is 5.
  - The distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is:

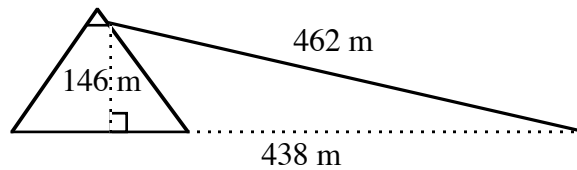
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- \*2.8** Since the Great Pyramid is 147 m tall, the ramp would have had to rise  $(147 - 1)$  or 146 m above the ground. Therefore, the “run” of the ramp can be found as follows:

$$\frac{146}{x} = \frac{1}{3}$$

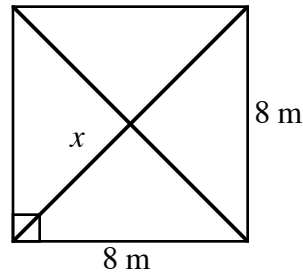
$$x = 3(146) = 438 \text{ m}$$

Using the Pythagorean theorem, the length of the ramp is  $\sqrt{438^2 + 146^2}$ , or approximately 462 m.



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- 2.9 Sample response: If the corners of the foundation are right angles, the lengths of the diagonals can be determined using the Pythagorean theorem.

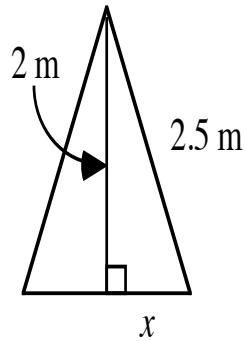


Since the sides are 8 m on each side,

$$8^2 + 8^2 = x^2$$

$$x = \sqrt{128} \approx 11.3 \text{ m}$$

- 2.10 a. The triangles formed are isosceles.  
 b. Since each triangle is isosceles, the altitude is perpendicular to the base and also bisects the base.



Using the Pythagorean theorem,

$$2^2 + x^2 = 2.5^2$$

$$x = 1.5 \text{ m}$$

Since  $x$  represents half the length of the base of the isosceles triangle, the length of the base is 3 m.

Therefore, the perimeter of the octagon is  $3 \cdot 8 = 24$  m. Adding the lengths of the diagonals, the total length of the wood necessary for the floor supports is  $24 + (4 \cdot 5) = 44$  m .

\* \* \* \* \*



### Activity 3

Students are introduced to the tangent (the first of the trigonometric ratios examined in this module) by investigating triangular ratios found in the casing blocks of the Great Pyramid. These casing blocks were cut with great accuracy using a constant slope. Although the ancient Egyptians might not have used trigonometry, they were aware of the ratios of a casing block's length to its height. This ratio essentially represents our concept of the cotangent of an angle.

#### Materials List

**Note:** The materials listed below are required for the research project.

- protractors (one per student)
- drinking straws (one per student)
- string
- paper clips (one per student)

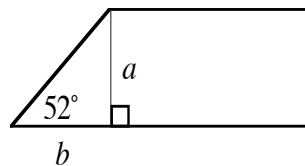
#### Technology

- geometry utility (students should set their geometry utility to use degree measure, not radian measure)
- spreadsheet

#### Teacher Note

You may wish to emphasize that the measure of a dihedral angle is the measure of the angle whose sides are the two rays formed by the intersections of the faces and a plane perpendicular to the edge (as shown in Figure 11).

Students may create paper representations of the cross sections of casing blocks, using various values of  $a$  from 10 to 30 cm.



These models can then be used to illustrate the smooth, continuous slope that casing blocks gave to the sides of a pyramid.

## Exploration

(page 46)

- a. Students construct a right triangle with an acute angle measure of  $52^\circ$ .
- b. For the angle measure of  $52^\circ$ , the ratio is approximately 1.28.
- c.
  - 1. The ratio remains constant for all these right triangles.
  - 2. Since all these triangles have congruent angle measures of  $90^\circ$ ,  $52^\circ$ , and  $38^\circ$ , the triangles are similar by the AAA Property.
- d–e. Students should realize that each acute angle measure is associated with a unique ratio. Sample table:

$m\angle A$	opposite/adjacent	$\tan\angle A$
$52^\circ$	1.28	1.28
$15^\circ$	0.268	0.268
$30^\circ$	0.577	0.577
$45^\circ$	1.00	1.00
$60^\circ$	1.73	1.73
$75^\circ$	3.73	3.73

- f. Sample response: The inverse tangent of each value from Part e is the measure of the corresponding angle in Table 2.

## Discussion

(page 50)

- a. Since the triangles generated are all similar to the first triangle, the corresponding ratios are equal.
- b. Students should realize that since they all started with the same angle measure, the ratio of opposite to adjacent (the tangent) is the same, regardless of the right triangle used.
- c. Sample response: As the measure of the angle gets larger, the tangent value gets larger.
- d. The tangent of an acute angle measure in a right triangle is the ratio of the opposite to adjacent, while the seqt is the ratio of adjacent to opposite. The tangent is the multiplicative inverse of the seqt.
- e.
  - 1. Sample response: I would find the value in the right-hand column of the table. The measure of the angle is the value in the left-hand column of the same row
  - 2. Sample response: The inverse tangent commands reports the angle measure which corresponds to a particular tangent value.
- f. Sample response: When the measure of an acute angle in a right triangle is  $45^\circ$ , the lengths of the two legs are equal. Since the lengths of the legs are equal, the ratio of opposite to adjacent is 1.

## Assignment

(page 51)

- \*3.1** a. Sample response: The right triangle in the casing block and the right triangle in the Great Pyramid both have angles that measure  $52^\circ$  and  $90^\circ$ . Therefore, the remaining angles in the two triangles also have the same measure. By the AAA Property, the triangles are similar.

- b. 1. Using proportions,

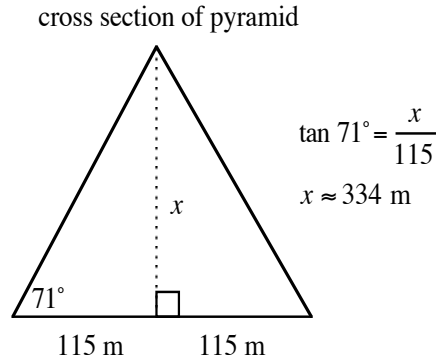
$$\frac{x}{60} = \frac{115}{147}$$
$$x \approx 47 \text{ cm}$$

2. Using the tangent ratio,

$$\tan 52^\circ = \frac{60}{x}$$
$$x \approx 47 \text{ cm}$$

- 3.2** a.  $x/50 = \tan 52^\circ$ ;  $x \approx 64$  cm  
b.  $120/x = \tan 52^\circ$ ;  $x \approx 94$  cm  
c.  $x^2 = 100^2 + 128^2$ ;  $x = \sqrt{26,384} \approx 162$  cm

- \*3.3** Sample response:



- 3.4** In their responses, students should describe the connections among  $\tan 90^\circ$ , dividing by 0, and the results displayed on their calculators.

- 3.5** a. Using the tangent ratio,

$$\tan 50^\circ = \frac{x}{94.5}$$
$$x \approx 113 \text{ m}$$

- b. Considering the right triangle formed by the height at which the  $50^\circ$  angle was abandoned:

$$\tan 50^\circ = \frac{73.5}{x}$$

$$x \approx 61.7 \text{ m}$$

The width of the pyramid where the angle changes is therefore about  $189 - 2(61.7)$  or 65.6 m.

The height of the upper part of the pyramid is:

$$\tan 37^\circ = \frac{x}{32.8}$$

$$x \approx 24.7 \text{ m}$$

The total height of Bent Pyramid is  $73.5 + 24.7 = 98.2 \text{ m}$ .

3.6 a.  $x = 10/\tan 30^\circ \approx 17 \text{ cm}$

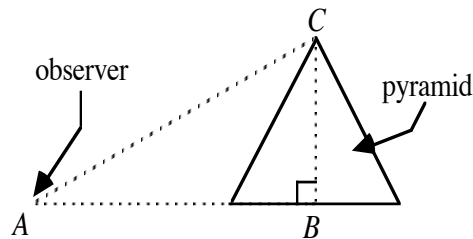
b.  $x = \tan^{-1}(20/20) = 45^\circ$

c.  $y = 125(\tan 70^\circ) \approx 343 \text{ m}$

d.  $y = \tan^{-1}(150/450) \approx 18^\circ$ ;  $x = \tan^{-1}(450/150) \approx 72^\circ$

- 3.7 Since the tangent of the dihedral angle is  $b/a$ , the Egyptians' ratio can be expressed as  $1/\tan \theta$ , where  $\theta$  is the measure of the dihedral angle.

- 3.8 Sample response: In the diagram below, the clinometer measures the angle of elevation at point A, where the observer is standing. The length of segment AB, the distance from the observer to the center of the pyramid, could be measured. The tangent ratio can then be used to calculate the length of segment BC—which is the height of the pyramid.



\*\*\*\*\*

3.9 a.  $x = \tan^{-1}(75/183) \approx 22^\circ$  and  $y = \tan^{-1}(183/75) \approx 68^\circ$

b.  $x = \frac{28 - 18}{2} \cdot \tan 63^\circ \approx 9.8 \text{ cm}$

3.10  $x = \tan^{-1}(2/5) \approx 22^\circ$

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## Research Project

(page 54)

Students may build clinometers like the one shown in Problem 3.8 or develop their own designs. Their responses should include a discussion of possible sources of error.

(page 54)

### Activity 4

In this activity, students explore two other right-triangle ratios that are constant for a given acute angle: sine and cosine.

#### Materials List

- protractors (one per student)

#### Technology

- geometry utility
- spreadsheet

#### Exploration

(page 54)

- a–c.** Answers will vary. Students list all ratios that they believe will remain constant for a given acute angle. The six possible ratios are  $a/b$ ,  $a/c$ ,  $b/a$ ,  $b/c$ ,  $c/a$ , and  $c/b$ .
- d–e.** To demonstrate that the ratios do not depend on the triangle used, you may wish to ask students to compare values for Table 3. Sample table:

$m\angle A$	opposite/hypotenuse	adjacent/hypotenuse	$\sin\angle A$	$\cos\angle A$
$5^\circ$	0.087	0.996	0.087	0.996
$15^\circ$	0.259	0.966	0.259	0.966
$30^\circ$	0.5	0.866	0.5	0.866
$45^\circ$	0.707	0.707	0.707	0.707
$60^\circ$	0.866	0.5	0.866	0.5
$75^\circ$	0.66	0.259	0.966	0.259

- f.** Students should determine that the minimum and maximum values of  $\sin\angle A$  and  $\cos\angle A$  are 0 and 1, respectively.
- g.** Students should observe that the appropriate inverse of each value from Part e is the measure of the corresponding angle in Table 3.

## Discussion

(page 56)

- a. In a right triangle,  $\sin \angle A = \text{opposite} / \text{hypotenuse}$  and  $\cos \angle A = \text{adjacent} / \text{hypotenuse}$ .
- b. Answers will vary. Some students will observe that, as  $m\angle A$  increases,  $\sin \angle A$  increases from 0 to 1 and  $\cos \angle A$  decreases from 1 to 0.
- c. When  $m\angle A = 45^\circ$ , both the sine and cosine of  $\angle A$  are approximately 0.707.
- d.
  1. Sample response: You would need to know the length of the side of the right triangle opposite the angle and the length of the hypotenuse.
  2. Sample response: You would need to know the length of the side of the right triangle adjacent to the angle and the length of the hypotenuse.
- e. Students should recognize that the side adjacent to  $\angle A$  is opposite  $\angle B$ , and vice versa.

$$\sin \angle A = \frac{BC}{AB} = \cos \angle B$$

## Assignment

(page 57)

- 4.1
  - a. cosine
  - b. sine
  - c. tangent
  - d. cosine
- \*4.2
  - a.  $h = 160 \cdot \sin 44^\circ \approx 111$  m ;  $l = 2(160 \cdot \cos 44^\circ) \approx 230$  m
  - b.  $x = 120 / \sin 52^\circ \approx 152$  cm
  - c. The length of the ramp is  $100 / \sin 25^\circ \approx 237$  m .
- 4.3  $x = \cos^{-1}(75/100) \approx 41^\circ$
- 4.4 Sample response:

$$\begin{aligned} \frac{\sin \angle A}{\cos \angle A} &= \frac{\text{opposite} / \text{hypotenuse}}{\text{adjacent} / \text{hypotenuse}} \\ &= \frac{\text{opposite}}{\text{hypotenuse}} \cdot \frac{\text{hypotenuse}}{\text{adjacent}} \\ &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \tan \angle A \end{aligned}$$

**4.5**  $x \approx 30$ ;  $y \approx 31$ ;  $z \approx 53^\circ$

$$\tan 37^\circ = \frac{x}{50} \qquad \cos 50^\circ = \frac{20}{y} \qquad \cos z^\circ = \frac{3}{5}$$

$$x \approx 30 \qquad y \approx 31 \qquad z \approx 253^\circ$$

**4.6 a.** By the definitions of sine and cosine,  $\sin 65^\circ$  and  $\cos 25^\circ$  in triangle  $ABC$  are both  $a/c$ , while  $\tan 65^\circ = a/b$  and  $\tan 25^\circ = b/a$ .

**b.** If  $\angle A$  and  $\angle B$  are the acute angles of a right triangle, then  $\sin \angle A = \cos \angle B$  and  $\tan \angle A = 1/\tan \angle B$ .

In other words,  $\sin \angle A = \cos(90^\circ - \angle A)$ ,  $\cos \angle A = \sin(90^\circ - \angle A)$ , and  $\tan \angle A = 1/\tan(90^\circ - \angle A)$ .

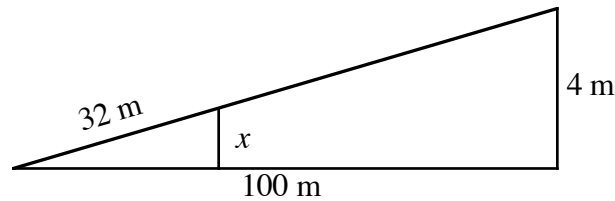
**4.7** Using the Pythagorean theorem, the side of the triangular face is approximately 187 m long. Designating one of the base angles as  $\angle A$ ,

$$\tan \angle A \approx \frac{187}{115}$$

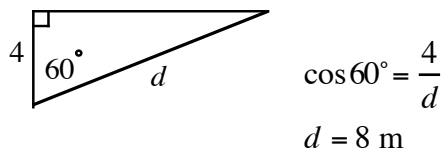
$$m\angle A \approx 58^\circ$$

**\*4.8 a.**  $\tan^{-1}(4/100) \approx 2.3^\circ$

**b.** The change in elevation is  $32 \cdot \sin(2.3^\circ) \approx 1.3$  m. Sample diagram (not drawn to scale):



**4.9** Using the diagram, the length ( $d$ ) of the crosspiece can be calculated by solving for  $d$  in



**4.10**  $\cos^{-1}(4/4.5) \approx 27^\circ$

**4.11 a.** The total length of cable is  $4(20/\sin 60^\circ) \approx 92$  m.

**b.** The distance from the bottom of the tower to each anchor point is:

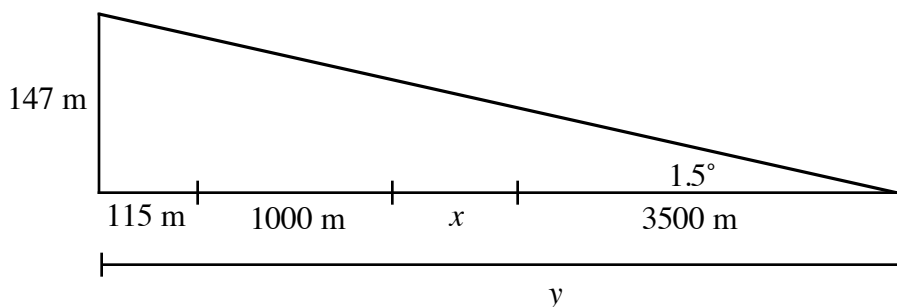
$$\frac{20}{\tan 60^\circ} \approx 11.5 \text{ m}$$

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## Answers to Summary Assessment

(page 61)

1. In the following sample diagram (not drawn to scale),  $x$  represents the width of the Nile and  $y$  represents the distance from the observer to the center of the pyramid's base.



Since  $\tan 1.5^\circ = 147/y$ ,  $y \approx 5614$  m. The width of the Nile, therefore, is about 999 m.

2. Since  $F$  is the center of the square base,  $DF$  is 115 m. Given that  $DE$  is 77 m, the Pythagorean theorem can be used to find the length of  $FE$  (the hypotenuse of right triangle  $FDE$ ):

$$115^2 + 77^2 = (FE)^2$$

$$FE \approx 138 \text{ m}$$

In right triangle  $CFE$ ,  $m\angle CEF$  can be found using the inverse tangent:

$$m\angle CEF = \tan^{-1}\left(\frac{147}{138}\right)$$

$$\approx 47^\circ$$

Since the measure of the dihedral angle is  $52^\circ$ , the angles have different measures. **Note:** This problem demonstrates the need for a specific definition of the measure of a dihedral angle (see Activity 3).

3. Sample response: Using the inverse tangent,  $m\angle B = \tan^{-1}(6/10) \approx 31^\circ$ . Since  $\angle C$  is a right angle,  $m\angle A \approx 59^\circ$ .

In right triangle  $BEF$ ,

$$\sin 31^\circ = \frac{EF}{5}$$

$$EF \approx 2.6 \text{ m}$$

In right triangle  $ACD$ ,

$$\sin 59^\circ = \frac{CD}{6}$$

$$CD \approx 5.1 \text{ m}$$



Using the Pythagorean theorem,  $AB$  can be found as follows:

$$6^2 + 10^2 = (AB)^2$$
$$(AB) \approx 11.7 \text{ m}$$

Again using the Pythagorean theorem,

$$(AD)^2 + 5.1^2 = 6^2$$
$$(AD) \approx 3.1 \text{ m}$$

and

$$(BE)^2 + 2.5^2 = 5^2$$
$$(BE) \approx 4.3 \text{ m}$$

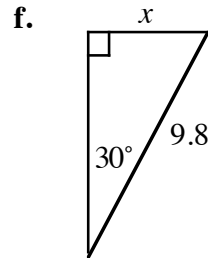
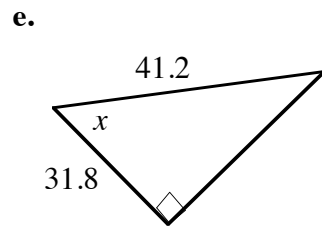
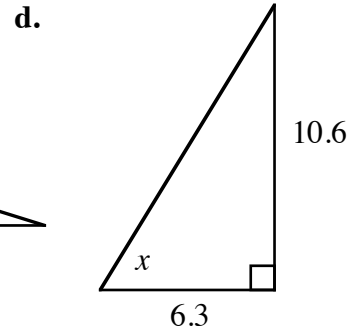
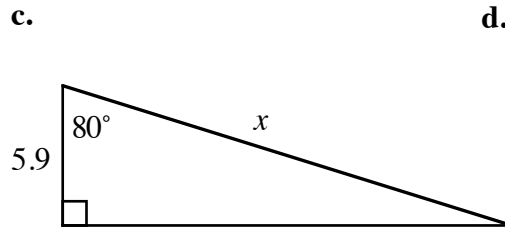
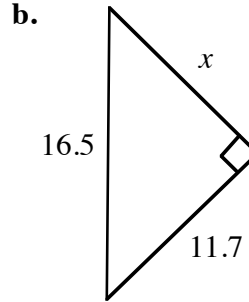
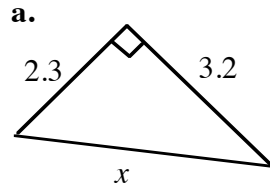
Therefore,  $DE$  is  $11.7 - 3.1 - 4.3 = 4.3 \text{ m}$ .

Since  $DE$  and  $EF$  are now known and  $DEF$  is a right triangle,  $DF$  can be found as follows:

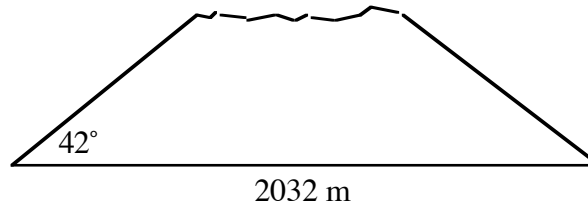
$$4.3^2 + 2.6^2 = DF^2$$
$$DF \approx 5 \text{ m}$$

## Module Assessment

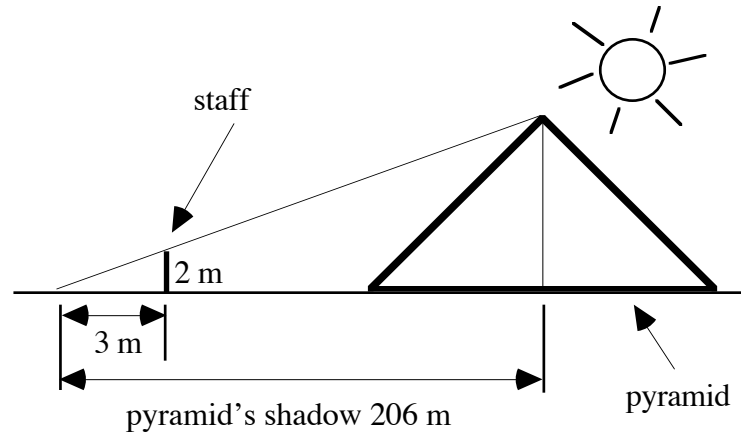
1. Determine the value of  $x$  in each of the following triangles.



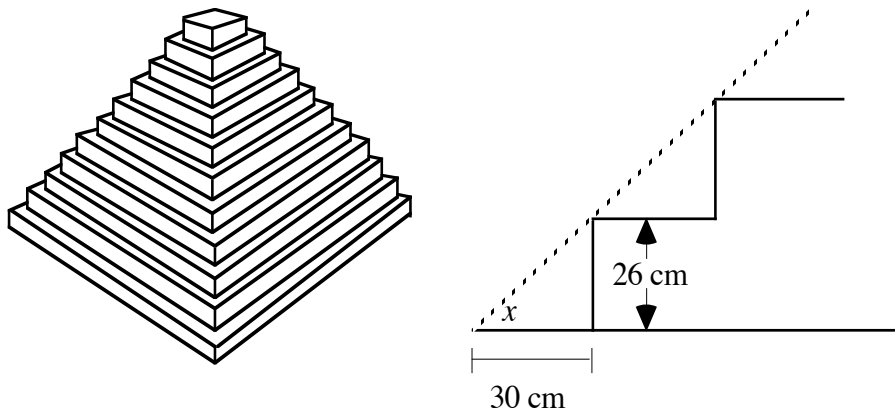
2. In a violent eruption, like that of Washington's Mount St. Helens in 1980, the peak of a volcano can be blown apart. Estimate the height of the volcano in the diagram below prior to its eruption. Its original profile appeared roughly triangular. Its base measures 2032 m and its sides form an angle of  $42^\circ$  with the horizon.



3. The second largest pyramid built at Giza was constructed for Khafra, son of Khufu. Out of respect, Khafra built his pyramid shorter than his father's. An old Egyptian was curious about the difference in heights. To measure the height of Khafra's pyramid, he held his staff at a right angle to the ground. He then found the point where the end of the staff's shadow coincided with the end of the pyramid's shadow. The height of the staff and the lengths of each shadow are given in the figure below. If Khufu's pyramid is 147 m high, how much shorter is Khafra's pyramid?



4. The Egyptians were not the world's only pyramid builders. The Mayans constructed a step pyramid, called El Castillo, in Mexico's Yucatan Peninsula. This pyramid was built as a temple of worship to the sun. It consisted of four sides, each representing one of the four seasons. To represent the number of days in a Mayan year, each side had 91 steps plus a top platform ( $4 \text{ sides} \cdot 91 \text{ steps} + 1 \text{ top} = 365$ ). If each step is 30 cm wide and 26 cm high, as shown in the diagram below, find the angle of ascent of a side of the pyramid.



## Answers to Module Assessment

1.
  - a.  $2.3^2 + 3.2^2 = x^2$ ;  $x \approx 3.9$
  - b.  $x^2 + 11.7^2 = 16.5^2$ ;  $x \approx 11.6$
  - c.  $\cos 80^\circ = 5.9/x$ ;  $x \approx 34$
  - d.  $\tan x = 10.6/6.3$ ;  $x \approx 59^\circ$
  - e.  $\sin x = 31.8/41.2$ ;  $x \approx 51^\circ$
  - f.  $\sin 30^\circ = x/9.8$ ;  $x \approx 4.9$
2. The height of the volcano before the eruption can be determined as follows:

$$\tan 42^\circ = \frac{x}{1016}$$
$$x \approx 915 \text{ m}$$

3. Using similar triangles, the height of Khafra's pyramid can be found by solving the following proportion:

$$\frac{2}{3} = \frac{x}{206}$$
$$x \approx 137.3 \text{ m}$$

This is approximately 10 m shorter than King Khufu's pyramid.

4. The angle of ascent can be found using the inverse tangent:

$$\tan x = \frac{26}{30}$$
$$x \approx 41^\circ$$

## **Selected References**

Badawy, A. *Architecture in Ancient Egypt and the Near East*. Cambridge, MA: M.I.T. Press, 1966.

Davidovits, J., and M. Morris. *The Pyramids*. New York: Hippocrene Books, 1988.

Eves, H., and H. Eves. *Introduction to the History of Mathematics with Cultural Connections*. Philadelphia, PA: Saunders College Publishing, 1990.

Macaulay, D. *Pyramid*. Boston, MA: Houghton Mifflin, 1975.

# Flashbacks

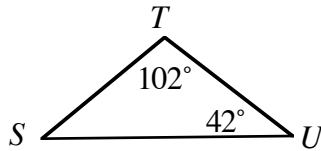
## Activity 1

- 1.1 When are two lines perpendicular?
- 1.2 When are two triangles similar?
- 1.3 Solve each of the following proportions for  $a$ .

a.  $\frac{12}{a} = \frac{306}{210}$

b.  $\frac{a}{9} = \frac{42}{3}$

- 1.4 Determine the  $m\angle S$  in the diagram below.



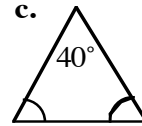
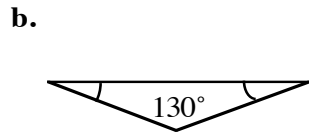
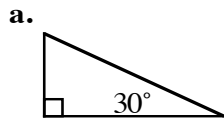
## Activity 2

2.1 Solve for  $x$  in each of the following equations.

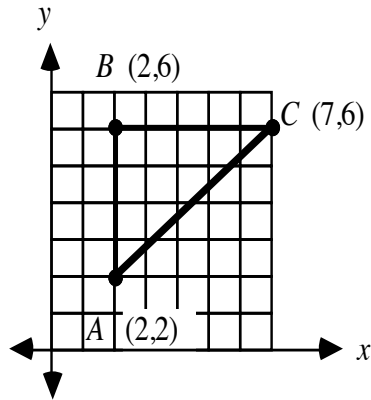
a.  $x^2 + 16 = 25$

b.  $12 + x^2 = 48$

2.2 Identify each of the following triangles as acute, obtuse, or right.



2.3 Determine the lengths of  $\overline{AB}$  and  $\overline{BC}$  in the following diagram.



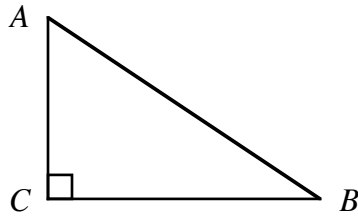
### Activity 3

3.1 Solve for  $x$  in each of the following equations.

a.  $36 = x/12$

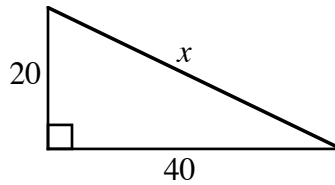
b.  $93 = 0.01/x$

3.2 Consider the right triangle in the following diagram.

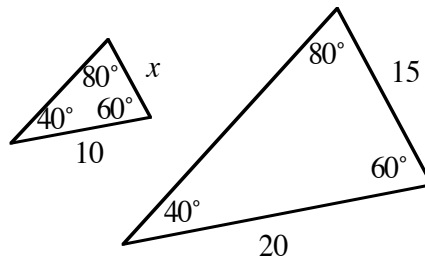


- Identify the leg opposite  $\angle A$ .
- Identify the leg adjacent to  $\angle A$ .
- Identify the hypotenuse.

3.3 Determine the value of  $x$  in the following triangle.



3.4 Determine the value of  $x$  in the diagram below.

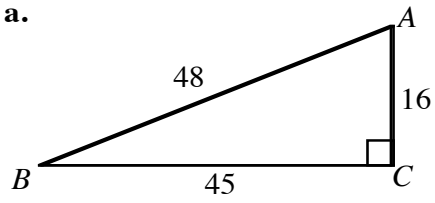




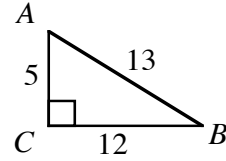
### Activity 4

4.1 Find the tangent of  $\angle B$  for each triangle below.

a.

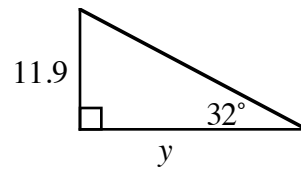


b.

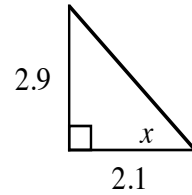


4.2 When are two angles complementary?

4.3 Find the value of  $y$  in the following diagram.



4.4 Determine the value of  $x$  in the diagram below.



## Answers to Flashbacks

### Activity 1

- 1.1 Sample response: Two lines are perpendicular when they intersect at a right or  $90^\circ$  angle.
- 1.2 Sample response: Two triangles are similar when all corresponding angles are congruent or when all corresponding sides are proportional.
- 1.3 a.  $a = \frac{12 \cdot 210}{306} \approx 8.2$
- b.  $a = 126$
- 1.4  $m\angle S = 180^\circ - 102^\circ - 42^\circ = 36^\circ$

### Activity 2

- 2.1 a.  $x = 3$  or  $-3$
- b.  $x = 6$  or  $-6$
- 2.2 a. right
- b. obtuse
- c. acute
- 2.3  $AB = 4$  and  $BC = 5$

### Activity 3

- 3.1 a.  $x = 432$
- b.  $x \approx 0.0001$
- 3.2 a.  $\overline{BC}$
- b.  $\overline{AC}$
- c.  $\overline{AB}$
- 3.3  $x = \sqrt{40^2 + 20^2} \approx 45$
- 3.4  $x = \frac{15 \cdot 10}{20} = 7.5$

#### Activity 4

4.1 a.  $\tan \angle B = 16/45 \approx 0.36$

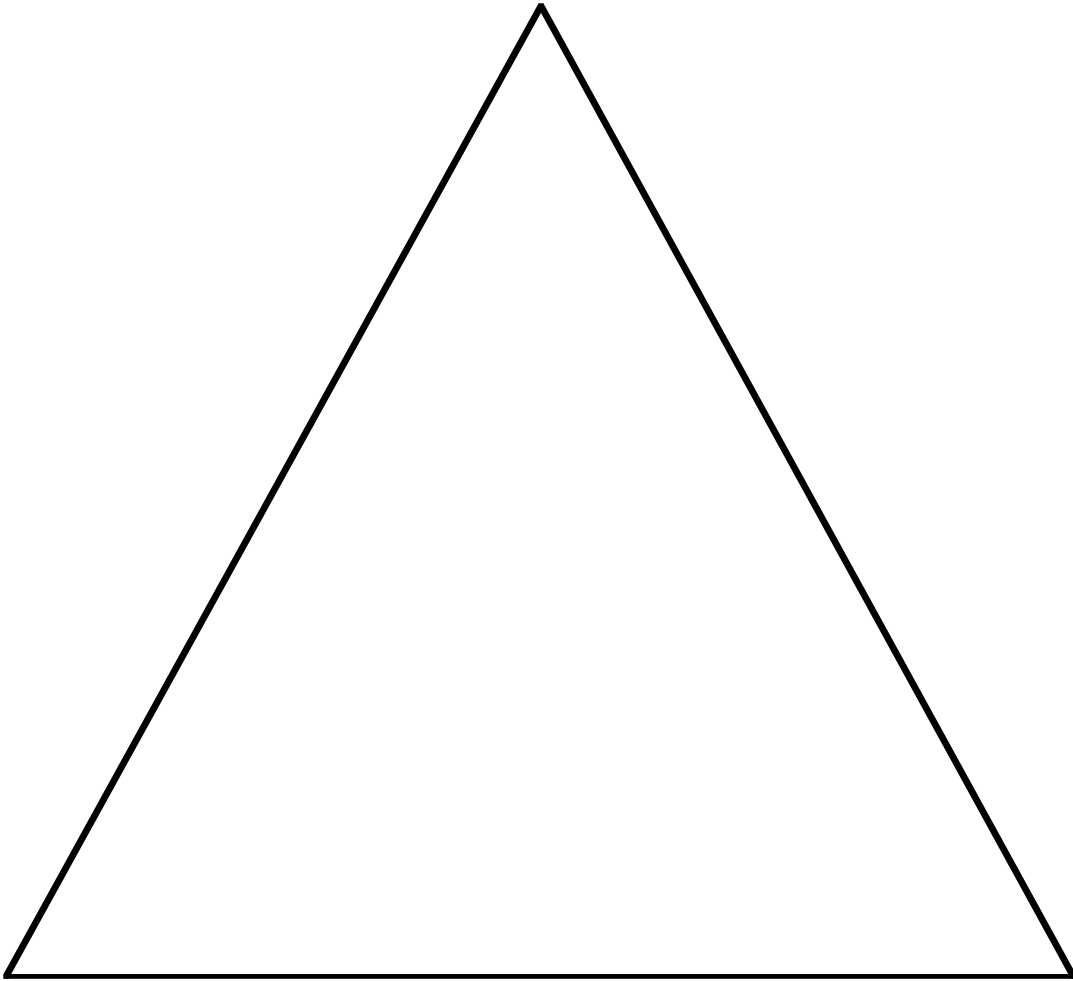
b.  $\tan \angle B = 5/12 \approx 0.42$

4.2 Two angles are complementary when the sum of their measures is  $90^\circ$ .

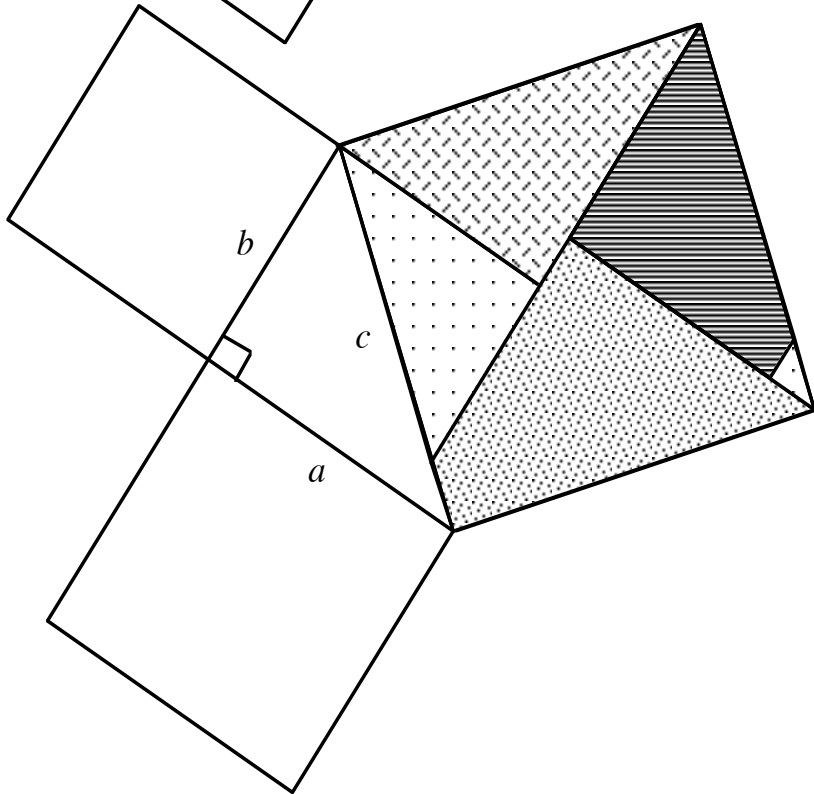
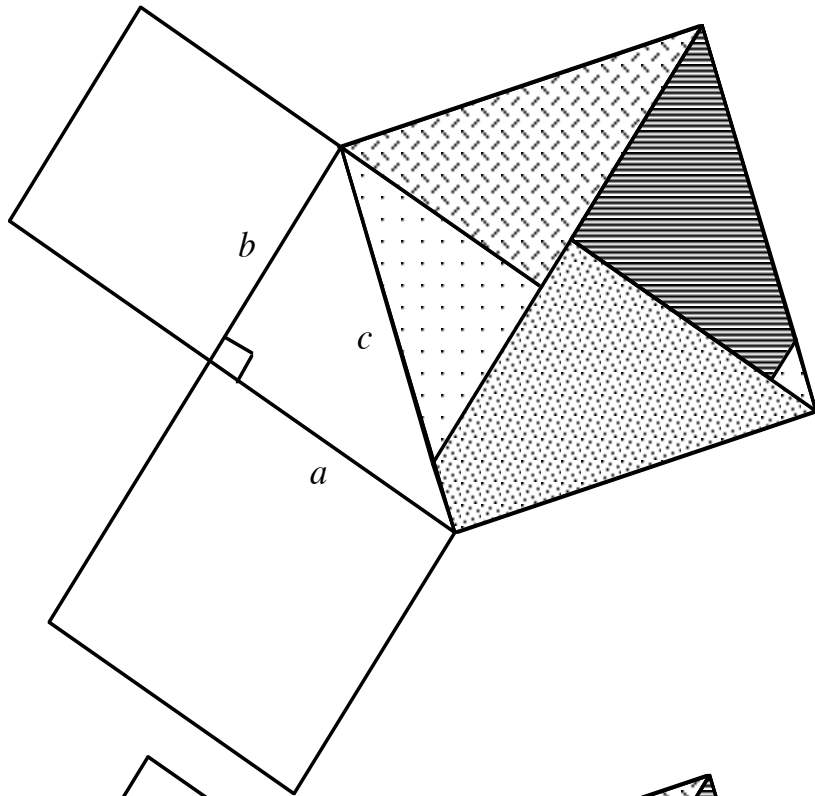
4.3  $y = \frac{11.9}{\tan 32^\circ} \approx 19$

4.4  $x = \tan^{-1}\left(\frac{2.9}{2.1}\right) \approx 54^\circ$

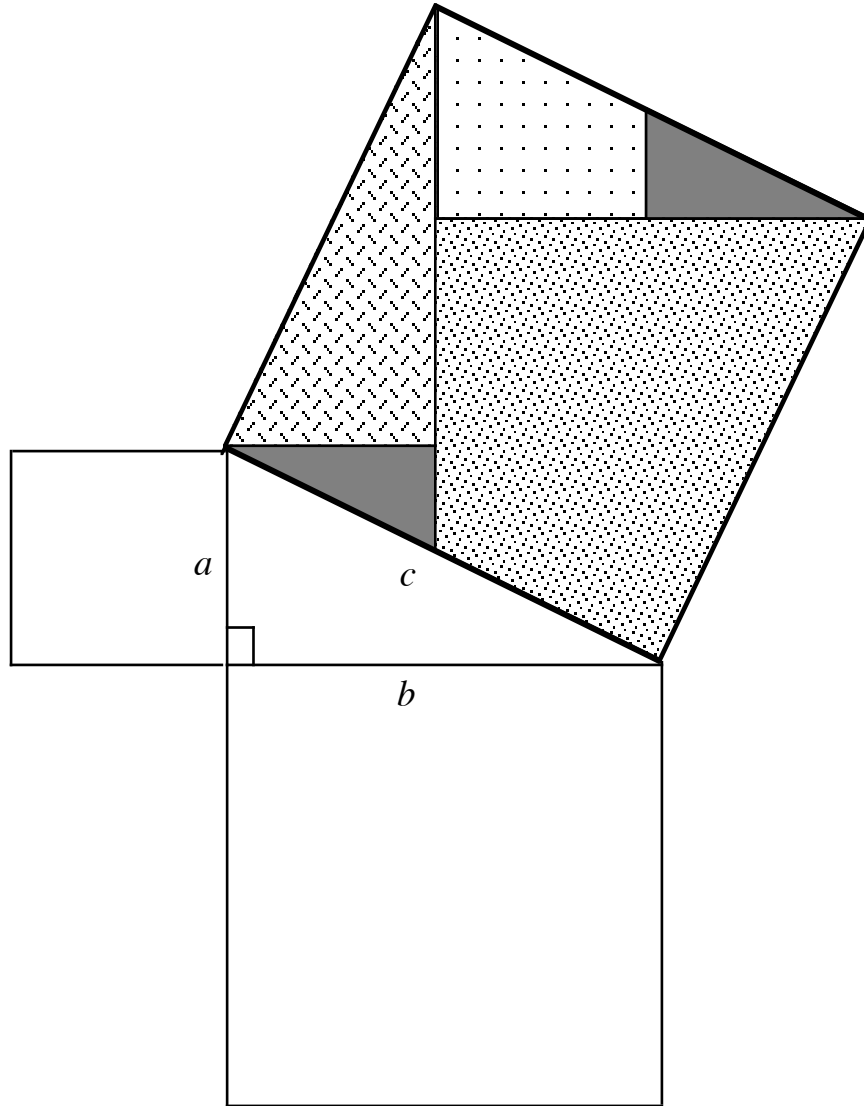
## Isosceles Triangle Template



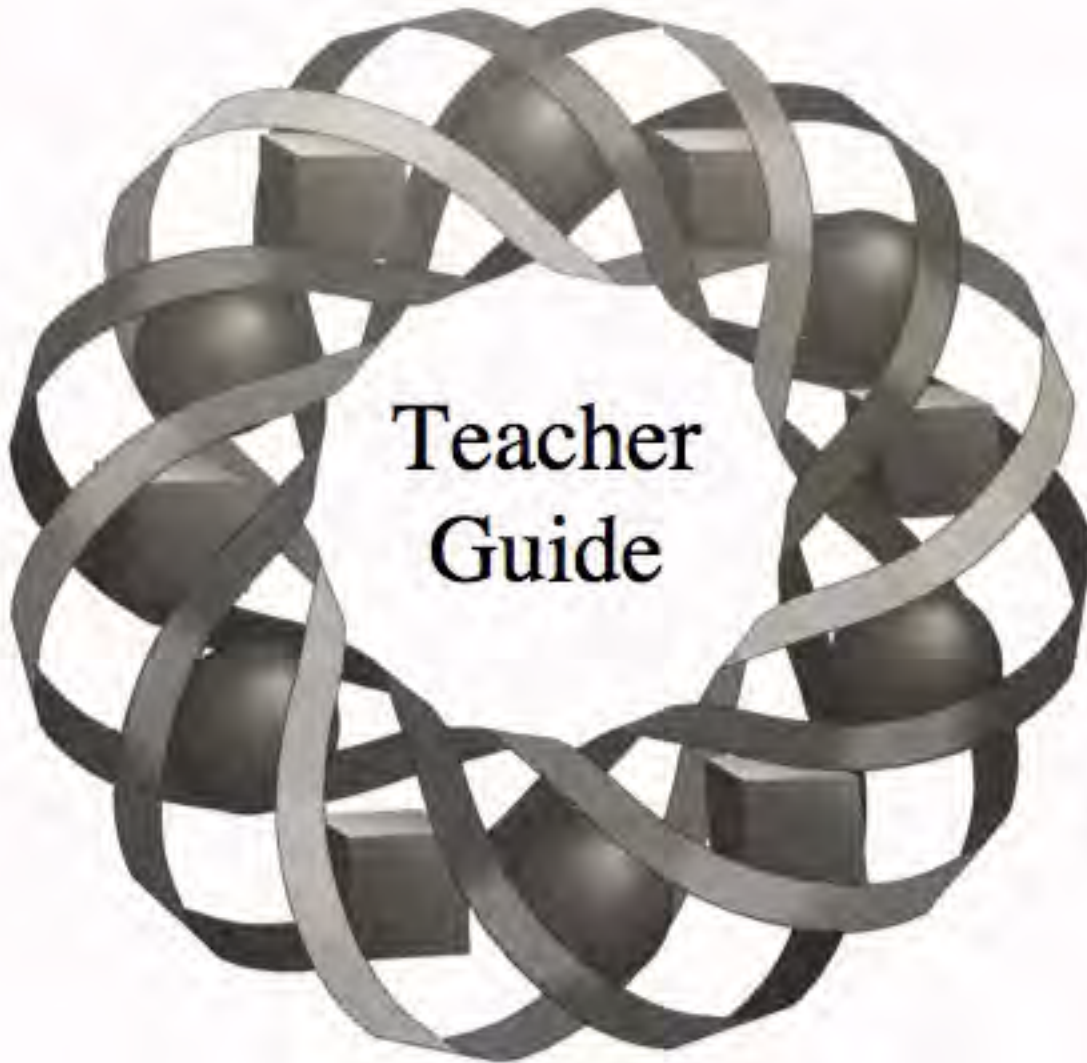
# Pythagorean Puzzle Template A



# Pythagorean Puzzle Template B



# When to Deviate from A Mean Task



You can't argue with statistics. Or can you? In this module, you look at several different strategies for presenting—and interpreting—statistical information.

*Randy Carspecken • Pete Stabio*



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## Teacher Edition

# When to Deviate from a Mean Task

### Overview

This module examines the statistics of a single variable. Students review and extend their knowledge of frequency tables, histograms, pie charts, stem-and-leaf plots, box-and-whisker plots, and measures of central tendency (mean, median, mode). Mean absolute deviation and standard deviation are introduced as measures of spread. Several problems address the potential use of statistics to mislead consumers.

### Objectives

In this module, students will:

- create a frequency table from raw data
- interpret data displayed in histograms
- interpret data displayed in pie charts
- interpret data displayed in stem-and-leaf plots
- interpret data displayed in box-and-whisker plots
- find measures of central tendency
- determine mean absolute deviation
- determine standard deviation.

### Prerequisites

For this module, students should be able to:

- find the mean, median, and mode of a set of data
- create histograms and pie charts
- create stem-and-leaf plots
- determine upper and lower quartiles
- create box-and-whisker plots
- work with percentages.



### Time Line

Activity	1	2	3	4	5	Summary Assessment	Total
Days	2	2	2	3	2	1	12

### Materials Required

Materials	Activity				
	1	2	3	4	5
metric rulers			X		
stopwatch		X			

### Technology

Materials	Activity				
	1	2	3	4	5
spreadsheet	X			X	X
graphing utility	X	X	X	X	

# When to Deviate from a Mean Task

## Introduction

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The context for much of this module is the analysis and interpretation of statistics gathered by a high school yearbook staff.

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## *Activity 1*

This activity reviews frequency tables, histograms, and pie charts (circle graphs).

## Teacher Note

Some graphing utilities may not allow users to graph histograms with non-integer data entries.

## Materials List

- none

## Technology

- graphing utility (capable of creating histograms)
- spreadsheet

## Exploration

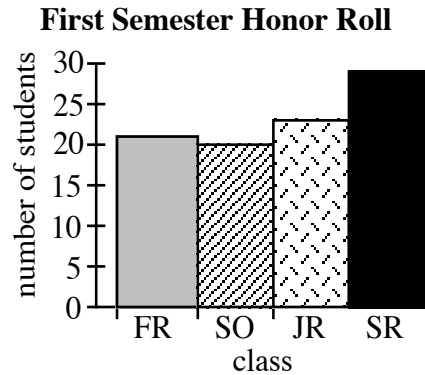
(page 68)

In this exploration, students create and interpret several histograms using the same set of data. They should begin to recognize some of the payoffs and pitfalls associated with displaying statistical information in graphical form.

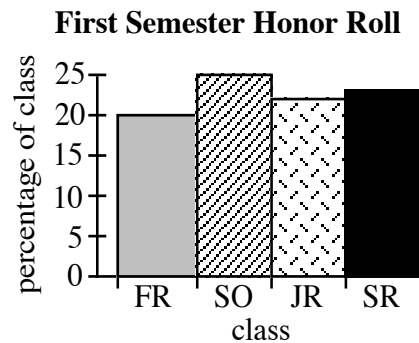
- a. Sample frequency table:

Class	Students on Honor Roll
Freshman	21
Sophomore	20
Junior	24
Senior	29
<b>Total</b>	94

- b. Students should recognize that the break in the vertical axis can visually exaggerate the difference between the number of seniors on the honor roll and the numbers from the other three classes. In the sample graph below, the differences among the classes are less prominent visually than in the original.



- c. The sample histogram below shows that the sophomore class had a larger percentage of its students on the honor roll than the seniors.



## Discussion

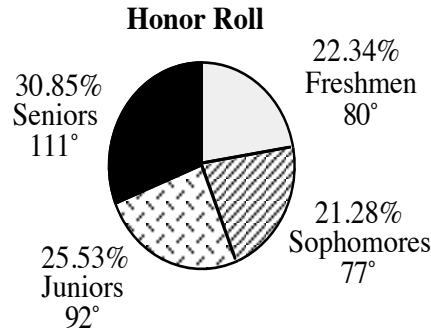
(page 69)

- a. Sample response: The graph is visually misleading because of the break in the vertical axis. This makes the differences among classes appear more substantial than they actually are.
- b. Since it allows a direct comparison among classes, students may argue that the graph that displays percentages is better. However, the histogram from Part **b** indicates exactly how many students from each class are on the honor roll, while the percentage histogram does not.
- c. Figure 2 suggests that the seniors had the best representation on the honor roll. The graph created in Part **b** of the exploration supports this same view but in a less exaggerated manner. The graph from Part **c** suggests that the sophomores had the best representation on the honor roll. The effect of that graph could be increased by adjusting the vertical axis as in Figure 2.
- d. Sample response: Round 27.50% down to 27% and round the other percentages to the nearest whole number.

## Assignment

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- \*1.1 a.** Sample graph:

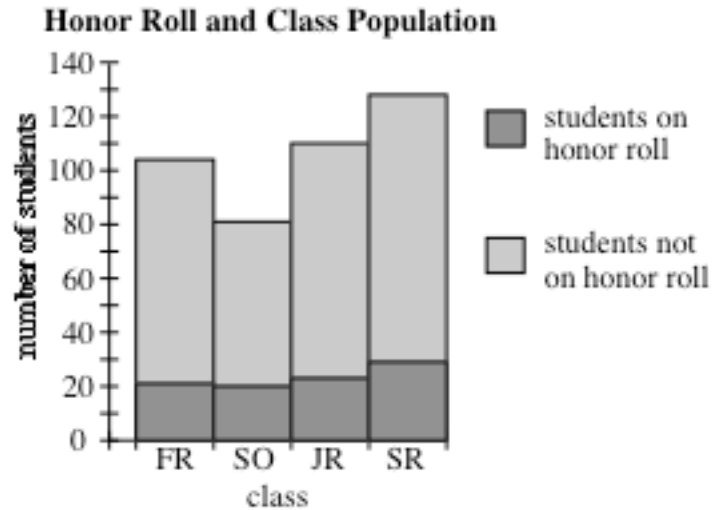


- b.** Sample response: The pie chart presents a more accurate picture of the comparison of classes. It does not exaggerate the number of seniors on the honor roll compared to the number of underclassmen.

Visually, it may be more difficult to compare categories with close percentage values on a pie chart. Because it involves calculating percentages and measuring the number of degrees in a sector, it is also more difficult to construct than a histogram.

- c.** No. The pie chart represents only the students who made the honor roll. The percentages are calculated with the number of students from each class who made the honor roll divided by the total number who made the honor roll. Since the number of sophomores on the honor roll is unchanged, this percentage remains constant.
- d.** The pie chart indicates that 31.18% of the students on the honor roll are seniors. This does not mean that 31.18% of all seniors made the honor roll.
- 1.2 a.** Answers will vary. Students may observe that the majority of the students surveyed were male, or that males had a greater tendency to attend more than 4 movies in September.
- b.** Sample response: It is not easy to compare the males because they do not have a common base line. The heights of the bars for the males takes time to calculate.

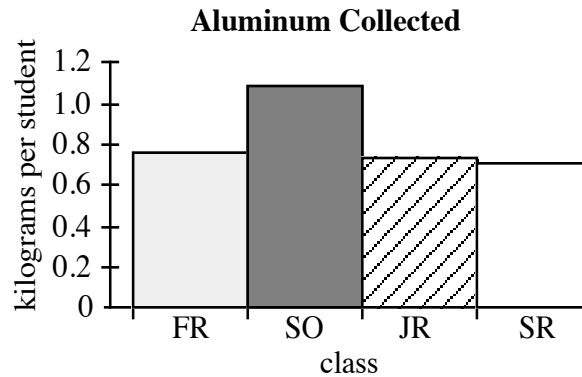
- 1.3 a. Sample graph:



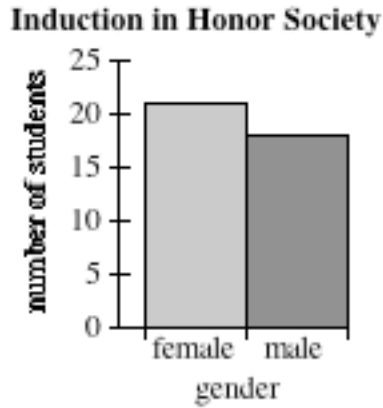
- b. In the sample graph, the class with the largest number on the honor roll can be determined by comparing the heights of the lower bars. This would not be as simple if the honor roll bars were displayed on the top.
- c. Although the percentages could be calculated, these values are not immediately evident from the sample graph.

\*1.4

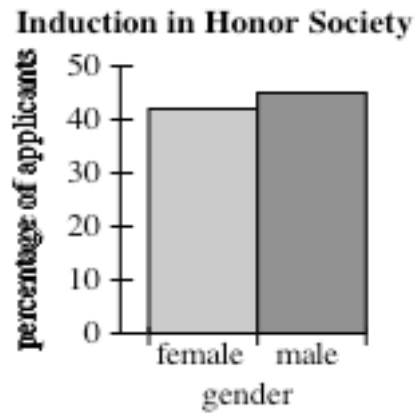
- a. Sample response: Redraw the vertical axis to include a complete scale from 0 to 92.
- b. The sample histogram below shows the kilograms of aluminum collected per student (by class). By this measure, the sophomores appear more productive than the seniors.



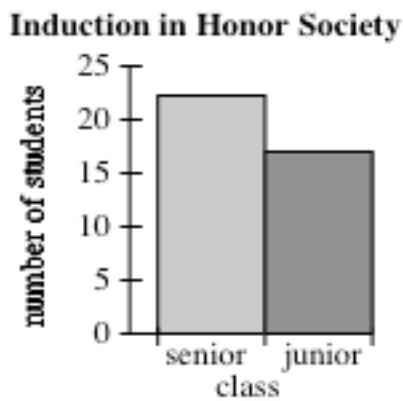
- 1.5 a. 1. The following histogram shows more females (21) were inducted in the Honor Society than males (18).



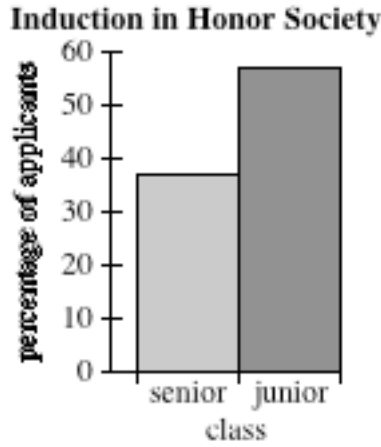
2. The following histogram shows that 45% of male applicants to the honor society were successful, while only 42% of female applicants were successful.



3. The following histogram shows that more seniors (22) were inducted than juniors (17).



4. The following histogram shows that almost 57% of junior applicants to the honor society were successful, while only about 37% of the senior applicants were successful.

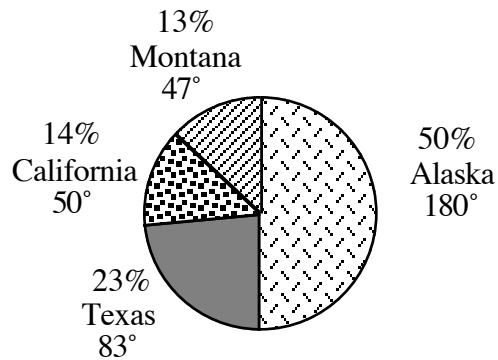


- b. Sample response: Yes. Using only the statistics in the table, it is possible to create graphs that argue for or against each statement in Part a.

\* \* \* \* \*

- 1.6 a. 1. Sample pie chart:

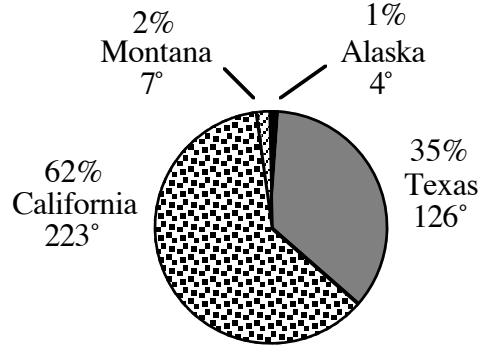
**Comparative Land Areas of Four States**



2. Judging from the pie chart, Alaska is about twice as big as Texas and about four times as big as California or Montana.

b. 1. Sample pie chart:

**Comparative Populations of Four States**

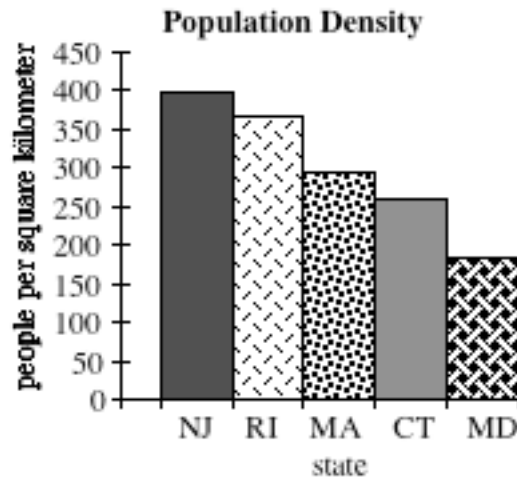


2. Sample response: Judging from the chart, the population of California is about 1.5 times larger than that of Texas. Visually, it is difficult to make this type of comparison with Montana or Alaska.

1.7 a. Jody's claim is not correct. The population densities for each state are shown in the table below. Rhode Island's population density is the second largest, after New Jersey.

State	Population Density (people/km <sup>2</sup> )
Connecticut	262
Maryland	189
Massachusetts	296
New Jersey	402
Rhode Island	371

b. Sample histogram:



1.8 Answers will vary. Students create histograms and pie charts that emphasize a characteristic of their home states.

\*\*\*\*\*



Students design surveys of no more than 10 questions to be administered to 30 people. You may wish to ask students to plan the survey and select questions as a class. Questions should provoke answers that lend themselves to statistical analysis—including histograms, pie charts, stem-and-leaf plots, box-and-whisker plots, and mean absolute deviation.

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**Activity 2**

Students review stem-and-leaf plots, range, and the measures of central tendency.

In the assignment, they examine situations in which measures of central tendency alone do not allow conclusive analysis of data.

**Materials List**

- stopwatch (or other timepiece that displays seconds; one per group)

**Technology**

- graphing utility

**Teacher Note**

The mode is the item that occurs most frequently in a data set. If no item occurs more than once, then the mode does not exist. If two items have the same, highest frequency, then the data set is bimodal. If three items share the highest frequency, the set is trimodal.

**Exploration**

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**Note:** If you do not wish to have students collect their own data, some sample data appears at the end of this teacher edition.

- a–c.** Students measure and record their own pulse rates three times during this exploration: while at rest, immediately after exercise, and after two minutes of recovery. For the second pulse reading, running in place for one minute is sufficient. Good results are obtained if students check the pulse in the neck area, count the rate for 15 sec, then multiply by 4.

- d. Students should save all data for the assignment. For comparison, you may want to ask students to separate data for males and females. The table below shows some sample data for 10 students.

**Heart rate (beats per minute)**

<b>Student</b>	<b>At Rest</b>	<b>Immediately After Exercise</b>	<b>Two Minutes After Exercise</b>
A	72	116	84
B	68	120	92
C	76	112	88
D	80	124	96
E	72	112	88
F	64	116	84
G	68	112	80
H	76	128	92
I	84	120	100
J	72	112	88
<b>Mean</b>	73.2	117.2	89.2
<b>Median</b>	72	116	88
<b>Mode</b>	72	112	88

- e. Students should select a representative number for each of the three pulse rates taken.

**Discussion**

(page 74)

- a. Students are likely to select the mean as a representative number for the class. Typical means will range from 70–80 beats/min for resting pulses, 115–125 beats/min for active pulses, and 85–95 beats/min for pulse rates after a two-minute rest.
- b. Sample response: The resting pulse rates produced the lowest mean. Exercise increased the mean. A two-minute rest lowered the mean but did not return it to the original resting rate.
- c. Height, weight, age, and general health often influence pulse rate.
- d. The class statistics are not likely to be representative of the entire school. During the teen years, pulse rates tend to decline with age. Seniors would be expected to have lower pulse rates than sophomores.

Students may also discuss whether or not their class is a representative sample of the entire student body in terms of physical fitness or male/female ratio.

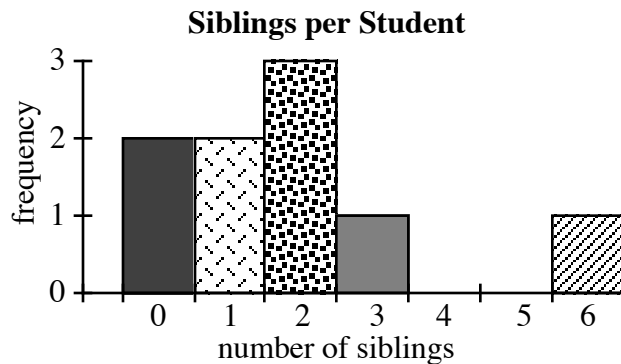
**Assignment**

(page 75)

- 2.1 a. Sample response: Either the median or mode of 2 would serve as a good representation of the number of siblings because it is a whole number. The mean of approximately 1.89 is not the best because you cannot have a part of a person.
- b. Sample frequency table:

Number of Siblings	Frequency
0	2
1	2
2	3
3	1
4	0
5	0
6	1

- c. Sample histogram:



- \*2.2 a. Sample table:

Occupation	Estimated Salary
teacher	\$25,000
superintendent	\$50,000
nurse	\$25,000
custodian	\$16,000
<b>Mean</b>	\$29,000
<b>Median</b>	\$25,000
<b>Mode</b>	\$25,000

b. Sample table:

<b>Occupation</b>	<b>Estimated Salary</b>
teacher	\$25,000
superintendent	\$50,000
nurse	\$25,000
custodian	\$16,000
baseball player	\$6,000,000
<b>Mean</b>	\$1,223,200
<b>Median</b>	\$25,000
<b>Mode</b>	\$25,000

- c. Sample response: The mean is a good representative number for the salaries in Part **a** because they are all relatively close in value.
- d. Sample response: The mean is not a good representative number for the incomes in the new group. It is distorted by the large difference in the baseball player's salary relative to the others.
- e. When some data items differ from the majority of the data by a large amount, the mean is most affected. In these situations, the median may be a better representative (although it still may not be a good one). In the sample data given above, the mean is greatly affected, while the median and mode are not affected at all.

2.3 If no element occurs more than once in a data set, then the mode does not exist.

2.4 Answers will vary. Some students may mention that the mode may not be representative for sets with multiple occurrences of very high or very low numbers. Neither the mean, the median, nor the mode may be representative when large differences exist within a data set. Measures of central tendency tend to be most useful when the values in a data set are closely grouped.

2.5 a. The range is 66 hr.

b. Since the range is the difference between the greatest data value and the least data value, it cannot be negative.

\*2.6 a. The mean is 167.7 cm; the median is 167.5 cm. There is no mode.

b. Sample stem-and-leaf plot:

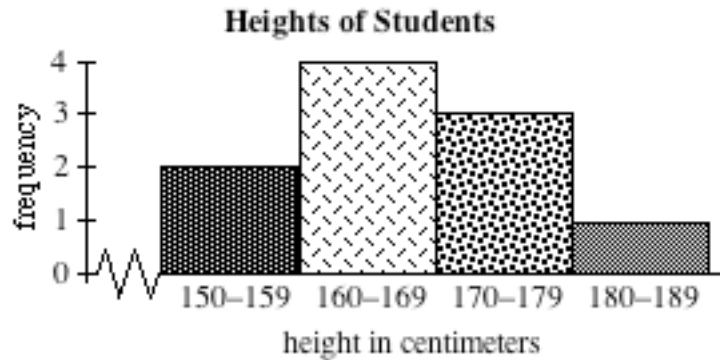
**Heights of Students**

```

15 | 37
16 | 3569
17 | 036
18 | 5
15 | 3 represents 153 cm

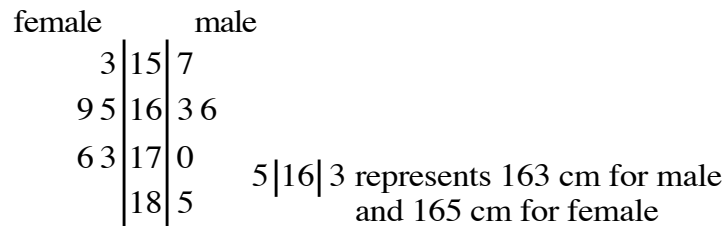
```

- c. The range of heights is  $185 - 153 = 32$  cm .  
 d. Sample histogram:

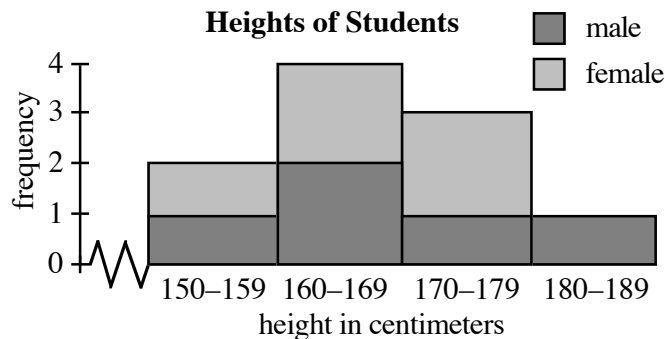


- e. Sample back-to-back stem-and-leaf plot:

**Heights of Students**



- f. Sample stacked histogram:



- 2.7
- a. Sample response: No. The advertiser could have driven 1000 new cars 1000 miles each. Most new cars do not break down during the first 1000 miles.
  - b. Sample response: Knowing the mean or median number of miles driven by each test car would provide a better measure of reliability.
- 2.8
- a. Answers will vary. Students may choose to claim that their class is in good, average, or poor physical condition.
  - b. Some possible weaknesses may include small sample size, lack of a truly random sample, unreliable measurements, or lack of knowledge about the connection between pulse rate and overall health.

\*\*\*\*\*

- 2.9** a. The following tables show the mean and median for land area, 1990 population, and population density (in people per square kilometer) for the New England states and Mountain states.

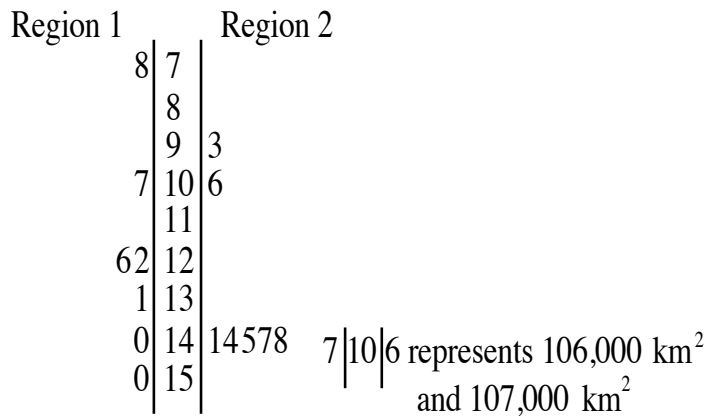
<b>New England States</b>			
	<b>Area (km<sup>2</sup>)</b>	<b>1990 Census</b>	<b>Population Density</b>
<b>Mean</b>	27,114	2,201,157	169
<b>Median</b>	21,766	1,168,590	155

<b>Mountain States</b>			
	<b>Area (km<sup>2</sup>)</b>	<b>1990 Census</b>	<b>Population Density</b>
<b>Mean</b>	277,417	1,707,347	6
<b>Median</b>	276,529	1,358,451	5

- b. Answers will vary. Students should use the data to support their preferences.

- 2.10** a. In the following stem-and-leaf plot, land areas are rounded to the nearest 1000 km<sup>2</sup>.

**State Land Area in 1000 km<sup>2</sup>**



- b. Sample response: The land areas in Region 1 tend to be more spread out, while the land areas in Region 2 tend to accumulate near 140,000 km<sup>2</sup>.

\*\*\*\*\*

## Activity 3

In this activity, students determine quartiles and create box-and-whisker plots.

### Materials List

- metric rulers

### Technology

- graphing utility (capable of creating box-and-whisker plots)

### Teacher Note

Some types of technology do not display the outliers on a box-and-whisker plot.

### Exploration

(page 79)

For the following exploration, you may wish to ask the class to collect and compile their own data for hand spans. This process, however, can prove time consuming.

- a.
1. 14.8, 15.4, 15.8, 16.4, 17.5, 17.6, 17.6, 18.1, 18.5, 19.3, 19.6, 19.9, 20.0, 20.7, 21.9, 22.3, 23.4, 23.4, 23.9, 24.1, 24.1, 25.2.
  2. The median is 19.75 cm.
  3. **Note:** When determining the upper and lower quartiles for a data set, the median is not counted as a data point.  
The upper quartile is 23.4 cm.
  4. The lower quartile is 17.6 cm.
  5. The interquartile range is  $23.4 - 17.6 = 5.8$  cm.
  6. Any outliers must be greater than  $23.4 + (5.8)(1.5) = 32.1$  cm, or less than  $17.6 - (5.8)(1.5) = 8.9$  cm. In this case, there are no outliers.

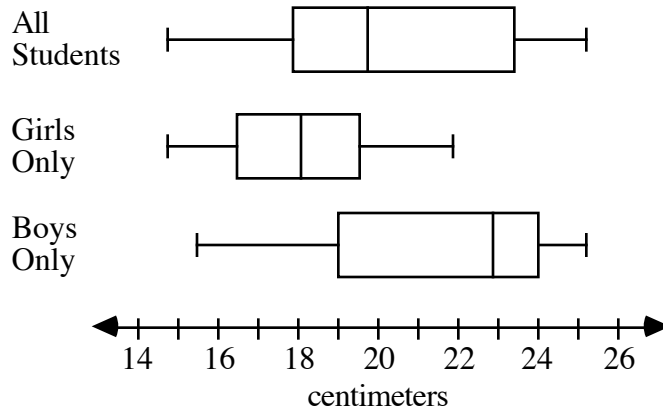
- b. In the five-point analysis for all 22 hand spans, the minimum is 14.8 cm, the lower quartile is 17.6 cm, the median is 19.75 cm, the upper quartile is 23.4 cm, and the maximum is 25.2 cm. The interquartile range is 5.8 cm. There are no outliers.

For the girls' hand spans, the minimum is 14.8 cm, the lower quartile is 16.4 cm, the median is 18.05 cm, the upper quartile is 19.6 cm, and the maximum is 21.9 cm. There are no outliers.

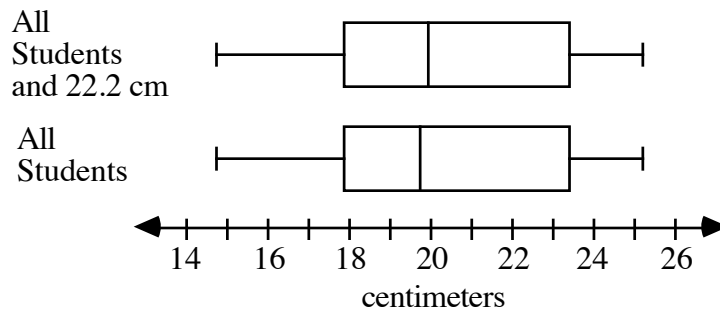
For the boys' hand spans, the minimum is 15.4 cm, the lower quartile is 19 cm, the median is 22.85 cm, the upper quartile is 24 cm, and the maximum is 25.2 cm. There are no outliers.

Sample box plots:

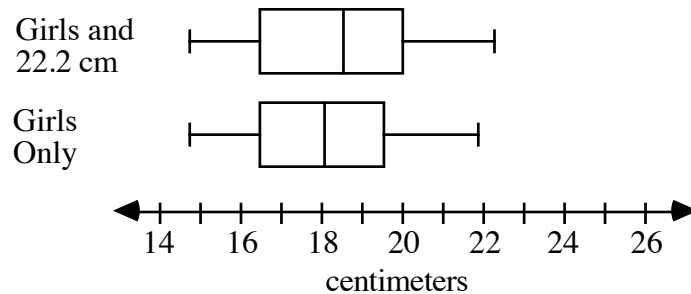
**Two Eagle Sophomore Hand Spans**



- c. A hand span is the distance from the end of the thumb to the end of the little finger when the two are stretched as far apart as possible.
- d. Answers will vary, depending on hand span. In the following sample plot, a hand span of 22.2 cm was added to the Two Eagle data. In this case, only the median changed (increasing from 19.75 cm to 19.9 cm).

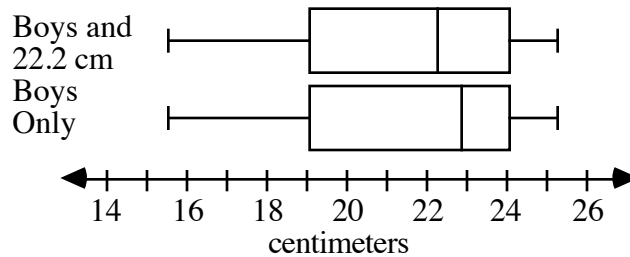


- e. Answers will vary. As shown in the following sample graph, adding a hand span of 22.2 cm to the girls' data increased the median from 18.05 cm to 18.5 cm and the upper quartile from 19.6 cm to 20 cm.





Adding a hand span of 22.2 cm to the boys' data lowered the median from 22.85 cm to 22.3 cm.



## Discussion

(page 81)

- a. Because the number of data points would change from odd to even or vice versa, adding an outlier might affect the median. It might also affect the nearer quartile, since it is determined in the same manner as the median. It is also possible that neither will be affected. If a new outlier has the same value as an existing one, the plot might not change at all. Otherwise, a new outlier would add an asterisk to the plot.
- b. Sample response: The median is less affected than the mean by the addition of extreme values.
- c. Sample response: No. A box-and-whisker plot only identifies the extreme values, the quartiles, and the median. This is not enough information to determine the mean of the data.
- d. The biggest advantage of box-and-whisker plots is that they allow visual comparison of several sets of data on the same graph. They also give a general view of the spread in a set of data. One disadvantage is that very little other information about the data set is readily apparent. Another is that there is no clear picture of the location of individual items within the data set.
- e. The five points identified graphically in a box plot are the minimum value, the maximum value, the median, the upper quartile, and the lower quartile.

## Assignment

(page 81)

- 3.1
  - a. Yes. The lengths of the whiskers are determined by the most extreme data points that are not outliers (in other words, the greatest and least values within 1.5 times the interquartile range of the two quartiles, respectively.)
  - b. Sample response: It is not possible to tell. It depends on the data set to which the value is added. See the answer to Discussion Part a.
  - c. The addition of a data point within the interquartile range may or may not affect the median or the nearer quartile. See the answer to Discussion Part a.

- 3.2** The following sample responses use a hand span of 22.2 cm.
- a. My hand span is between the median (19.8 cm) and the upper quartile (23.4 cm) of the Two Eagle data. It is above the mean (20.0 cm) and below the two modes (23.4 cm and 24.1 cm).
  - b. My hand span is greater than the largest value (21.9 cm) in the Two Eagle girls' data. It is above both the mean (18.1 cm) and the median (18.1 cm).
  - c. My hand span is between the median (22.9 cm) and the lower quartile (19 cm) of the Two Eagle boys' data. It is above the mean (21.5 cm) and below the two modes (23.4 cm and 24.1 cm).

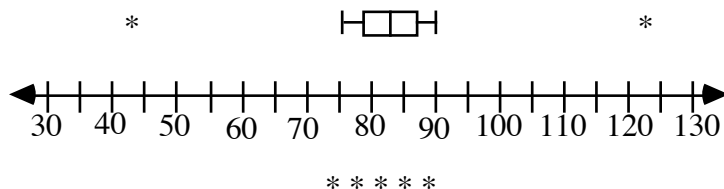
**3.3** Sample response: Yes, this measurement is an outlier for the Two Eagle High data. It is more than 1.5 times the interquartile range greater than the upper quartile.  $37 > 23.4 + (5.8)(1.5)$

- 3.4**
- a. Class III
  - b. Class I (smallest range)
  - c. Class III
  - d. Since box plots give little information about the distribution of individual data points, this cannot be determined from the graph. By assuming that the data is evenly distributed, some students may guess Class II, since its lower quartile has the highest value.
  - e. Since class size is not displayed on the box plots, it is not possible to identify the largest class.

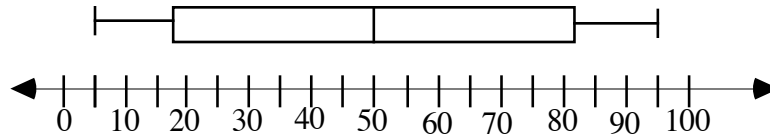
**\*3.5** Answers and methods will vary. Sample response: I chose the value that the mean and median would share (83), then picked seven values equidistant on either side of this value: 76, 78, 79, 80, 82, 82, 82, and 84, 84, 84, 86, 87, 88, 90. This guarantees that the middle element is also the mean.

To choose an outlier, I determined the interquartile range for these 15 values, then multiplied the interquartile range by 1.5 and used it to help select the value for one outlier (43). Then I selected a second outlier an equal distance from the mean in the other direction (123).

My final data set has 17 items: 43, 76, 78, 79, 80, 82, 82, 82, 83, 84, 84, 84, 86, 87, 88, 90, 123.



- 3.6 a. Sample response: The students at university A tend to be older than those at university B. Of those surveyed, about 75% of the students from university A are older than the oldest students at university B.
- b. Sample response: The box plots do not show how many people were in the survey. It also doesn't allow you to see the actual ages of the students.
- 3.7 a–b. As shown below, the box plots for these two data sets are the same.



- c. Sample response: It can be misleading because box plots do not show information about individual data points. A box plot only gives a general picture of the spread in a data set.
- d. A back-to-back stem-and-leaf plot might better represent the data.

second set		first set
7 6 5	0	5 5 5 5 5
	1	
8	2	
7	3	0 0 0 0 0
1	4	
	5	
8 1	6	
6	7	0 0 0 0 0
9	8	
5 2	9	5 5 5 5 5

7|3|0 represents 30 in the first data set  
and 37 in the second data set

\* \* \* \* \*

(page 83)

### Activity 4

In this activity, students examine data sets that are not adequately described by measures of central tendency. **Note:** Because of its ease of calculation and its direct link with distance from the mean, mean absolute deviation is introduced before standard deviation.

#### Materials List

- none

## Technology

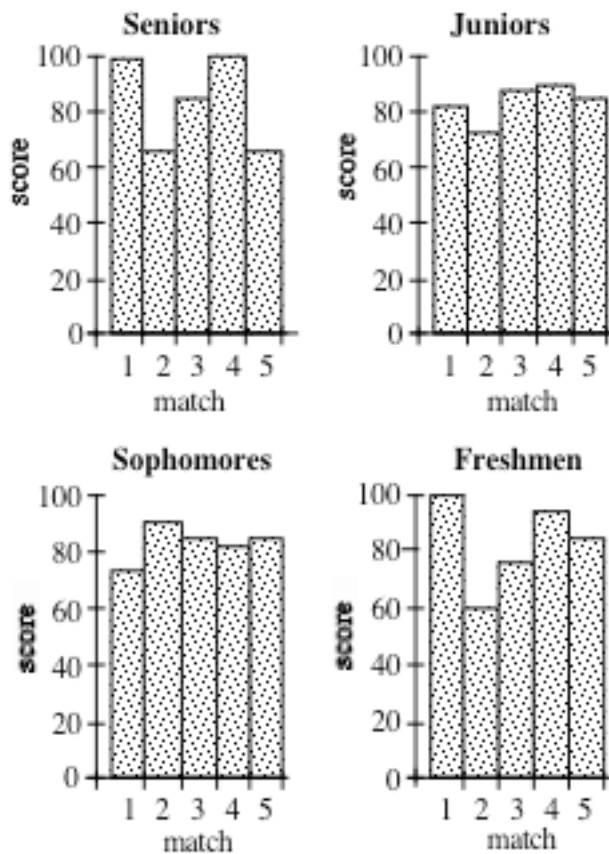
- spreadsheet
- graphing utility (capable of creating histograms)

## Exploration 1

(page 83)

Students discover that the graphs of several data sets with identical means and medians may still reveal some differences. The notion of spread as a measure of variability in a data set is introduced.

- Note:** Students should save their spreadsheets for use in Exploration 2.
- Sample histograms:



- The mean for all teams is 83.2. The median for all teams is 85.
- Using only the histograms and measures of central tendency, students may propose a number of methods for selecting a winner. The remainder of the activity, however, is designed to focus students on the spread in the data.

## Discussion 1

(page 84)

- a. Answers will vary. Using measures of central tendency, no clear winner emerges. Considering other criteria, each team won one match. All—except the seniors—also tied for the lead in one match. The seniors, however, have two of the three highest scores.
- b. Judging from the histograms, the ranges in the scores for the juniors and sophomores are less than the ranges for the seniors and freshmen.
- c. Sample response: If the variation can be quantified, this might be used as a criterion for selecting a winner. This would reward the team that performed most consistently.
- d. Two of the rules might affect the spread of scores—the one that allows teams to work cooperatively, and the one that permits teams to change members after each match.

## Exploration 2

(page 84)

This exploration introduces students to mean absolute deviation as a measure of spread.

- a. Students may use a spreadsheet to determine the mean absolute deviation for each team's scores. (The mean for all teams was 83.2.)

Seniors' Scores	Distance from Mean	Juniors' Scores	Distance from Mean
99	15.8	82	1.2
66	17.2	72	11.2
85	1.8	88	4.8
100	16.8	89	5.8
66	17.2	85	1.8
<b>Mean Absolute Deviation</b>	13.76	<b>Mean Absolute Deviation</b>	4.96

Sophomores' Scores	Distance from Mean	Freshmen's Scores	Distance from Mean
74	9.2	100	16.8
90	6.8	60	23.2
85	1.8	77	6.2
82	1.2	94	10.8
85	1.8	85	1.8
<b>Mean Absolute Deviation</b>	4.16	<b>Mean Absolute Deviation</b>	11.76

- b. Any set of five, equal scores will work. For example, the following set has a mean of 50 and a mean absolute deviation of 0: {50, 50, 50, 50, 50}.

- c. Two sets of scores produce a mean absolute deviation of 24. These sets are  $\{50, 50, 50, 100, 100\}$  and  $\{50, 50, 100, 100, 100\}$ . In both sets of scores, the distance from each data point to the mean is as great as possible; therefore, the mean absolute deviation is as great as possible.

## Discussion 2

(page 85)

- a. Since distances cannot be negative, the absolute value function is used to express the difference between each data item and the mean as a non-negative value.
- b. The expression  $|x_i - \mu|$  is the distance from the mean for each data point. The sum of these distances, divided by the number of data points ( $n$ ), is the average distance from the mean.
- c. The distance from the top of each bar to the line is the distance between the data point and the mean. The mean of these distances is the mean absolute deviation.
- d. Sample response: The histograms whose bars vary more in height have larger mean absolute deviations.
- e. Answers will vary, depending on which team students initially chose as the winner. Adding mean absolute deviation as a criterion would make the sophomores a reasonable selection.
- f. Students might reasonably expect the sixth score to fall within a range of values equal to the mean plus or minus the mean absolute deviation.
- g. This question encourages students to consider the spread in a data set. Since the scores for the sophomores are clustered more closely around the mean than the scores from the other teams', a score of 95 seems more unlikely for the sophomores.

## Assignment

(page 86)

- 4.1 a. Measures of central tendency are not adequate to differentiate between these two sets of scores. Although the graphs reveal some differences, the mean (85), median (85), and mode (85) are identical for both teams.
- b. The scores for students 3 and 4 on the junior team are 75 and 95, respectively. The distance from the mean score of 85 is 10 for both.
- c. The mean absolute deviation would be reduced. The original set of scores has a mean absolute deviation of 5.0; the new set has a mean absolute deviation of 2.5. This occurs because the two revised scores are both closer to the mean.

- d. Adding another score of 85 increases  $n$  in the formula for mean absolute deviation from 4 to 5, without increasing the sum of the distances from the mean. Therefore, the mean absolute deviation would be reduced.

- 4.2 a. The mean absolute deviation for each team's scores is shown below.

Class	Mean Absolute Deviation
Seniors	16.5
Juniors	5
Sophomores	0
Freshmen	8

- b. Sample response: All the scores are the same: 85. As a result, the mean is 85 and the distance from each score to the mean is 0. The mean absolute deviation, therefore, is also 0.
- c. Sample response: Since the seniors' scores are widely spread (mean absolute deviation 16.5), they probably worked independently. The same is likely to be true for the juniors, although the spread in their scores is not as great (mean absolute deviation 5.0). Since the sophomores' scores are identical, they appear to have worked as a cooperative group. The freshmen may have worked independently, or in two small groups since two pairs of scores are the same (mean absolute deviation 8).

- 4.3 a. Sample response: The score in match 5 for the seniors and the scores in matches 1 and 2 for the juniors all have a deviation greater than "an average amount."
- b. Sample response: Since mean absolute deviation is the average of the deviations from the mean, I used it to determine the "average amount."
- c. Sample response: The score in match 4 equals the mean score. The "missing" bar is actually a bar with a height of 0.

- 4.4 a. The upper end of the interval was determined by adding 13.8 to 83.2; the lower end of the interval was found by subtracting 13.8 from 83.2.
- b. The mean absolute deviation is 0 when the data point has the same value as the mean.
- c. Sample response: The interval that contains scores 2 mean absolute deviations from the mean is  $[69.4 - 13.8, 97 + 13.8]$  or  $[55.6, 110.8]$ . As it is not possible to score more than 100 points on the test, the interval would have to be  $[55.6, 100]$ .

- \*4.5**
- a. The mean absolute deviation is 11.84 for the seniors and 14 for the juniors.
  - b. Since the mean absolute deviation for the juniors is higher, the juniors' scores are spread more widely than the seniors'.
  - c. The seniors' score of 51 is more than 2 deviations from the mean. The juniors' scores of 50 and 100 are more than 1 mean absolute deviation from the mean.

**4.6** Sample response: I would select the sophomores because they have the smallest mean absolute deviation and therefore score the most consistently around the mean of 83.2.

\* \* \* \* \*

- 4.7**
- a. Sample response: Since the mean absolute deviation for Rita's scoring average is much smaller than Bernard's, she is the more consistent offensive player.
  - b. The range of values for Rita is from 20 to 30 points; for Bernard, the range is from 10 to 40 points.
  - c. Sample response: Rita's performance in the last game is more impressive because it is out of her range of "typical scores" (established by the mean, plus or minus the mean absolute deviation). Her typical range of points would be between 20 and 30 points per game, so 35 is an outstanding performance. Bernard's "typical scores" range from 10 points to 40 points. For him, a 35-point game is within the expected range.
  - d. If both teams average 60 points per game, Bernard's teammates must score in the range from 20 to 50 points, while Rita's teammates must score in the range from 30 to 40 points.

**4.8** Sample response: Runner 1 should receive the award because her time in the sixth race is the only one more than 1 mean absolute deviation from her mean time for the previous five races.

	<b>Runner 1</b>	<b>Runner 2</b>	<b>Runner 3</b>
<b>Race 1</b>	1530 sec	1210 sec	1335 sec
<b>Race 2</b>	1425 sec	1190 sec	1520 sec
<b>Race 3</b>	1455 sec	1290 sec	1215 sec
<b>Race 4</b>	1325 sec	1185 sec	1465 sec
<b>Race 5</b>	1270 sec	1120 sec	1210 sec
<b>Mean</b>	1401 sec	1199 sec	1349 sec
<b>Mean Absolute Deviation</b>	82.8 sec	40.8 sec	114.8 sec

\* \* \* \* \*



## Activity 5

In this activity, students investigate a more commonly encountered measure of variance—standard deviation.

### Materials List

- none

### Technology

- spreadsheet

### Teacher Note

Both mean absolute deviation and standard deviation can be treated as functions. If the derivative of the function exists, the deviation can be minimized. However, the absolute value function is not everywhere differentiable. For this reason, statisticians prefer standard deviation rather than mean absolute deviation as a measure of spread.

In this activity, students calculate standard deviation for data sets. The formula given in the mathematics note is for the standard deviation of a population ( $\sigma$ ), not the standard deviation for a sample ( $s$ ). In the formula for sample standard deviation, the sum of the squared differences is divided by  $(n - 1)$ , rather than  $n$ .

The built-in functions in your technology may handle this calculation in different ways. For example, the TI-82 and TI-83 will determine either sample or population standard deviation. However, Microsoft Excel calculates only sample standard deviation. You may wish to discuss these capabilities with students before assigning Problem 5.2.

### Discussion

(page 90)

- a. Sample response: Both formulas use the differences between the mean and the data points, along with the number of data points in the set, to determine spread. They are different in that mean absolute deviation uses absolute value, while standard deviation uses the squares of the differences, then takes the square root.
- b. Sample response: The differences are kept non-negative by squaring them.
- c. The square root is taken to eliminate the effect of squaring the differences. It also returns the value of the deviation to its proper units.
- d. Sample response: Since standard deviation is a measure of spread, it must be non-negative. A “negative spread” would have no meaning.

## Assignment

(page 90)

**5.1** The mean is 83.2. From

$$\sigma = \sqrt{\frac{(99 - 83.2)^2 + (66 - 83.2)^2 + (85 - 83.2)^2 + (100 - 83.2)^2 + (66 - 83.2)^2}{5}},$$

the standard deviation is approximately 15.01.

- 5.2**
- Many calculators use  $\bar{x}$  or  $\mu$  to represent the mean. The mean score for all three teams is 83.2.
  - To represent standard deviation, calculators may use  $\sigma$ ,  $\sigma_x$ , or  $\sigma_x$ . The standard deviation for the juniors' scores is 6.1, for the sophomores' scores is 5.3, and for the freshmen's scores is 14.0.
- 5.3**
- There are 5 audiences more than 1 standard deviation from the mean.
  - Sample response: No. Since the standard deviation is 66, the audience would have to have been less than 18 or greater than 282 to be more than 2 standard deviations from the mean.

\* \* \* \* \*

- \*5.4**
- The mean mass is 76.96 kg.
  - The standard deviation is 11.3 kg.
  - The interval of values within 1 standard deviation of the mean is [65.66, 88.26]. The masses of 9 of the 25 players, or 36%, fall outside this interval.
  - The mass of 54 kg is more than 2 standard deviations from the mean.
- 5.5**
- The mean, 340, is the midpoint of the interval [230, 450].
  - From Part **a**, one standard deviation equals  $450 - 340$  or 110. The interval of values within 2 standard deviations of the mean is  $340 \pm 2(110)$  or [120, 560].
  - The interval of values within 3 standard deviations of the mean is  $340 \pm 3(110)$  or [10, 670].

\* \* \* \* \*

## Answers to Summary Assessment

(page 92)

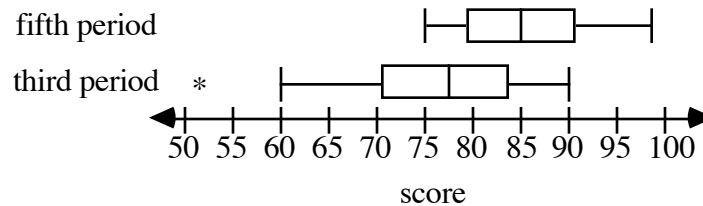
1. Sample graph:

fifth period		third period
	5	2
	6	0 6
9 8 7 6	7	1 2 3 5 7 8
8 5 5 4 3 2 0 0	8	0 2 4 5 8 9
9 6 6 2 2 0 0 0	9	0

6|7|1 represents a score of 76 in fifth period and a score of 71 in third period

2. The mean score for the third-period class is approximately 76. The mean score for the fifth-period class is approximately 86.
3. For the third-period class, the minimum is 52, the maximum is 90, the lower quartile is 71.5, the median is 77.5, and the upper quartile is 84.5. The score of 52 is an outlier.

For the fifth-period class, the minimum is 76, the maximum is 90, the lower quartile is 80, the median is 85, and the upper quartile is 91.



Sample observations: Third-period test results are definitely below those in fifth period. The range of scores in third period is greater than the range of scores in fifth period.

4. a. The standard deviation for the third-period scores is approximately 9.9. The standard deviation for the fifth-period scores is approximately 6.7.
- b. For the third-period class, the interval that contains scores within 1 standard deviation of the mean is [66.1, 86.7]. Eleven of the 16 scores, or about 69%, are in this interval.

For the fifth-period class, the interval that contains scores within 1 standard deviation of the mean is [79.4, 92.8]. Thirteen of the 20 scores, or about 65%, are in this interval.

## *Module Assessment*

- 1.** An average weekday for a student at Two Eagle High includes 6.5 hr of school, 8.5 hr of sleep, 1.0 hr of travel to and from school, 1.5 hr of television viewing, 0.75 hr of eating, and 5.75 hr of other activities.
  - a.** Determine the measure of the central angle needed for each sector in a pie chart of this data.
  - b.** Display the data in a pie chart.
  
- 2.** The heights in meters of the 10 tallest buildings in Chicago are 443, 346, 343, 295, 274, 262, 259, 236, 222, and 213.

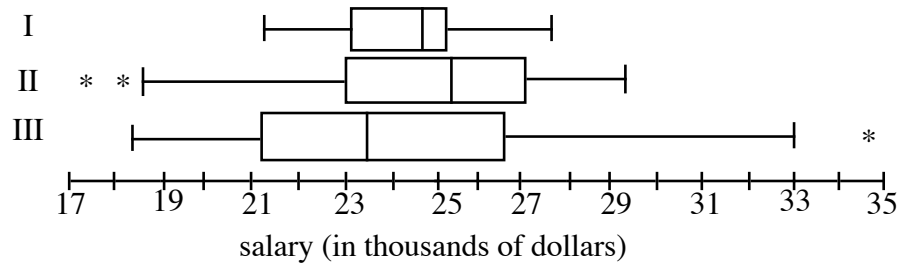
The heights in meters of the 10 tallest buildings in New York are 419, 381, 319, 290, 279, 274, 259, 248, 246, and 241.

  - a.** Create a histogram that compares the heights of the 10 tallest buildings in each city.
  - b.** Create a histogram that exaggerates the differences in the heights.
  
- 3.** The heights, in centimeters, of the Two Eagle High girls' basketball team are 173, 160, 191, 173, 165, 180, 178, 168, 178, 178, and 178.

The heights, in centimeters, of the boys' basketball team are 170, 201, 175, 193, 178, 193, 180, 191, 180, 188, and 183.

  - a.** Create a back-to-back stem-and-leaf plot of this data.
  - b.** Find the mean, median, mode, range, mean absolute deviation, and standard deviation for the heights of the girls' team.
  - c.** Find the mean, median, mode, range, mean absolute deviation, and standard deviation for the heights of the boys' team.
  - d.**
    - 1.** To compare the heights of the two teams, create box plots for both sets of data on the same scale.
    - 2.** Write a summary of the information displayed in the box plots.

4. The box-and-whisker plots shown below represent the starting salaries for three occupations that require a college degree.



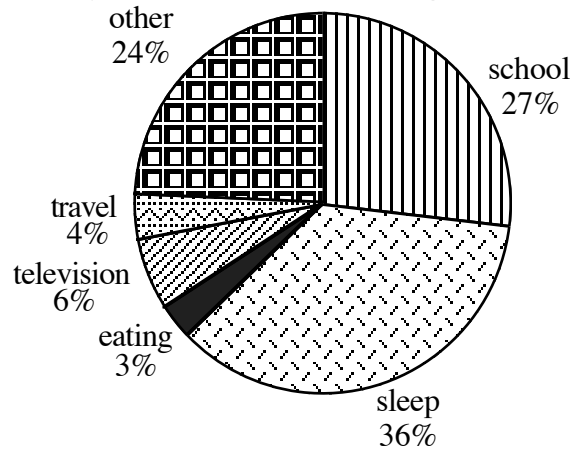
- Which occupation has the greatest range of starting salaries?
- Which occupation has the smallest interquartile range of salaries?
- If starting salaries were your only concern, which occupation would you choose to prepare for in college? Which occupation would be your second choice? Explain your choices.

## Answers To Module Assessment

1. a. In the pie chart shown below,  $127.5^\circ$  represents 8.5 hr,  $97.5^\circ$  represents 6.5 hr,  $86.25^\circ$  represents 5.75 hr,  $22.5^\circ$  represents 1.5 hr,  $15^\circ$  represents 1 hr, and  $11.25^\circ$  represents 0.75 hr.

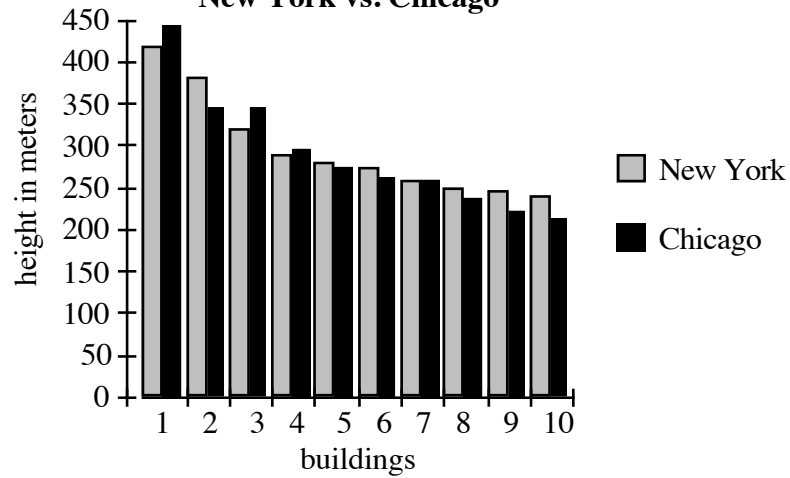
- b. Sample graph:

**A Day in the Life of a Two Eagle Student**

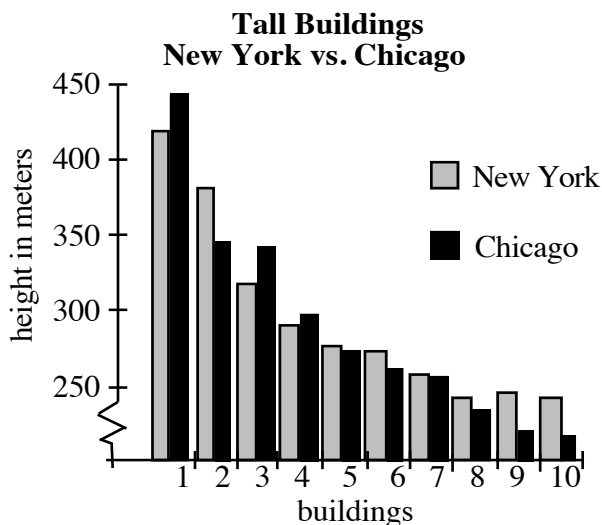


2. a. Sample graph:

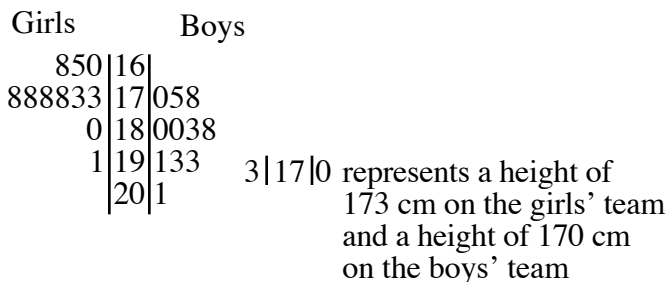
**Tall Buildings  
New York vs. Chicago**



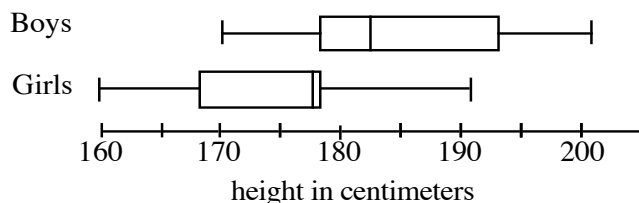
b. Sample graph:



3. a. Sample graph:



- b. For the girls' team, the mean is 174.7, the median is 178, and the mode is 178. The range is 31. The mean absolute deviation is 6.3 and the standard deviation is 7.97.
- c. For the boys' team, the mean is 184.7 and the median is 183. The data is bimodal at 180 and 193. The range is 31, the mean absolute deviation is 7.7, and the standard deviation is 8.82.
- d. 1. There are no outliers in either data set. Sample graph:



2. Sample response: The data for the boys' team has a greater interquartile range, suggesting that their heights vary more than the girls'. The upper 75% of the boys are as tall or taller than the upper 25% of the girls.

4.
  - a. Occupation III
  - b. Occupation I
  - c. Answers will vary. Occupation III has the highest potential starting salary. Occupation II has the highest median. Occupation I has the least range in salaries.



# Flashbacks

## Activity 1

- 1.1** One day at the beach, the lifeguard counts 6 red, 10 blue, 5 orange, and 3 green umbrellas.
- Draw a histogram that shows the frequency of umbrellas of each color.
  - What percentage of the total number of umbrellas is represented by each color?
  - Draw a histogram that shows the percentage of umbrellas of each color.
- 1.2** Draw a large circle. Divide and label the circle so that 25% of it is red, 38% is blue, 18% is green and 19% is orange.

## Activity 2

- 2.1** Create a stem-and-leaf plot of the following set of data: 12, 18, 24, 24, 30, 32, 34, 36, 36, 36, 40, 42, 48.
- 2.2**
- Determine the mean, median, and mode of the data set in Flashback **2.1**.
  - What percentage of the data is greater than the mean?
    - What percentage of the data is less than the mean?

## Activity 3

- 3.1** Consider the following set of data: 12.2, 28.6, 29.4, 30.6, 30.8, 32.2, 34.8, 36.2, 36.2, 38.5, 40.8, 48.9, 56.5.
- Determine the mean, median, and mode of the data set.
  - Determine the upper and lower quartiles of the data.

#### Activity 4

- 4.1**    **a.** Create a box-and-whisker plot of the following set of data:  
2162, 1118, 2124, 2264, 1230, 1198, 1132, 2284, 2136, 1256,  
2106, 2250.
- b.** What is the interquartile range of the data?
- 4.2**    Simplify each of the following expressions.
- a.**  $|12 - 5|$
- b.**  $|12 - 15|$
- c.**  $|-21 - 10|$
- d.**  $|-2312 + 1355|$

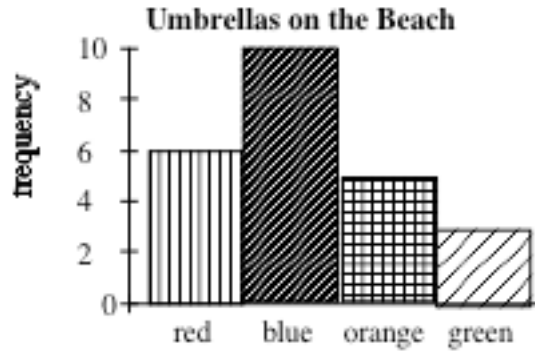
#### Activity 5

- 5.1**    Consider the following set of data: 21, 32, 43, 54, 65, 32, 23, 34, 45,  
8, 126, 32, 65.
- a.** Find the mean of the data.
- b.** Calculate the mean absolute deviation of the data.

# Answers to Flashbacks

## Activity 1

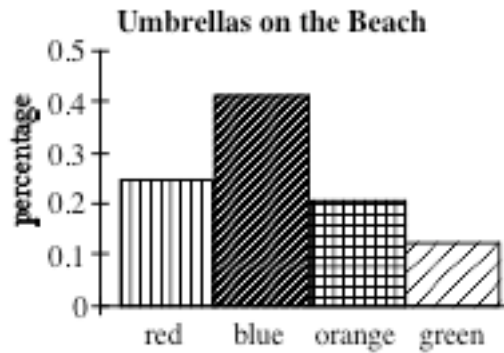
1.1 a. Sample histogram:



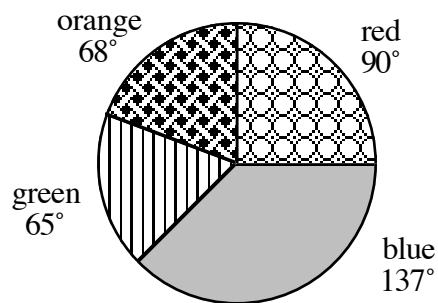
b. The percentages are shown in the table below.

Color	Number	Percentage
red	6	25
blue	10	42
orange	5	21
green	3	12

c. Sample histogram:



1.2 Sample drawing:



## Activity 2

2.1 Sample graph:

```
1 | 2 8
2 | 4 4
3 | 0 2 4 6 6 6
4 | 0 2 8
```

2|4 represents 24

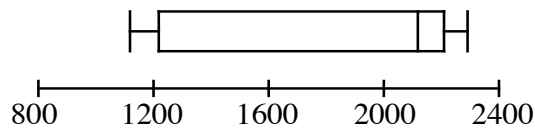
- 2.2
- The mean is approximately 31.7, the median is 34, and the mode is 36.
  - Approximately 62% of the data falls above the mean.
    - Approximately 38% of the data falls below the mean.

## Activity 3

- 3.1
- The mean is approximately 35.1, the median is 34.8, the mode is 36.2.
  - The upper quartile is 39.7; the lower quartile is 30.

## Activity 4

4.1 a. Sample graph:



- b. The upper quartile is 2207; the lower quartile is 1214. The interquartile range is 993.
- 4.2
- 7
  - 3
  - 31
  - 957

## Activity 5

- 5.1
- The mean is 44.6.
  - The mean absolute deviation is 20.3.

## Heart Rate Data

(beats per minute)

<b>Student</b>	<b>At Rest</b>	<b>Immediately After Exercise</b>	<b>Two Minutes After Exercise</b>
A	72	116	84
B	68	120	92
C	76	112	88
D	80	124	96
E	72	112	88
F	64	116	84
G	68	112	80
H	76	128	92
I	84	120	100
J	72	112	88

## Heart Rate Data

(beats per minute)

<b>Student</b>	<b>At Rest</b>	<b>Immediately After Exercise</b>	<b>Two Minutes After Exercise</b>
A	72	116	84
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H	76	128	92
I	84	120	100
J	72	112	88

# Who Gets What and Why?



Why do some U.S. states have more senators than House representatives? In this module, you examine the process—and the politics—of apportionment.

*Patricia Bean • Dean Preble • Teri Willard*



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## Teacher Edition

# Who Gets What and Why?

### Overview

The mathematics of apportionment is used to assign seats in the U.S. House of Representatives. In this module, students investigate several historical methods of apportionment. As part of their investigations, they round and order decimals, use and interpret inequalities, simplify rational expressions, and apply problem-solving techniques.

### Objectives

In this module, students will:

- use and interpret inequalities involving rational expressions
- be introduced to geometric mean
- simplify rational expressions.

### Prerequisites

For this module, students should know:

- how to use scientific notation
- how to use proportions
- rounding techniques
- the greatest integer function
- how to use a spreadsheet
- how to simplify basic algebraic expressions.

### Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	2	2	2	2	10

## Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
map of Antarctica	X				

## Teacher Note

A blackline master of the map template appears at the end of the teacher edition for this module.

## Technology

Software	Activity				
	1	2	3	4	Summary Assessment
spreadsheet	X	X		X	X
symbolic manipulator			X		



# Who Gets What and Why?

## Introduction

(page 97)

The introduction presents a scenario in the late 21st century. In this hypothetical situation, Antarctica must be fairly apportioned among seven global regions.

## Discussion

(page 97)

- a. Students should recognize that border disputes are a common occurrence and influence many world events.
- b–c. Border disputes might involve religious differences, resource availability, population characteristics, or strategic geographic features.

(page 98)

## *Activity 1*

In the exploration, students consider the apportionment of both land and representatives.

## Materials List

- map of Antarctica (one copy per student)

## Technology

- spreadsheet

## Teacher Note

A map template appears at the end of this teacher edition. You may wish to obtain a more detailed representation of the Antarctic continent.

## Exploration

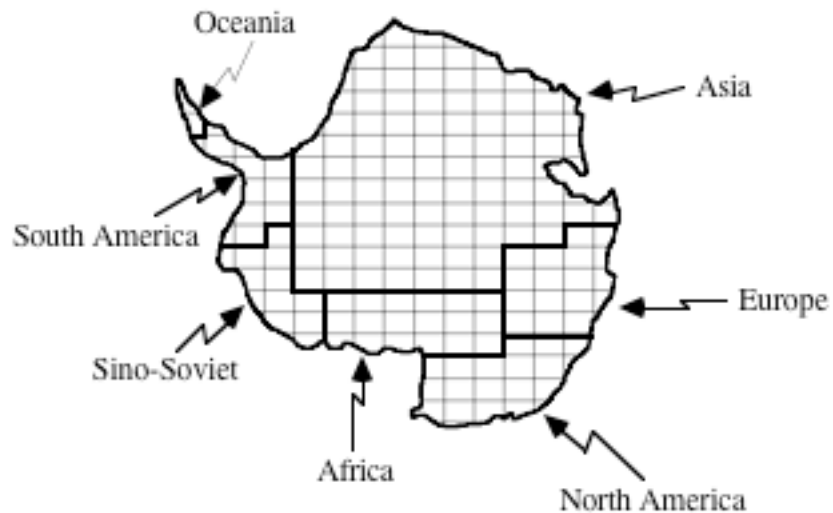
(page 98)

**Note:** Two of the regions listed in Table 1 may be unfamiliar to some students. The Sino-Soviet region contains the countries of the former U.S.S.R. Oceania describes lands of the central and South Pacific, including Micronesia, Australia, and New Zealand.

a. Sample response:

Region	People/km <sup>2</sup>
Asia	357
Africa	69
North America	55
South America	53
Europe	327
Oceania	10
Sino-Soviet	41

b. Students may use the grid on the map template to estimate area.  
Sample response:



c. Methods will vary. If students base the number of delegates on population, Asia should receive the most delegates, while Oceania should receive the least. Sample response:

Region	Delegates
Asia	28
Africa	6
North America	4
South America	3
Europe	5
Oceania	1
Sino-Soviet	2
<b>Total</b>	<b>49</b>

## Discussion

(page 99)

- a. Answers will vary. Students may choose to divide the continent based on population density. For example, since the most densely populated region is Asia, they may give Asia the largest portion of land.
- b. Sample response: Land can be divided into fractions of units (such as square kilometers), whereas people cannot be divided into fractional parts.
- c. Answers will vary. Students should recognize that there are many ways to divide both land and representatives. In practice, differences often arise due to the needs and ambitions of individual regions. Each region typically would attempt to obtain the most advantageous portion.

## Assignment

(page 99)

- 1.1
  - a. Answers will vary. Sample response: No. Some nations pay more to support the activities of the United Nations than others, yet all receive one vote.
  - b. Some students may suggest basing the number of representatives from each member country on its financial contribution. Others may argue that representation should be determined by the ratio of the member country's population to the world's population. (You may wish to ask students how the United States would fare under various plans.)
- 1.2 Sample response: Having two senators from each state helps give each state an equal voice in government. Under this system, a state like Montana—with a relatively small population and only one seat in the House of Representatives—can have the same impact in the Senate as any other state.

\* \* \* \* \*

- 1.3 a–b. Sample table:

Region	Population	Area km <sup>2</sup>
Asia	$6 \cdot 10^9$	$2 \cdot 10^7$
Africa	$1 \cdot 10^9$	$2 \cdot 10^7$
North America	$9 \cdot 10^8$	$2 \cdot 10^7$
South America	$6 \cdot 10^8$	$1 \cdot 10^7$
Europe	$1 \cdot 10^9$	$3 \cdot 10^6$
Oceania	$5 \cdot 10^7$	$5 \cdot 10^6$
Sino-Soviet	$6 \cdot 10^8$	$1 \cdot 10^7$
<b>Total</b>	$1.015 \cdot 10^{10}$	$8.7 \cdot 10^7$

c. Sample table:

<b>Region</b>	<b>Fraction of Total Population</b>	<b>Fraction of Total Area</b>
Asia	$3/5$	$2/9$
Africa	$1/10$	$2/9$
North America	$9/100$	$2/9$
South America	$3/50$	$1/9$
Europe	$1/10$	$1/30$
Oceania	$1/200$	$1/18$
Sino-Soviet	$3/50$	$1/9$

d. Answers may vary slightly, due to rounding. The sums of both the population fractions and the area fractions should be approximately 1.

e. 1. The following sample response divides the delegates according to population:

<b>Region</b>	<b>Delegates</b>
Asia	29
Africa	5
North America	4
South America	3
Europe	5
Oceania	1
Sino-Soviet	2
<b>Total</b>	49

2. Answers will vary. The apportionment made in the exploration also was based on population.

1.4 Answers will vary. Sample response: To find the number of books each student should receive, divide the total number of books, 957, by the total number of years of post-secondary education, 27. This results in approximately 35.44 books for each year of education. Using this method and rounding to the nearest integer, Elizabeth should receive 354 books, Bill 142 books, Isabella 213 books, and Kareem 248 books.

\* \* \* \* \*

## Activity 2

Students compare three historical methods of apportionment for the U. S. House of Representatives: the Jefferson method, the Webster method, and Hamilton method.

### Materials List

- none

### Technology

- spreadsheet

### Exploration

(page 100)

- a. The standard divisor is  $2945/10 = 294.5$  people/representative .
- b. Sample table:

Region	Population	Standard Quota
A	813	2.76
B	652	2.21
C	1385	4.70
D	95	0.32
<b>Total</b>	2945	9.99

- c. Answers will vary. If students round each quota down to the previous whole number (assigning 1 to region D), they will assign a total of 9 representatives. If students round to the nearest integer, they may assign a total of 11 representatives (more than the available 10).
- d. **1-2.** The following table shows the results of rounding each quota down, according to the Jefferson method.

Region	Standard Quota	Rounded Quota	Representatives
A	2.76	2	2
B	2.21	2	2
C	4.70	4	4
D	0.32	0	1
<b>Total</b>	9.99	8	9

3. No. Not all 10 representatives were assigned.

- e. Using the Jefferson method and a divisor between 272 and 277 results in the assignment of all 10 representatives as follows:

Region	Rounded Quota	Representatives
A	2	2
B	2	2
C	5	5
D	0	1
<b>Total</b>	9	10

## Discussion

(page 102)

- a. Since the standard divisor in this case is  $P/d$ , the standard quota for a region can be expressed as follows:

$$\frac{R}{P/d} = \frac{R \cdot d}{P}$$

In other words, the standard quota for a region can be found by multiplying the total number of representatives by the fraction of the population residing in the region.

- b. Sample response: The standard quota for a region may not be a whole number. The number of delegates, however, must be a whole number. Depending on the method of rounding, the standard quota may not assign at least one representative to each region.
- c. Sample response: Yes. The method was based on population and each region received at least one representative.
- d. Sample response: Technology can round each quota down using the greatest integer function.
- e. Sample response: Use a spreadsheet to experiment with different values of the divisor until the rounded quotas add up to the exact number of representatives.

## Assignment

(page 102)

- 2.1 Because the use of the standard divisor does not assign all 12 representatives, students must modify the divisor. A value of 255 for example, results in the following rounded quotas.

Region	Population	Rounded Quota	Representatives
A	813	3	3
B	652	3	3
C	1385	5	5
D	95	0	1
<b>Total</b>	2945	11	12

2.2 a–b. Sample table:

Region	Standard Quota	Rounded Quota	Representatives
A	2.76	3	3
B	2.21	2	2
C	4.70	5	5
D	0.32	0	1
<b>Total</b>	9.99	10	11

c. Using a strategy similar to that described in Part e of the discussion, one possible modified divisor is 310.

Region	Rounded Quota	Representatives
A	3	3
B	2	2
C	4	4
D	0	1
<b>Total</b>	9	10

2.3 a. The Hamilton method assigns the first nine representatives as shown in the table below.

Region	Standard Quota	Rounded Quota	Representatives
A	2.76	2	2
B	2.21	2	2
C	4.70	4	4
D	0.32	0	1
<b>Total</b>	9.99	8	9

The remaining seat is assigned by ranking decimal parts of the standard quota from greatest to least. Since region A has the greatest decimal part, it receives the additional seat.

b. The approximate number of people per delegate in each region is shown below.

Region	People per Delegate
A	271
B	326
C	346
D	95

- 2.4 a. Sample response: The methods are similar since all three require finding a standard divisor and standard quotas. Each method deals with the decimal portion of the quotas in a different way. The Jefferson method truncates the quotas and modifies the divisor until the exact number of delegates is assigned. The Webster method rounds the quotas to the nearest integer and modifies the divisor until the exact number of delegates is assigned. The Hamilton method truncates the quotas, then assigns any remaining representatives according to a ranking of the decimal portions of the quotas. In all three methods, each region must receive at least one delegate.
- b. The following table shows the number of representatives assigned to each region using the three methods. In this case, the Webster and Hamilton methods result in the same apportionment.

Method	Number of Representatives			
	Region A	Region B	Region C	Region D
Jefferson	2	2	5	1
Webster	3	2	4	1
Hamilton	3	2	4	1

- \*2.5 The standard divisor is approximately  $2.20 \cdot 10^8$ . This results in the standard quotas shown in the table below.

Region	Population	Standard Quota
Asia	$6.32 \cdot 10^9$	28.64
Africa	$1.35 \cdot 10^9$	6.11
North America	$8.60 \cdot 10^8$	3.89
South America	$6.04 \cdot 10^8$	2.73
Europe	$1.04 \cdot 10^9$	4.71
Oceania	$5.40 \cdot 10^7$	0.24
Sino-Soviet	$5.84 \cdot 10^8$	2.64
<b>Total</b>	$1.08 \cdot 10^{10}$	48.96



Using the Jefferson method, a possible modified divisor is  $2.30 \cdot 10^8$ . Using the Webster method, a possible modified divisor is  $2.10 \cdot 10^8$ . The following table shows the number of delegates assigned to each region using the three methods.

Region	Method of Apportionment		
	Jefferson	Webster	Hamilton
Asia	30	27	28
Africa	6	6	6
North America	4	5	5
South America	4	4	4
Europe	2	3	3
Oceania	1	1	1
Sino-Soviet	2	3	2
<b>Total</b>	49	49	49

\* \* \* \* \*

- \*2.6 The following table shows the number of representatives assigned to each region using the three methods. In this case, the Webster and Hamilton methods result in the same apportionment.

Method	Number of Representatives			
	Region A	Region B	Region C	Region D
Jefferson	5	4	10	1
Webster	6	4	9	1
Hamilton	6	4	9	1

- 2.7
- $1.58 \cdot 10^4$
  - The standard divisor is 683,544.
  - The corresponding standard quotas are shown in the table below.

Region	Standard Quota
Asia	9245.93
Africa	1975.00
North America	1258.14
South America	883.63
Europe	1521.48
Oceania	79.00
Sino-Soviet	854.37

- d. The following table shows an apportionment made using the Jefferson method. The modified divisor must be in the interval  $[684,131, 684,168]$ .

Region	Allotment ( $1 \cdot 10^6$ kL)
Asia	9237
Africa	1973
North America	1257
South America	882
Europe	1520
Oceania	78
Sino-Soviet	853

The table below shows the results using the Webster method. The modified divisor must be in the interval  $[684,316, 684,390]$ .

Region Name	Allotment ( $1 \cdot 10^6$ kL)
Asia	9235
Africa	1973
North America	1257
South America	883
Europe	1520
Oceania	79
Sino-Soviet	853

\* \* \* \* \*

## **Research Project**

(page 104)

This research project allows students to explore the relationship between methods of apportionment and the course of U.S. history. In the past, interest groups and political parties have often used the mathematics of apportionment to manipulate the assignment of representatives. In the presidential election of 1876, for example, Samuel L. Tilden (who won the popular vote) was defeated in the electoral college 185 to 184 by Rutherford B. Hayes. Hayes' one-vote margin of victory may be traced to the assignment of nine seats by political expediency rather than the Hamilton method.

### Activity 3

In this activity, students simplify rational expressions and are introduced to the geometric mean. They examine a method for minimizing the impact of assigning another representative to a region by comparing “relative differences.”

#### Materials List

- none

#### Technology

- symbolic manipulator (optional)

#### Discussion 1

(page 105)

- a. The number of people represented by each delegate in Washington can be expressed as  $W/w$ , where  $W$  is Washington’s population and  $w$  its number of representatives.
- b. The number of people represented by each delegate in Massachusetts is less than the number of people represented by each delegate in Washington, as shown below.

$$\frac{M}{m} = \frac{6,029,050}{10} \approx 602,905 < \frac{W}{w} = \frac{4,887,945}{8} \approx 610,993$$

Therefore, the people of Massachusetts were better represented in the U.S. House.

- c. The relative difference can be calculated as follows:

$$\frac{610,993 - 602,905}{602,905} \approx 0.013$$

- d. If  $M/m$  is greater than  $W/w$ , then the number of people per delegate is higher for Massachusetts than for Washington. The relative difference can be expressed as follows:

$$\frac{\frac{M}{m} - \frac{W}{w}}{\frac{W}{w}}$$

## Exploration

(page 106)

- a. 1. If Massachusetts receives the additional seat, then the number of people represented by each delegate would be:

$$\frac{M}{m+1} = \frac{6,029,050}{11} \approx 548,095$$

The relative difference would be:

$$\frac{610,993 - 548,095}{548,095} \approx 0.115$$

2. If Washington receives the additional seat, then the number of people represented by each delegate would be:

$$\frac{W}{w+1} = \frac{4,887,945}{9} = 543,105$$

The relative difference would be:

$$\frac{602,905 - 543,015}{543,015} \approx 0.110$$

- b. To minimize the effect of adding another representative, Washington should receive the additional seat.

- c. 1. If New Jersey receives 14 representatives, then the number of people represented by each delegate would be  $7,748,634/14 \approx 553,474$ .

The relative difference would be:

$$\frac{610,993 - 553,474}{553,474} \approx 0.104$$

2. If New Jersey does not receive an additional seat, the number of people represented by each delegate would be  $7,748,634/13 \approx 596,049$ .

The relative difference would be:

$$\frac{596,049 - 543,105}{543,105} \approx 0.097$$

3. The relative difference would be:

$$\frac{602,905 - 553,474}{553,474} \approx 0.089$$

4. The relative difference would be:

$$\frac{596,049 - 548,095}{548,095} \approx 0.087$$

- d. Sample response: Using relative difference, Massachusetts would be favored over New Jersey to receive the final representative. However, since Washington is favored over both Massachusetts and New Jersey, Washington should receive the final representative.
- e. 1. When determining which of 4 states should receive an additional representative, 12 relative differences would need to be calculated.
2. When determining which of 5 states should receive an additional representative, 20 relative differences would need to be calculated.

**Note:** In general, when determining which of  $n$  states should receive an additional representative, the number of relative differences to be calculated would be  ${}_n P_2$ .

## Discussion 2

(page 107)

- a. The expression represents the relative difference between Washington and Massachusetts if Massachusetts receives an additional seat.
- b. In this case,  $M/m$  is greater than  $W/(w+1)$ . The relative difference can be expressed as follows:

$$\frac{\frac{M}{m} - \frac{W}{w+1}}{\frac{W}{w+1}}$$

- c. 1.  $0.115 > 0.110$
2. Sample response:

$$\frac{\frac{W}{w} - \frac{M}{m+1}}{\frac{M}{m+1}} > \frac{\frac{M}{m} - \frac{W}{w+1}}{\frac{W}{w+1}}$$

## Assignment

(page 107)

- 3.1 a.  $21/20$
- b.  $39/80$
- c.  $207/40$
- d.  $\frac{4(x+1)}{6} - 1 = \frac{2x-1}{3}$

**\*3.2**

- a. 1. Sample response: Subtraction requires fractions with a common denominator. In the original inequality, the difference was written as one fraction with the same denominator.
2. Sample response: When a fraction has the same value in the numerator as the denominator, the fraction is equivalent to 1.
- b. The right-hand side of the inequality can be simplified as follows:

$$\frac{\frac{M}{m} - \frac{W}{w+1}}{\frac{W}{w+1}} = \frac{\frac{M}{m}}{\frac{W}{w+1}} - 1$$

- c. By adding 1 to both sides, the inequality can be written as follows:

$$\frac{\frac{W}{m}}{\frac{M}{m+1}} > \frac{\frac{M}{m}}{\frac{W}{w+1}}$$

- d. 1. Since the existing denominators are both positive, their product (the common denominator) also is positive.
2.  $\frac{W^2}{w(w+1)} > \frac{M^2}{m(m+1)}$
- e. 1. The numerators of both fractions are positive. Since the original denominators and the common denominator are all positive, their products (the resulting denominators) also are positive. Because both numerators and both denominators are positive, both fractions are positive.
- 2–3. Since the fractions are both positive, the inequality can be written as shown below by taking the principle square root of both sides.

$$\frac{W}{\sqrt{w(w+1)}} > \frac{M}{\sqrt{m(m+1)}}$$

4. Sample response: If the fractions were negative, the square root would be undefined.
- f. 1. The two inequalities simplify as follows:

$$\frac{\frac{W}{m} - \frac{M}{m+1}}{\frac{M}{m+1}} > \frac{\frac{M}{m} - \frac{W}{w+1}}{\frac{W}{w+1}} \qquad \frac{W}{\sqrt{w(w+1)}} > \frac{M}{\sqrt{m(m+1)}}$$

$0.115 > 0.110$   $576,050 > 574,847$

2. Sample response: The inequality that compares priority numbers is easier to use since there are no complex fractions.

**\*3.3** a. The arithmetic mean of two numbers is found by adding the two numbers, then dividing the sum by 2.

b. 1.  $\sqrt{8 \cdot 32} = 16$

2.  $\sqrt{4 \cdot 32} \approx 11.31$

**3.4** a. The geometric mean is found in the denominators of the inequality that compares priority numbers:

$$\frac{W}{\sqrt{w(w+1)}} > \frac{M}{\sqrt{m(m+1)}}$$

b. Sample response: The state with the greater priority number would be assigned the additional delegate.

\*\*\*\*\*

**3.5** a. Since  $\angle BCD$  and  $\angle BDA$  are both complementary to  $\angle BDC$ , their measures are equal. By the AAA Property,  $\triangle ABD$  is similar to  $\triangle DBC$ . Since  $\triangle ABD \sim \triangle DBC$ , the following proportion is true:

$$\frac{a}{h} = \frac{h}{b}$$

Solving for  $h$  yields:  $h^2 = a \cdot b$ ;  $h = \sqrt{a \cdot b}$ .

b. 1.  $h \approx 3.27$

2.  $b \approx 3.11$

3.  $a \approx 8.70$

\*\*\*\*\*

(page 110)

## Activity 4

In this activity, students are introduced to the Huntington method of apportionment, which uses “priority numbers” to assign representatives. These priority numbers are related to the relative differences developed in Activity 3.

### Materials List

- none

### Technology

- spreadsheet

## Exploration 1

(page 110)

- a. Sample table:

Region	Population	Rounded Quota	(Rounded Quota) + 2
A	264	3	5
B	481	5	7
C	330	4	6
D	1652	17	19
E	1442	14	16
F	477	5	7
<b>Total</b>	4646	48	60

- b. As shown in the table above, the sum of the values in the right-hand column is 60. Some students may argue that this means 60 priority numbers must be calculated. However, since each of the 6 regions must receive at least 1 representative, the actual number of priority numbers to be calculated is  $60 - 6 = 54$ .

## Discussion 1

(page 112)

- a. Column C displays the possible numbers of representatives for the six regions. In each row, the possible number of representatives is the current number of representatives plus one additional seat.
- b. 1. Sample response: The rounded quota for region A is 3. Adding 2 to the rounded quota gives a value of 5. This is the most representatives possible for that region. Since each region must receive at least one representative, priority numbers have to be calculated for the other four possible representatives.
2. Since the rounded quota is 4, the most representatives possible is  $4 + 2 = 6$ . Since no priority number is necessary to award the first representative, 5 priority numbers should be calculated for region C.
- c. Students should base the spreadsheet formula on the formula given in the previous historical note. Using Microsoft Excel, for example, the following formula calculates the priority number in cell E2:
- $$=D2/(SQRT(B2*C2))$$
- d. The highest priority number in the portion of the spreadsheet shown is 340.12 for region B. Students may recognize that priority numbers for a given number of representatives are larger for regions with larger populations. Since the priority numbers for regions D, E, and F are not shown on the partial spreadsheet, 340.12 is not the largest priority number among the six regions.



## Exploration 2

(page 112)

a. Sample spreadsheet:

	<b>A</b>	<b>B</b>	<b>C</b>	<b>D</b>	<b>E</b>
<b>1</b>	<b>Region</b>	<b>Current No. of Reps.</b>	<b>Possible No. of Reps.</b>	<b>Pop.</b>	<b>Priority Number</b>
<b>2</b>	A	1	2	264	186.676
<b>3</b>	A	2	3	264	107.778
<b>4</b>	A	3	4	264	76.210
<b>5</b>	A	4	5	264	59.032
<b>6</b>	B	1	2	481	340.118
<b>7</b>	B	2	3	481	196.367
<b>8</b>	B	3	4	481	138.853
<b>9</b>	B	4	5	481	107.555
<b>10</b>	B	5	6	481	87.818
<b>11</b>	B	6	7	481	74.220
<b>12</b>	C	1	2	330	233.345
<b>13</b>	C	2	3	330	134.722
<b>14</b>	C	3	4	330	95.263
<b>15</b>	C	4	5	330	73.790
<b>16</b>	C	5	6	330	60.249
<b>17</b>	D	1	2	1652	1168.140
<b>18</b>	D	2	3	1652	674.426
<b>19</b>	D	3	4	1652	476.891
<b>20</b>	D	4	5	1652	369.398
<b>21</b>	D	5	6	1652	301.613
<b>22</b>	D	6	7	1652	254.909
<b>23</b>	D	7	8	1652	220.758
<b>24</b>	D	8	9	1652	194.690
<b>25</b>	D	9	10	1652	174.136
<b>26</b>	D	10	11	1652	157.512
<b>27</b>	D	11	12	1652	143.788
<b>28</b>	D	12	13	1652	132.266
<b>29</b>	D	13	14	1652	122.454
<b>30</b>	D	14	15	1652	113.999
<b>31</b>	D	15	16	1652	106.636
<b>32</b>	D	16	17	1652	100.167
<b>33</b>	D	17	18	1652	94.439
<b>34</b>	D	18	19	1652	89.330
<b>35</b>	E	1	2	1442	1019.648
<b>36</b>	E	2	3	1442	588.694
<b>37</b>	E	3	4	1442	416.270
<b>38</b>	E	4	5	1442	322.441
<b>39</b>	E	5	6	1442	263.272

40	E	6	7	1442	222.505
41	E	7	8	1442	192.695
42	E	8	9	1442	169.941
43	E	9	10	1442	152.000
44	E	10	11	1442	137.489
45	E	11	12	1442	125.510
46	E	12	13	1442	115.452
47	E	13	14	1442	106.888
48	E	14	15	1442	99.507
49	E	15	16	1442	93.081
50	F	1	2	477	337.290
51	F	2	3	477	194.734
52	F	3	4	477	137.698
53	F	4	5	477	106.660
54	F	5	6	477	87.088
55	F	6	7	477	73.603

- b. After each region receives one representative, the remaining 39 representatives are assigned by selecting the region with the highest priority number, the region with the next highest priority number, and so on, until all remaining seats have been assigned.

Students may sort the entire spreadsheet according to priority number, greatest to least, then assign representatives to the regions in the first 39 rows. **Note:** Depending on how the spreadsheets were created, students may lose some cell values when sorting. On some technology, this may be avoided by selecting manual calculation rather than automatic calculation, so that the spreadsheet does not recalculate after sorting.

The table below shows the results of an apportionment by the Huntington method.

Region	Population	Representatives
A	264	3
B	481	5
C	330	3
D	1652	15
E	1442	14
F	477	5
<b>Total</b>	4646	45

**Discussion 2**

(page 113)

- a. If  $m$  and  $w$  represent the current number of representatives assigned to Massachusetts and Washington, respectively, the priority numbers can be expressed as follows:

$$\frac{M}{\sqrt{m(m+1)}} \qquad \frac{W}{\sqrt{w(w+1)}}$$

- b. The inequality includes the expressions for both priority numbers. The denominator of each priority number is the geometric mean  $\sqrt{ab}$ , where  $a$  is the current number of representatives and  $b$  is the possible number of representatives if one additional seat is assigned.
- c. The following table shows the standard quotas for each region along with the number of representatives assigned using the Huntington method.

Region	Population	Standard Quota	Representatives
A	264	2.557	3
B	481	4.659	5
C	330	3.196	3
D	1652	16.001	15
E	1442	13.967	14
F	477	4.620	5
<b>Total</b>	4646	45.000	45

In this case, region D received fewer representatives (15) than its standard quota (16). In *Fair Representation: Meeting the Ideal of One Man, One Vote*, Balinski and Young estimate that the Huntington method will result in some regions not meeting quota in 29 of 10,000 apportionments. On the other hand, the Webster method results in some regions not meeting quota in only 6 out of 10,000 apportionments.

**Assignment**

(page 113)

- 4.1 a–b. As shown in the table below, the Hamilton method assigns one more representative to region D, and one less to region A.

Region	Number of Representatives	
	Huntington method	Hamilton method
A	3	2
B	5	5
C	3	3
D	15	16
E	14	14
F	5	5

4.2 The resulting apportionment is shown in the following table.

Region	Representatives
A	3
B	5
C	4
D	18
E	15
F	5
<b>Total</b>	50

\*4.3 The use of the Huntington method results in the apportionment below.

Region	Population	Representatives
Asia	$6.32 \cdot 10^9$	27
Africa	$1.35 \cdot 10^9$	6
North America	$8.60 \cdot 10^8$	4
South America	$6.04 \cdot 10^8$	3
Europe	$1.04 \cdot 10^9$	5
Oceania	$5.40 \cdot 10^7$	1
Sino-Soviet	$5.84 \cdot 10^8$	3

\* \* \* \* \*

4.4 If one person moves as indicated, region D would gain one representative, while region F would lose one.

\* \* \* \* \*

## Answers to Summary Assessment

(page 115)

1. Canada's total population in 1991 was 27,296,859. The table below shows the standard quota for each region and the results of the four apportionment methods.

Province	Standard Quota	Jeff.	Web.	Ham.	Hunt.
Alberta	27.510	27	27	27	28
British Columbia	35.470	36	35	35	34
Manitoba	11.801	12	12	12	12
New Brunswick	7.823	7	8	8	8
Newfoundland	6.144	6	6	6	6
Nova Scotia	9.726	9	10	9	10
Ontario	108.988	110	109	109	109
P. E. I.	1.402	1	1	1	1
Quebec	74.525	75	74	75	74
Saskatchewan	10.687	10	11	11	11
N. W. T.	0.623	1	1	1	1
Yukon	0.300	1	1	1	1
<b>Total</b>	295.000	295	295	295	295

**Note:** None of these apportionments match the actual representation of each region in the House of Commons, shown in the table below.

Province	Delegates	Province	Delegates
Alberta	26	Ontario	99
British Columbia	32	P. E. I.	4
Manitoba	14	Quebec	75
New Brunswick	10	Saskatchewan	14
Newfoundland	7	N. W. T.	2
Nova Scotia	11	Yukon	1

2. Some students may argue that the Hamilton method is best because it requires the fewest calculations. Others may argue that the Jefferson and Webster methods are better because it is not necessary to consider the ranking of the fractional parts of the standard quotas. Still others may argue that the Huntington method is best because there is no need to modify the divisor. However, this method results in British Columbia receiving one less seat than its standard quota.

## ***Module Assessment***

Center City is governed by a city council with 12 seats. These seats are shared among five districts, and each district must receive at least one seat. The population of each district is shown in the table below.

<b>District</b>	<b>Population</b>
North	12,015
East	1512
West	6218
South	5001
Central	4587

1.
  - a. Determine the standard divisor in this situation.
  - b. Calculate the standard quota for each region.
2. Use either the Jefferson, Webster, or Hamilton method to apportion the 12 seats.
3.
  - a. Use the Huntington method to apportion the 12 seats.
  - b. Describe how the geometric mean is used to calculate priority numbers. Include an example in your response.

## Answers to Module Assessment

1. a. The standard divisor is  $29,333/12 \approx 2444.42$ .
- b. The following table shows the standard quota for each region.

District	Population	Standard Quota
North	12,015	4.92
East	1512	0.62
West	6218	2.54
South	5001	2.05
Central	4587	1.88
<b>Total</b>	<b>29,333</b>	<b>12.01</b>

2. As shown in the table below, the apportionment is the same for each of the three methods. For the Jefferson method, one possible modified divisor is 2100. For the Webster method, one possible modified divisor is 2600.

District	Number of Representatives		
	Jefferson	Webster	Hamilton
North	5	5	5
East	1	1	1
West	2	2	2
South	2	2	2
Central	2	2	2

3. a. Using the Huntington method, the apportionment is the same as shown in the table above.
- b. Sample response: The geometric mean,  $1/\sqrt{ab}$ , is used to calculate the multiplier for a region's population. For example, the multiplier for a possible fourth seat is:

$$\frac{1}{\sqrt{3 \cdot 4}}$$

## Selected References

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# Flashbacks

## Activity 1

- 1.1** Write each of the following numbers in decimal form.
- $2.54 \cdot 10^6$
  - $5.67 \cdot 10^{-3}$
  - $7.04 \cdot 10^0$
- 1.2** Write each of the following numbers using scientific notation.
- 0.0045
  - 1,230,000,000
- 1.3** Calculate each of the following. Express the result in both scientific notation and decimal notation.
- $\frac{(6.34 \cdot 10^5)}{(3.65 \cdot 10^2)}$
  - $(2.4 \cdot 10^3) \cdot (3.7 \cdot 10^7)$
  - $(2.3 \cdot 10^3) + (4.5 \cdot 10^3) + (6.1 \cdot 10^4)$

## Activity 2

- 2.1** Round each of the following numbers down to the previous integer.
- 36.589
  - 6987.34
- 2.2** Round each of the numbers below to the nearest integer.
- 2.65
  - 0.957
- 2.3** Rewrite the following expression using multiplication by a fraction instead of division by a fraction.

$$A \div \frac{B}{C}$$

- 2.4** Solve for  $x$  in each of the following equations.
- $\frac{24}{x} + \frac{16}{x} = 10$
  - $\frac{12}{x} + \frac{18}{x} + \frac{30}{x} = 10$

### Activity 3

- 3.1** In the expression below,  $a = 2$ ,  $b = 4$ ,  $c = 6$ ,  $d = 8$ ,  $e = 10$ , and  $f = 12$ . Write this expression as a fraction in simplest terms.

$$\frac{\frac{a}{b} - \frac{e}{f}}{\frac{c}{d}}$$

- 3.2** Calculate each of the following:
- $(\sqrt{234})^2$
  - $\frac{(\sqrt{234})^2}{4 \cdot 3}$
- 3.3** Solve each of the following inequalities and check your solutions.
- $6x > 12$
  - $-6x > 12$
  - $6x > -12$

### Activity 4

- 4.1** In Parts **a** and **b** below, use estimation to determine which of the two values is larger.
- $326 \cdot \frac{1}{\sqrt{4 \cdot 3}}$  or  $1243 \cdot \frac{1}{\sqrt{6 \cdot 5}}$
  - $12,326 \cdot \frac{1}{\sqrt{7 \cdot 6}}$  or  $3246 \cdot \frac{1}{\sqrt{5 \cdot 4}}$
- 4.2** A total population of 2564 people in several regions must elect 12 delegates to a governing body.
- Find the standard divisor in this situation.
  - One of the regions has a population of 1052. Determine the standard quota for this region.

## Answers to Flashbacks

### Activity 1

- 1.1 a. 2,540,000  
b. 0.00567  
c. 7.04
- 1.2 a.  $4.5 \cdot 10^{-3}$   
b.  $1.23 \cdot 10^9$
- 1.3 a.  $1.74 \cdot 10^3$ ; 1740  
b.  $8.88 \cdot 10^{10}$ ; 88,800,000,000  
c.  $6.78 \cdot 10^4$ ; 67,800

### Activity 2

- 2.1 a. 36  
b. 6987
- 2.2 a. 3  
b. 1
- 2.3  $A \div \frac{B}{C} = A \cdot \frac{C}{B}$
- 2.4 a.  $x = 4$   
b.  $x = 6$

### Activity 3

- 3.1  $-4/9$
- 3.2 a. 234  
b. 19.5
- 3.3 a.  $x > 2$   
b.  $x < -2$   
c.  $x > -2$

#### Activity 4

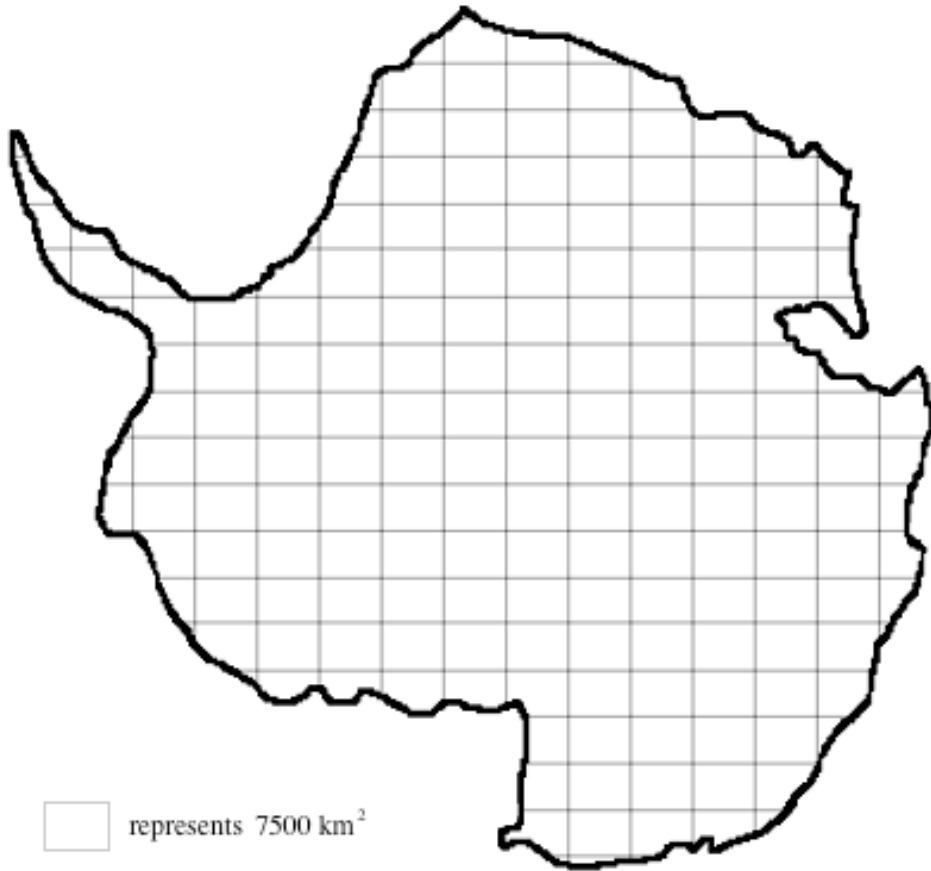
4.1 a.  $1243 \cdot \frac{1}{\sqrt{6 \cdot 5}}$

b.  $12,326 \cdot \frac{1}{\sqrt{7 \cdot 6}}$

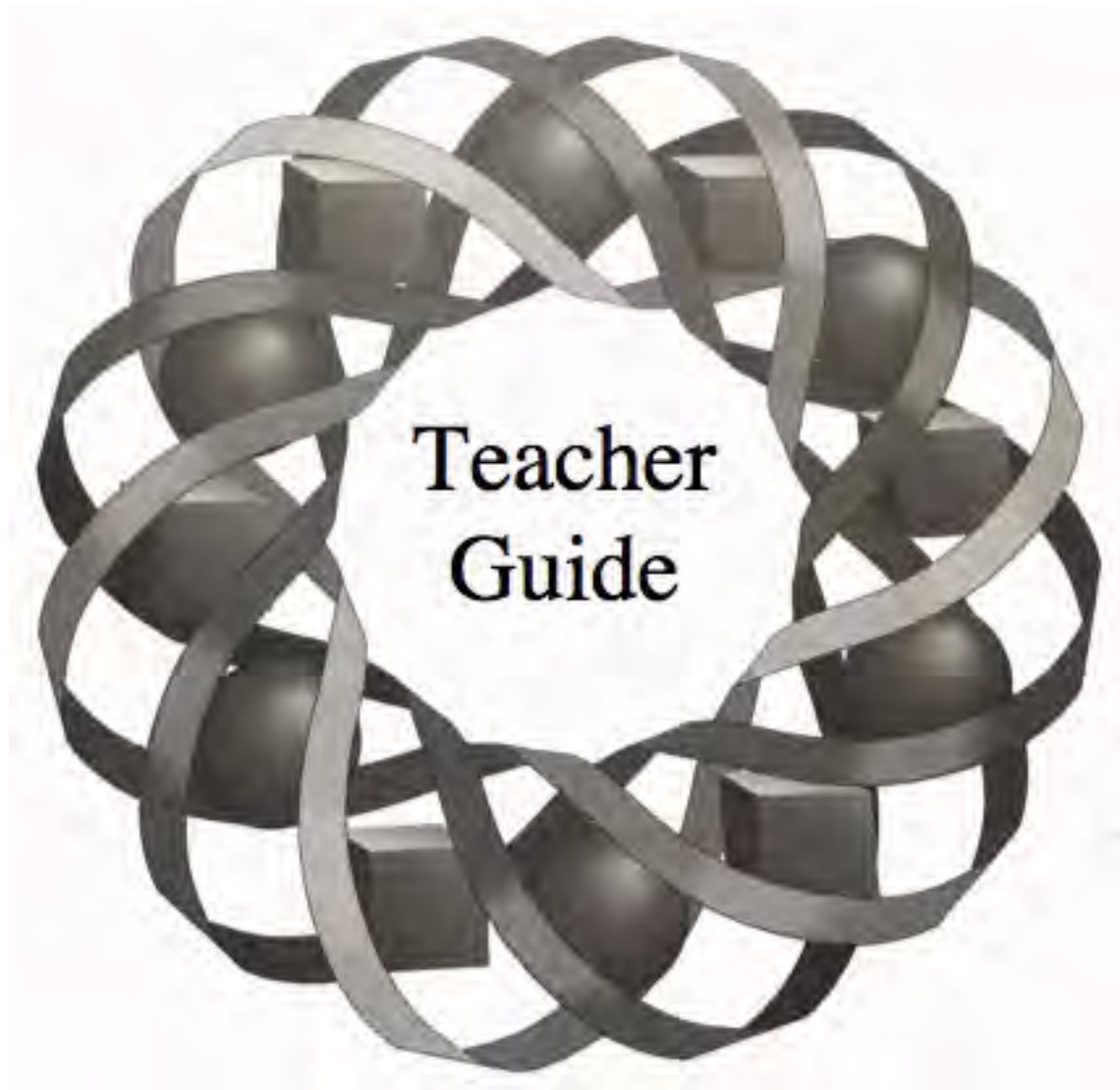
4.2 a. The standard divisor is 213.67.

b. The standard quota for this region is 4.92.

# Map of Antarctica



# What Are My Child's Chances?



Did you ever wonder why your eyes match your mother's? Or why you have your father's chin? In this module, you study the genetics of several inherited traits.

*Patricia Bean • Bonnie Eichenberger*



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## Teacher Edition

# What Are My Child's Chances?

### Overview

Students examine various probability models and their applications to the study of genetics. They use probability to solve problems involving genetic traits and disorders.

### Objectives

In this module, students will:

- collect data and calculate experimental probabilities
- use Punnett squares and tree diagrams to determine sample spaces
- determine the theoretical probabilities of events
- compare experimental probabilities and theoretical probabilities
- investigate formulas for determining the probability of two or more events
- identify complementary, independent, dependent, and mutually exclusive events
- simulate a situation involving independent events.

### Prerequisites

For this module, students should know:

- how to determine sample spaces
- how to make tree diagrams
- how to use the fundamental counting principle
- how to draw Venn diagrams
- how to calculate expected value.

### Time Line

Activity	1	2	3	4	5	Summary Assessment	Total
Days	2	2	2	2	2	1	11

## Materials Required

Materials	Activity				
	1	2	3	4	5
coins					X

## Teacher Note

This module discusses human characteristics from a biological perspective and uses the vocabulary of genetics. You may wish to present this module in coordination with a biology class or with the cooperation of a science teacher.

The science of genetics is extremely complex. To simplify both the mathematical and scientific focuses of this module, the five traits examined in Activity 1 are assumed to be independent of each other. Throughout the module, each allele in each parent's genotype is assumed to have an equal chance of being passed on to the offspring.



# What Are My Child's Chances

## Introduction

(page 121)

The introduction provides an elementary explanation of genetics and genetic disorders.

(page 121)

## *Activity 1*

Students use five inherited characteristics to explore experimental probability.

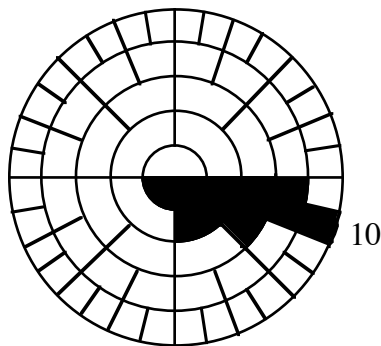
## Materials List

- none

## Exploration

(page 121)

- a–d. Students determine and record their personal numbers. For example, a personal number of 10 is reached by moving through the rings as shown in the diagram below:



- e. Sample data:

Trait	No. with Trait	No. without Trait
mid-digit hair	18	7
widow's peak	17	8
free earlobes	18	7
rolled tongue	19	6
folded hands	20	5

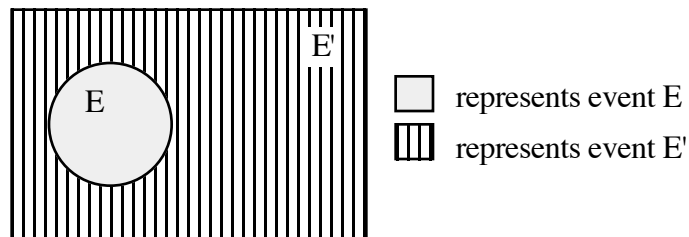
- f. The experimental probabilities in the table below were calculated using the sample data given in Part e.

Trait	Experimental Probability
mid-digit hair	$18/25$
widow's peak	$17/25$
free earlobes	$18/25$
rolled tongue	$19/25$
folded hands	$20/25 = 4/5$

## Discussion

(page 123)

- a. Answers will vary, depending on the class data. Sample response: No. Since the probability of having each trait is not the same as the probability of having each of the other traits, the traits are not equally likely.
- b. For each of the five traits, there are two possibilities: yes or no. Therefore, the number of possible outcomes is  $2^5$ , or 32.
- c. Since order is unimportant in this situation, there are 10 different pairs of traits. Using the letters A, B, C, D, and E to designate each trait, the 10 pairs are: AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE.
- d. Sample Venn diagram:



- e. Given the experimental probability of having a specific trait, the experimental probability of not having that trait can be calculated by subtracting the known probability from 1. For example, if the experimental probability of having a widow's peak is  $17/25$ , the experimental probability of not having this trait is  $1 - 17/25 = 8/25$ .
- f. Sample response: If the class is representative of the school population, the product of the school population and the experimental probability of a trait would give a prediction for the number of students with that trait.

## Assignment

(page 124)

- \*1.1 a.** Answers will vary, depending on class data. The probabilities in the table below were calculated using the sample data given in the exploration.

Trait	Experimental Probability of Having Trait	Experimental Probability of Not Having Trait
mid-digit hair	18/25	7/25
widow's peak	17/25	8/25
free earlobes	18/25	7/25
rolled tongue	19/25	6/25
folded hands	20/25 = 4/5	5/25 = 1/5

- b.** Answers will vary. Students should multiply the experimental probability of having free earlobes by the number of students in the school population.
- c.** Because the two events are complementary, the sum of their probabilities is 1.
- 1.2** Answers will vary. Students may or may not believe that their class is representative of the national population. Some may argue that the sample size is inadequate to make predictions.
- 1.3** Since the two events are complementary, the probability of not having a widow's peak in this class is  $1 - 0.7 = 0.3$ .
- \*1.4** The event that is complementary to the trait "folded hands" is "not having folded hands," which means that the right thumb falls naturally on top when hands are folded so that fingers interlace.

\* \* \* \* \*

**1.5 a.** 
$$\frac{542,000 + 62,000 + 36,000}{2,041,000} \approx 0.31$$

**b.** 
$$\frac{62,000 + 36,000}{2,041,000} \approx 0.05$$

**c.** 
$$1 - \frac{428,000}{2,041,000} \approx 0.79$$

**\*1.6 a.** 
$$\frac{46,078}{12,232,000} \approx 0.004$$

- b.** Since the total number of deaths shown in the table is 2,972,965, the percentage was:

$$\frac{1,716,056}{2,972,965} \approx 0.577$$

- c. Since the total number of men represented in the table is 74,253,000, the probability is:

$$\frac{2,972,965}{74,253,000} \approx 0.040$$

- d. Sample response:

$$1 - \frac{3278}{21,564,000} \approx 0.9998$$

\*\*\*\*\*

(page 126)

## Activity 2

Students examine the theoretical probabilities of the inherited traits encountered in Activity 1.

### Materials List

- none

### Teacher Note

Some students may find it helpful to develop a reference sheet of unfamiliar biological terms, such as *allele* and *genotype*. These terms are described in brief in the module summary of the student edition.

### Exploration

(page 126)

- a. Answers will vary. The table below shows the possible genotypes for each cross.

Cross	Possible Genotypes
FF × FF	FF
FF × ff	Ff
FF × Ff	FF, Ff
Ff × ff	Ff, ff
Ff × Ff	FF, Ff, ff
ff × ff	ff

- b. Some students may have used tree diagrams to show all the possibilities. Others may have listed the combinations in some pattern in order to consider all possibilities.

- c. Answers will vary. The following sample response shows the Punnett square for the cross  $FF \times Ff$ . The two possible genotypes are  $FF$  and  $Ff$ . The only possible phenotype is free earlobes.

	<b>Ff</b>	
	<b>F</b>	<b>f</b>
<b>FF</b>	<b>F</b>	<b>Ff</b>
	<b>F</b>	<b>Ff</b>

**Note:** To ensure that all possible crosses are investigated, you may wish to assign specific crosses to each student or group.

- d. Answers will vary. For the sample response given in Part c, the probability of each of the two possible genotypes is  $1/2$ .
- e. Answers will vary. For the sample response given in Part c, the probability of the phenotype free earlobes is 1.

### Discussion

(page 128)

- a. 1. Answers will vary. Sample response: For the cross  $Ff \times Ff$ , there are three possible genotypes:  $FF$ ,  $Ff$ , and  $ff$ .
2. Sample response: For the cross  $Ff \times Ff$ , there are two possible phenotypes: free earlobes and attached earlobes.
- b. Sample response: The genotypes of the parents determine the possible genotypes and phenotypes for the child. For example, in the cross  $ff \times ff$ , the only possible genotype is  $ff$  and the only possible phenotype is attached earlobes.
- c. The parents' genotypes determine the sample space of possible genotypes for the child. These genotypes, in turn, determine the possible phenotypes for the child. The frequency of each phenotype in the sample space determines the probability of that phenotype.

### Assignment

(page 129)

- \*2.1 a. The Punnett square for this cross is shown below.

	<b>ff</b>	
	<b>f</b>	<b>f</b>
<b>FF</b>	<b>F</b>	<b>Ff</b>
	<b>F</b>	<b>Ff</b>

- b. The only genotype possible from this cross is  $Ff$ . Its probability is 1.
- c. The only phenotype possible is free earlobes, the dominant trait.
- d. The probability of free earlobes is 1.

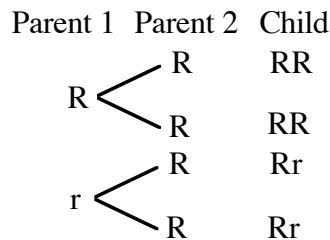
- 2.2 a. The Punnett square for this cross is shown below.

	<b>ff</b>	
	<b>f</b>	<b>f</b>
<b>Ff</b>	<b>Ff</b>	<b>Ff</b>
	<b>f</b>	<b>ff</b>

- b. The two possible genotypes are Ff and ff. Each has a probability of 1/2.
- c. Since F is the dominant trait, the genotype Ff results in the phenotype free earlobes. The genotype ff results in the phenotype attached earlobes.
- d. Each phenotype has a probability of 1/2.
- 2.3 a. As shown in the following Punnett square, the probability of the genotype RR is 1/2.

	<b>RR</b>	
	<b>R</b>	<b>R</b>
<b>Rr</b>	<b>RR</b>	<b>RR</b>
	<b>r</b>	<b>Rr</b>

- b. As shown in the tree diagram below, the probability of the genotype rr is 0.



- c. Because both of the possible genotypes result in the same phenotype, the probability that a child of these parents has the ability to roll the tongue is 1.
- \*2.4 a. Since having a widow's peak is a dominant trait, the father's genotype is Ww.
- b. As shown in the following Punnett square, the two possible genotypes are Ww and ww.

	<b>ww</b>	
	<b>w</b>	<b>w</b>
<b>Ww</b>	<b>Ww</b>	<b>Ww</b>
	<b>w</b>	<b>ww</b>

- c. Since the genotype ww results in a child with no widow's peak, the probability of this phenotype is  $2/4 = 1/2$ .

- 2.5 a. As shown in the Punnett square below, the probability of a child receiving one dominant and one recessive allele (genotype Ww) is  $2/4 = 1/2$ .

		<b>Ww</b>	
		<b>W</b>	<b>w</b>
<b>Ww</b>	<b>W</b>	WW	Ww
	<b>w</b>	Ww	ww

- b. Since having a widow's peak is a dominant trait, both Ww and WW result in the same phenotype.
- c. The three outcomes that result in a widow's peak are WW, Ww, and Ww. The genotype ww results in no widow's peak. The ratio of the probabilities is 3/1.

\* \* \* \* \*

- \*2.6 a. Attached earlobes is a recessive trait. If F represents the allele for free earlobes, and f represents the allele for attached earlobes, then Deirdre's genotype must be ff. The only possible cross involving parents with free earlobes that results in this genotype is  $Ff \times Ff$ . As shown in the following Punnett square, the possible genotypes are FF, Ff, and ff.

		<b>Ff</b>	
		<b>F</b>	<b>f</b>
<b>Ff</b>	<b>F</b>	FF	Ff
	<b>f</b>	Ff	ff

- b. The probability that a child of Deirdre's parents has attached earlobes is  $1/4$ .

- 2.7 a. The Punnett square for the cross of two carriers is shown below.

		<b>Cc</b>	
		<b>C</b>	<b>c</b>
<b>Cc</b>	<b>C</b>	CC	Cc
	<b>c</b>	Cc	cc

- b. The probability is  $1/4$ .
- c. Since the probability of being a carrier is  $1/25$ , the probability of not being a carrier is  $1 - 1/25 = 24/25$ .

\* \* \* \* \*

### Activity 3

Students explore probabilities for more than one event. The mathematics note introduces a formula for finding the probability of multiple independent events.

#### Materials List

- none

#### Discussion 1

(page 131)

- a.
1. Since the father has two different alleles, and each has an equal chance of being passed on, the probability is  $1/2$ .
  2. Since the mother has two identical alleles, the probability is 1.
  3. The probability that a child of this couple will have the genotype RR is  $1/2$ .
- b. Sample response: No. The probability does not change because the allele passed on by the father is not influenced by the mother's genotype.
- c. Since receiving a particular allele from the father is independent of receiving a particular allele from the mother, the probability of a child receiving a given genotype equals the product of the probabilities for receiving each allele from each parent.

For example, the probability that this child will receive the genotype RR can be calculated as follows:

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

#### Exploration

(page 132)

Students examine the relationship among  $P(A)$ ,  $P(B)$ , and  $P(A \text{ and } B)$  when A and B are independent events.

- a–b. The following table shows sample data for the traits folded hands and rolled tongue.

Trait A: folded hands		
Trait B: rolled tongue		
Total Number in Class:		
Trait	Frequency	Experimental Probability
A	20	$20/25$
B	19	$19/25$
A and B	16	$16/25$



- c. Students repeat Part **b** for each of the 10 possible pairs of two traits.
- d. For each pair of traits, students should use experimental probabilities to approximate theoretical probabilities. If  $P(A) \cdot P(B) \approx P(A \text{ and } B)$ , then they should observe that traits A and B appear to be independent.

## Discussion 2

(page 132)

- a.
  - 1. Answers may vary, depending on class data.
  - 2. If  $P(A) \cdot P(B)$  is not approximately equal to  $P(A \text{ and } B)$ , students should observe that traits A and B appear to be dependent.
- b.
  - 1. Sample response: Having blond hair and being male appear to be independent events. This could be demonstrated by using the experimental probability of each trait to approximate the theoretical probabilities. If  $P(A) \cdot P(B) \approx P(A \text{ and } B)$ , then the traits would appear to be independent.
  - 2. Sample response: Receiving an A in every class and getting on the honor roll are dependent events. This could be demonstrated by using the experimental probability of each event to approximate the theoretical probabilities. If  $P(A) \cdot P(B) \neq P(A \text{ and } B)$ , then the events would appear to be dependent.
- c. Sample response: The product of the probability of that trait and the number of students in the class gives the number of students you could expect to have the trait.

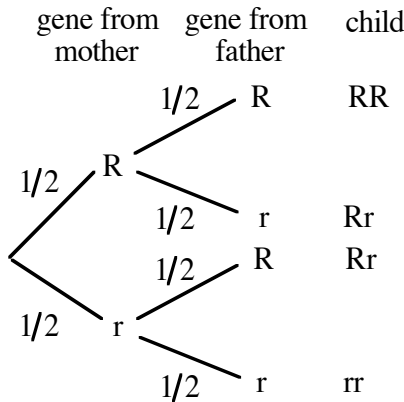
## Assignment

(page 133)

- 3.1 Answers may vary, depending on class data. Using the sample data given in Activity 1, the probability of having free earlobes and the ability to roll the tongue and no widow's peak is:

$$\frac{18}{25} \cdot \frac{19}{25} \cdot \frac{8}{25} = \frac{2736}{15,625}$$

- \*3.2 a.** The probability for each branch is shown on the tree diagram below.



- b.** Since receiving an allele from one parent is independent of receiving an allele from the other parent, the probability for the genotype rr can be found by multiplying the probabilities on the two corresponding branches:

$$P(r) \cdot P(r) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

The same method can be used to find the probability for genotype RR:

$$P(R) \cdot P(R) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Similarly, for each occurrence of the genotype Rr:

$$P(R) \cdot P(r) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

However, since the genotype Rr occurs twice in the cross, the probabilities of the two occurrences must be added:

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

- c.** The Punnett square for this cross is shown below.

		<b>Rr</b>	
		<b>R</b>	<b>r</b>
<b>Rr</b>	<b>R</b>	RR	Rr
	<b>r</b>	Rr	rr

Using a Punnett square, the probabilities for rr and RR are 1/4 and the probability for Rr is 1/2.

- d.** The probabilities found for each genotype are the same using either method.

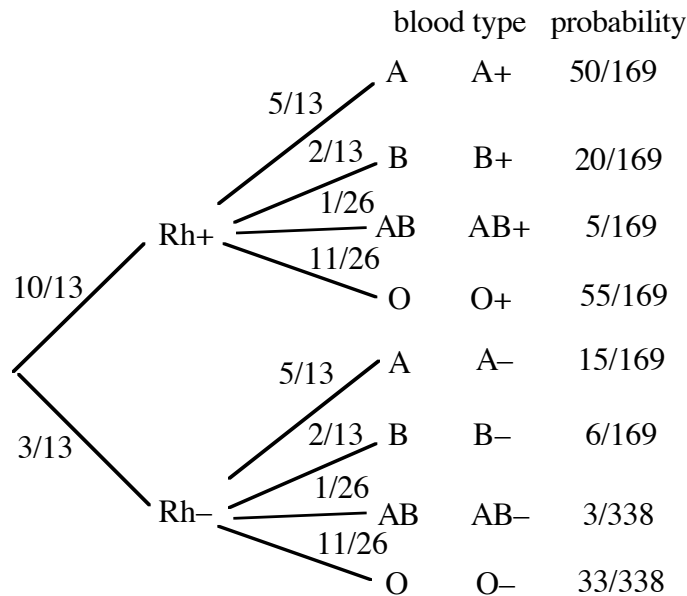
**3.3** The cross  $Mm \times Mm$  yields the outcomes  $MM, Mm, Mm,$  and  $mm$ . Since 3 of the 4 result in the phenotype mid-digit hair, the probability that a child of these parents has mid-digit hair is  $3/4$ .

The cross  $WW \times Ww$  yields the sample space  $WW, WW, Ww,$  and  $Ww$ . Since all of these outcomes result in a widow's peak,  $P(\text{widow's peak}) = 1$ .

Therefore, the probability that a child of these parents has both traits is:

$$\frac{3}{4} \cdot 1 = \frac{3}{4}$$

- \*3.4**
- The experimental probability for type A is  $10/26 = 5/13$ , for type B is  $4/26 = 2/13$ , for type AB is  $1/26$ , and for type O is  $11/26$ .
  - The experimental probability for the presence of the Rh factor is  $20/26 = 10/13$ ; the experimental probability for the absence of the Rh factor is  $6/26 = 3/13$ .
  - Sample tree diagram:



\*\*\*\*\*

**3.5** Students may use a Punnett square or draw a tree diagram to find the sample space  $hH, hH, hh, hh$ . The genotype  $hH$  results in curly hair, so the probability of a child having curly hair is  $1/2$ . The probability that two children of these parents have curly hair is  $1/2 \cdot 1/2 = 1/4$ .

**\*3.6** Assuming that two people chosen at random are unrelated, the probability that both are carriers of cystic fibrosis is:

$$\frac{1}{25} \cdot \frac{1}{25} = \frac{1}{625}$$

- 3.7
- The probability that a person chosen at random prefers cola is  $0.30 + 0.25 + 0.10 = 0.65$ .
  - Assuming that preference for soft drink is independent of preference for snack food, the probability that a person chosen at random prefers cola and pretzels is  $0.65 \cdot 0.10 = 0.065$ .
  - The probability that a person chosen at random prefers root beer is 0.15. Therefore, the probability that a person chosen at random does not prefer root beer is  $1 - 0.15 = 0.85$ .
- \*3.8
- The Punnett square for this cross is shown below.

	<b>Hh</b>	
	<b>H</b>	<b>h</b>
<b>h</b>	Hh	hh
<b>h</b>	Hh	hh

- $P(hh) = 1/2$
- Since the births are independent events, the probability that a child is a carrier is the same for each birth:  $P(Hh) = 1/2$ .
- Since the births are independent events,

$$P(hh \text{ and } hh) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

\* \* \* \* \*

## Research Project

(page 135)

Students may contact a local hospital or the March of Dimes Birth Defects Foundation for more information on genetic counseling.

There are more than 2000 diseases with some identifiable genetic component. The following paragraphs briefly describe the diseases listed in the student edition.

- Cystic fibrosis is associated with the failure of the chlorine pump in cell membranes, resulting in a build up of mucous outside cells. The accumulation of mucous in the pancreas, lungs, and digestive tract makes victims susceptible to pneumonia and other infections.

Untreated victims typically die by age 4 or 5. Treatment with special diets low in chlorine and daily doses of antibiotics can increase life expectancy to an average of 27 years.

In the United States, cystic fibrosis occurs in about 1 of every 2500 births. About 1 in every 25 people is a carrier. The disease occurs in individuals with double recessive genotypes.

- Tay-Sachs disease is caused by the failure of an enzyme that breaks down excess lipids, resulting in a build up of lipids in the brain. Infants develop seizures and blindness, accompanied by a decline in mental and motor skills. There is currently no treatment. Death occurs a few years after birth.

Tay-Sachs occurs in about 1 of every 360,000 births worldwide. It is more frequent (about 1 in every 3600) in Jewish people whose ancestors came from Central Europe. The disease occurs in individuals with double recessive genotypes.

- Huntington's disease is characterized by a breakdown in the tissues of the nervous system. Victims experience a steady decline in mental and motor skills and typically do not manifest symptoms until middle age (between 35 and 45 years old). There is currently no treatment for the disease.

Huntington's disease occurs in individuals with one dominant and one recessive allele. Fetuses with double dominant genotypes are spontaneously aborted.

- Sickle-cell anemia is caused by deformed hemoglobin in red blood cells. The malformed red blood cells result in blood clotting, lower blood oxygen levels, and other symptoms of anemia. There is currently no treatment. Although usually not fatal, the disease limits activity and can cause chronic health problems.

In the United States, sickle-cell anemia is most frequent among people of African origins, occurring in about 1 of every 400 births (about 1 in 10 are carriers). Sickle-cell anemia is a co-dominant disease, which means that carriers may have mild symptoms. In tropical regions, carriers have displayed an increased resistance to malaria.

- Phenylketonuria (PKU) is characterized by the inability to metabolize the amino acid, phenylalanine. As a result, phenylpyruvate accumulates in the bloodstream, eventually reaching toxic levels. Maintaining a diet low in phenylalanine has shown to be an effective treatment.

PKU occurs in about 1 of every 15,000 births in the United States. By using a simple blood test, infants can be screened for the disease before they leave the hospital. PKU occurs in individuals with double recessive genotypes.

## Activity 4

Students identify mutually exclusive events. Given two events A and B, they use appropriate formulas to find the probability of A or B.

### Materials List

- none

### Exploration

(page 136)

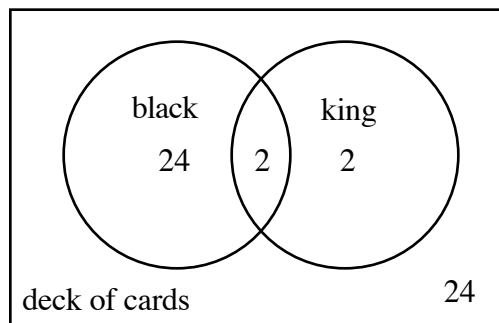
- a. Sample data:

<b>Trait A:</b> left-handedness		
<b>Trait B:</b> right-handedness		
<b>Total Number in Class:</b> 25		
Trait	Frequency	Experimental Probability
A	10	$10/25 = 2/5$
B	15	$15/25 = 3/5$
A and B	0	$0/25 = 0$
A or B	25	$25/25 = 1$

- b. Students repeat Part **a** for two other pairs of traits.

### Teacher Note

You may wish to use the following Venn diagram to help explain the example given in the mathematics note.



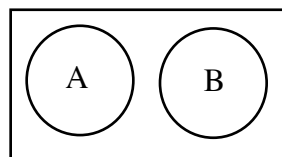
- c–d.** The calculated values for  $P(A \text{ or } B)$  should be equal to those recorded in the tables. For example, using the sample data given in Part **a**:

$$\begin{aligned}P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\&= \frac{10}{25} + \frac{15}{25} - 0 \\&= \frac{25}{25} = 1\end{aligned}$$

## Discussion

(page 137)

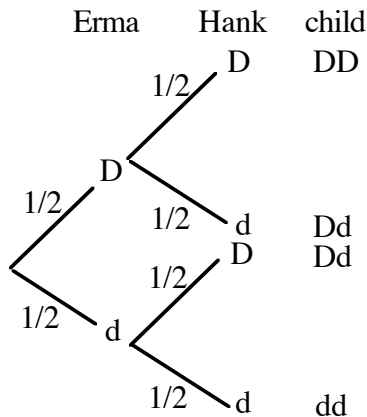
- a.** Sample response: Yes. The probabilities determined by the formula were equal to those observed in the class.
- b.** For all classes,  $P(A \text{ and } B)$  should equal 0 for left-handedness and right-handedness. Depending on the class data, this also may be true for other pairs of traits.
- c.**
1. For all classes, the preferences for writing with the left hand or the right hand should be mutually exclusive. Depending on the class data, other pairs of traits also may be mutually exclusive.
  2. The preferences for writing with the left hand or the right hand are always mutually exclusive since it is not possible to prefer both at the same time. In the general population, the other pairs of traits are not mutually exclusive.
- d.** Sample response: Since  $P(A \text{ and } B) = 0$  for mutually exclusive events, that part of the formula can be omitted without affecting the result.
- e.** Yes. Since the complement of event  $E$  is defined as the event that  $E$  does not occur, it is impossible for an event and its complement to occur at the same time.
- f.** No. On a single roll of a die, for example, a roll of six and a roll of three are mutually exclusive. However, they are not complementary. The probability of rolling a six is  $1/6$ . The probability of rolling a three is also  $1/6$ . The sum of their probabilities is  $1/3$ , not 1.
- g.** When mutually exclusive traits  $A$  and  $B$  are considered as sets, they do not intersect, as shown in the diagram below. Since there is no intersection, the probability of  $A$  and  $B$  occurring at the same time is 0.



## Assignment

(page 138)

- 4.1**
- Because these two events cannot occur at the same time (and are complementary), they are mutually exclusive.
  - Since a person can have both traits at the same time, these two events are not mutually exclusive.
  - Since a person with two blue eyes cannot also have two brown eyes, these two events are mutually exclusive.
- \*4.2** One way to determine the probability that a child of this couple is either mildly affected or not affected at all is to use a tree diagram.



Since the genotype Dd occurs twice, the probability that a child is mildly affected is:

$$\left(\frac{1}{2} \cdot \frac{1}{2}\right) + \left(\frac{1}{2} \cdot \frac{1}{2}\right) = \frac{1}{2}$$

Since the genotype dd occurs only once, the probability that a child is not affected at all is  $1/4$ . Using the formula for  $P(A \text{ or } B)$ ,

$$\begin{aligned} P(\text{Dd or dd}) &= \frac{1}{2} + \frac{1}{4} - 0 \\ &= \frac{3}{4} \end{aligned}$$

This probability can also be determined using a Punnett square.

	<b>Dd</b>	
	<b>D</b>	<b>d</b>
<b>Dd</b>	<b>D</b>	<b>d</b>
	DD	Dd
	<b>d</b>	<b>d</b>
	Dd	dd

Since Dd occurs twice in a sample space of 4 genotypes, its probability is  $1/2$ . Since dd occurs once, its probability is  $1/4$ . As shown above,  $P(\text{Dd or dd}) = 3/4$ .



- 4.3** a. For the cross  $Mm \times Mm$ , the probability of the phenotype mid-digit hair is  $\frac{3}{4}$ . For the cross  $WW \times Ww$ , the probability of the phenotype widow's peak is 1. The probability that a child of these parents has mid-digit hair or a widow's peak is:

$$\frac{3}{4} + 1 - \left(\frac{3}{4} \cdot 1\right) = 1$$

**Note:** Students may solve this problem simply by reasoning that since the probability of a widow's peak is 1, the probability of a widow's peak or mid-digit hair is also 1.

- b. Since the probability of either a widow's peak or mid-digit hair is 1, all five children would be expected to have one trait or the other.

\* \* \* \* \*

- 4.4** Since these events are mutually exclusive, the probability can be calculated as follows:

$$\frac{3}{20} + \frac{1}{20} = \frac{4}{20} = \frac{1}{5}$$

- \*4.5** a. 0.008  
 b.  $0.008 \cdot 250 = 2$   
 c. 1.  $0.008 \cdot 0.008 = 6.4 \cdot 10^{-5}$   
 2.  $0.008 + 0.008 - (0.008 \cdot 0.008) \approx 0.016$

- 4.6** a. Since there are 10 digits on each tumbler, the number of possible combinations for a three-tumbler lock is  $10 \cdot 10 \cdot 10 = 1000$ . Therefore, the probability of guessing the right combination on one guess is  $1/1000$ .
- b. Sample response: Yes. The number of possible combinations for a five-tumbler lock is  $10^5 = 100,000$ . This lock has 100 times more possible combinations than a three-tumbler lock, which decreases the probability that someone could guess the actual combination.

\* \* \* \* \*

## Activity 5

In this activity, students discover that the gender of each child in a family is an independent event. They apply this fact to finding the probabilities of specific family combinations of boys and girls. (This presents a somewhat simplified view of the determination of gender. While it is true that the gender of each child is an independent event, the probabilities of having a boy or girl for any particular family may not be exactly  $1/2$ , due to a range of biological and genetic factors.)

### Materials List

- coins (two per group)

### Exploration

(page 140)

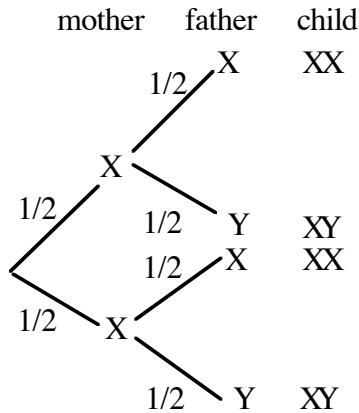
Students use coins to simulate the random selection of the chromosomes that determine gender. (Since the coin that represents the mother's chromosomes always results in an X, students may realize that only the toss of the coin representing the father's chromosomes actually determines the gender of the child.)

- Answers may vary. Some students may correctly predict  $1/2$ .
- Students simulate the chromosomes by labeling two coins.
- Students simulate the combination of male and female chromosomes by tossing the two coins. The experimental probability that the combination results in a female child should be approximately  $1/2$ .
- As shown below, either method can be used to analyze the cross of genotype XY (male) and genotype XX (female). The sample space from this cross contains four equally likely outcomes: two each of XX and XY. Therefore, the theoretical probability of each outcome is  $1/2$ .

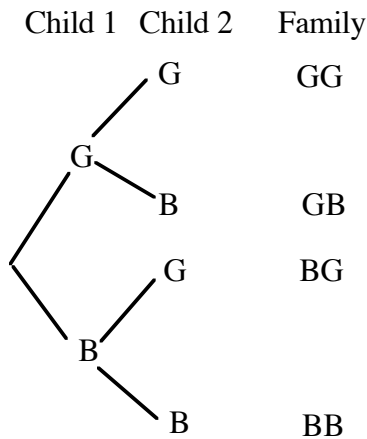
Sample Punnett square:

		<b>XX</b>	
		<b>X</b>	<b>X</b>
<b>XY</b>	<b>X</b>	XX	XX
	<b>Y</b>	XY	XY

Sample tree diagram:



- e. Answers may vary. Some students may correctly predict  $1/2$ .
- f. The experimental probability that two independent combinations of male and female chromosomes result in one male and one female child should be approximately  $1/2$ .
- g. As shown in the following tree diagram, 2 of the 4 possible outcomes result in one male and one female child. The theoretical probability, therefore, is  $1/2$ .



- h. The experimental probabilities calculated using the class data should be approximately  $1/2$ .

## Discussion

(page 141)

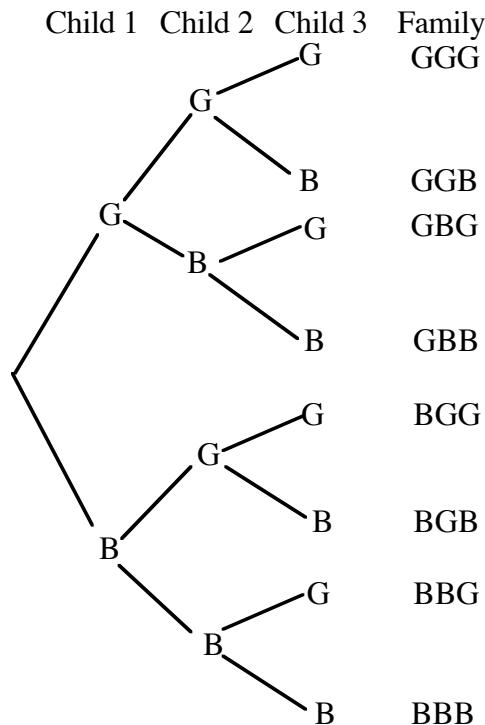
- a. Sample response: They are independent events because the outcome of one coin does not affect the outcome of the other coin.
- b. Sample response: Yes. They are equally likely since the chance of getting an X on the father coin is the same as the chance of getting a Y on the father coin:  $1/2$ . Because the chance of getting an X on the mother coin is 1, the toss of the mother coin does not affect the outcome.

- c. 1. Answers may vary. As the number of trials increases, the experimental probability is more likely to be close to  $1/2$ .
2. The experimental probabilities calculated using the class results are more likely to be close to the theoretical probabilities.
- d. 1. The sample space contains 4 items: GG, GB, BG, and BB, where G represents girl and B represents boy.
2. Since there are 2 possibilities for each child, the number of possible outcomes for 3 children is  $2 \cdot 2 \cdot 2 = 2^3 = 8$ .
3.  $2^n$

### Assignment

(page 142)

- \*5.1 a. Sample tree diagram:



- b. Given that a boy and a girl are equally likely, the probability on each branch in the tree diagram is  $1/2$ . Therefore,  $P(GGB) = 1/8$ .
- c. Two of the three children are girls in 3 of the possible outcomes: GGB, GBG, and BGG. Since the probability of each outcome is  $1/8$ , the total probability is  $3/8$ .
- d. The answers to Parts b and c differ because the event in Part b considers the specific order GGB, while the event in Part c does not. There are 3 different outcomes that satisfy Part c, but only one that satisfies Part b. Therefore, their probabilities are different.

**5.2** Assuming that the probability of having a boy or girl was equally likely for each of the previous four children, the fifth child is as likely to be a boy as a girl.

**5.3** If students assume that each birth is an independent event with the same probability,  $1/2$ , they should agree with this method.

\* \* \* \* \*

**5.4** a.  $1/50$

b. Assuming that the outcome of one car rental has no effect on the outcome of another, the probability can be calculated as follows:

$$(1/50)^5 = 0.000000032 \approx 0$$

c. They can expect  $250(1/50) = 5$  cars to be involved in accidents.

**5.5** a. The probability that the fourth digit is a 7 is  $1/10$ .

**b–d.** The probability that any one sequence of six digits is the winning number is  $(1/10)^6 = 1/1,000,000$ .

**\*5.6** a. When rolling two dice, there are 36 equally likely outcomes. Since 6 of them result in a roll of 7 and 2 result in a roll of 11, the probability of winning is:

$$\frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

b. Since each roll is an independent event, the probability of winning on three rolls in a row is:

$$\frac{2}{9} \cdot \frac{2}{9} \cdot \frac{2}{9} = \frac{8}{729}$$

\* \* \* \* \*

## Answers to Summary Assessment

(page 143)

1. a. The possible genotypes for each blood type are shown below.

Blood Type	A	B	AB	O
Possible Genotype	AA, AO	BB, BO	AB	OO

- b. Assuming that each genotype is equally likely, the probability of each phenotype is listed below.

Blood Type	A	B	AB	O
Probability	$2/6 = 1/3$	$2/6 = 1/3$	$1/6$	$1/6$

**Note:** The actual distribution of human blood types varies by population.

2. Assuming that the genotypes of blood donors are distributed as in Problem 1, the expected amounts can be found by multiplying each probability by 75 L and rounding to the nearest 0.5 L. The total should not exceed 75 L.

Blood Type	A	B	AB	O
Expected Amount (L)	25	25	12.5	12.5

3. Assuming that blood type and Rh factor are independent events, the expected amount of each blood type can be multiplied by 85% to determine the expected amount of Rh+ blood, and by 15% to determine the amount of Rh- blood, with the results rounded appropriately. As in Problem 2, the total should not exceed 75 L.

Blood Type	A+	A-	B+	B-	AB+	AB-	O+	O-
Expected Amount (L)	21	4	21	4	10.5	2	10.5	2

4. As shown in the table in Problem 3, the expected amount for each Rh- blood type is less than 5 L. The table below shows the amounts needed and the number of donors required for each.

Blood Type	A-	B-	AB-	O-
Amount Needed (L)	1	1	3	3
Donors Needed	2	2	6	6

## *Module Assessment*

The following table shows the frequencies of some inherited traits in a class of 24 students. Use this data to complete Problems 1–6 below.

Trait(s)	Number of Students
mid-digit hair	18
widow's peak	14
free earlobes	11
widow's peak and free earlobes	6
widow's peak or free earlobes	19

1. In this class, what is the experimental probability of having a widow's peak?
2. In this class, what is the experimental probability of not having mid-digit hair?
3. Describe the relationship between the events "widow's peak" and "no widow's peak."
4. Determine the sum of the probabilities of having free earlobes and having attached earlobes.
5.
  - a. In this class, what is the experimental probability of having both a widow's peak and free earlobes?
  - b. Why can this probability be no more than the experimental probability of having a widow's peak alone?
6. Explain why the experimental probability of having a widow's peak or free earlobes does not equal the sum of the probabilities of having each trait.
7. Having curly hair (C) is a dominant trait. Having straight hair (c) is a recessive trait. Consider the cross  $CC \times cc$ .
  - a. Use a Punnett square to determine the sample space for this cross.
  - b. Find the probability that a child of these parents has curly hair.
  - c. Draw a tree diagram that shows the sample space of this cross.
  - d. Find the probability that a child of these parents has straight hair.
8.
  - a. A couple plans to have four children. What is the probability that they have a boy, then a girl, then a boy, then a girl?
  - b. What is the probability that they have two girls and two boys, in any order?

9. Carriers of cystic fibrosis have the genotype  $Cc$ . If a father and mother who are both carriers decide to have two children, what is the probability that both children will have the genotype  $cc$ ?
10. Describe an example of each of the following:
- a. two independent events
  - b. two dependent events
  - c. two mutually exclusive events
  - d. two events that are not mutually exclusive
  - e. two complementary events.



## Answers to Module Assessment

- $14/24 = 7/12$
- $1 - 18/24 = 6/24 = 1/4$
- They are complementary events.
- Since these two traits are complementary events, the sum of their probabilities must be 1:

$$\frac{11}{24} + \frac{13}{24} = 1$$

- $6/24 = 1/4$
  - Sample response: This can be no more than the probability of having a widow's peak since any student who has a widow's peak and free earlobes will also be counted in the column for widow's peak alone.
- Sample response: Having a widow's peak and having free earlobes are not mutually exclusive events. In fact, six students in the class have both traits. The probability of having a widow's peak or free earlobes is calculated by subtracting the probability of having both traits from the sum of the probabilities of having each trait:

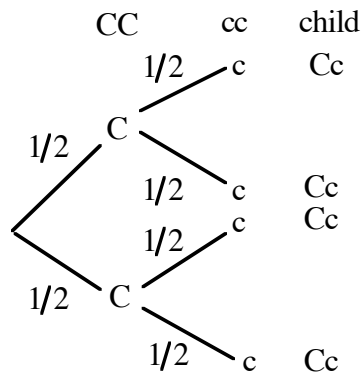
$$\frac{14}{24} + \frac{11}{24} - \frac{6}{24} = \frac{19}{24}$$

- Sample Punnett square:

	<b>cc</b>	
	<b>c</b>	<b>c</b>
<b>CC</b>	<b>C</b>	<b>Cc</b>
	<b>C</b>	<b>Cc</b>

- Since the genotype Cc results in curly hair, the probability of curly hair is  $4/4 = 1$ .

- Sample tree diagram:



- d. Since only the genotype cc results in straight hair, the probability is 0.
8. a. Assuming that the birth of each child is an independent event and that the probabilities of having a boy and a girl are the same:

$$P(\text{BGBG}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

where B represents boy and G represents girl.

- b. The total number of outcomes is 16. Of these, 6 include two boys and two girls: BBGG, BGBG, BGGB, GBBG, GBGB, and GGBB. Therefore, the probability is  $6/16 = 3/8$ .
9. The following Punnett square shows the possible genotypes for this cross:

	Cc	
	C	c
C	CC	Cc
c	Cc	cc

The probability of having one child with cystic fibrosis is  $1/4$ . Assuming that the birth of each child is an independent event, the probability of having two children with the genotype cc is:

$$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

10. Answers may vary from the following sample responses.
- Two independent events are having a widow's peak and having mid-digit hair.
  - Two dependent events are being female and being a mother.
  - Two mutually exclusive events are being male and being female.
  - Two events that are not mutually exclusive are having brown eyes and having brown hair.
  - Two complementary events are being able to roll your tongue and not being able to roll your tongue.

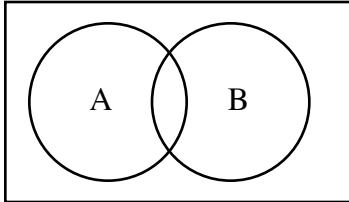
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- Schraer, W. D., and H. J. Stoltze. *Biology: The Study of Life*. Needham, MA: Prentice-Hall, 1991.
- Note:** To order copies of the March of Dimes pamphlet, call 1-800-367-6630, or write the March of Dimes, P.O. Box 1657, Wilkes-Barre, PA 18703.

# Flashbacks

## Activity 1

- 1.1** Shade the portion of the Venn diagram below that represents all elements not in set A.



- 1.2** Given that  $b = 1 - a$ , find the ratio  $a/b$  for each of the following values of  $a$ .
- $a = 0.76$
  - $a = 3/8$
- 1.3** A bag contains 20 blue marbles, 16 green marbles, and 12 red marbles. If you select one marble at random from the bag, what is the probability that the marble is:
- red?
  - green?
  - green or red?
- 1.4**
- If a fair coin is tossed 40 times, how many times would you expect it to land heads up?
  - If a die is tossed 60 times, how many times would you expect it to show a 5?

## Activity 2

- 2.1** A bag contains four marbles: one blue, one red, one green, and one orange. Consider an experiment which involves drawing the marbles from the bag one at a time and recording the order in which they appear.
- How many outcomes are there in the sample space for this experiment?
  - In 6 of the possible outcomes, the blue marble is the second one selected. What is the probability that the blue marble will be the second one selected?
- 2.2** Consider an experiment which involves turning over three cards—an ace, a king, and a queen—one at a time.
- Describe the sample space for this experiment.
  - What is the probability that the ace is the first card turned over?
  - What is the probability that the king is the second card turned over?
  - What is the probability that the queen is the first card turned over and the ace is the second?

## Activity 3

- 3.1** Consider an experiment in which three coins are tossed, one after the other.
- Create a tree diagram of all the possible outcomes. List a probability for each branch.
  - What is the probability that the second coin is heads?
  - What is the probability that the first coin and third coin are heads?
  - What is the probability of obtaining three heads?
- 3.2** If the experiment in Flashback **3.1** were repeated 80 times, how many times would you expect each of the following events to occur?
- the second coin is heads
  - the first and third coins are heads
  - all three coins are heads

#### Activity 4

- 4.1** In a standard deck of 52 playing cards, what is the probability of randomly selecting each of the following:
- an eight?
  - a diamond?
- 4.2** Consider an experiment in which three cards—an ace, a king, and a queen—are placed face down. One card is then turned over. At the same time, one marble is drawn from a bag containing four marbles: one blue, one red, one green, and one orange.
- List the sample space for this experiment.
  - What is the probability that the card is a king and the marble is red?
  - What is the probability that the card is a king or the marble is red?

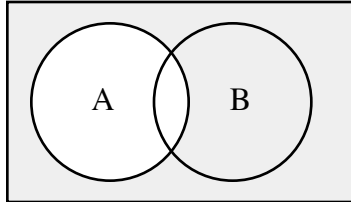
#### Activity 5

- 5.1** Evaluate each of the following expressions.
- $2^3$
  - $5^{10}$
  - $(2/3)^{10}$
- 5.2** Consider an experiment which involves tossing coins and determining the resulting arrangement of heads and tails. Determine the number of elements in the sample space when tossing each of the following numbers of coins.
- 2
  - 3
  - 4
  - 5

## Answers to Flashbacks

### Activity 1

- 1.1 The shaded region in the diagram below represent the elements not in set A.



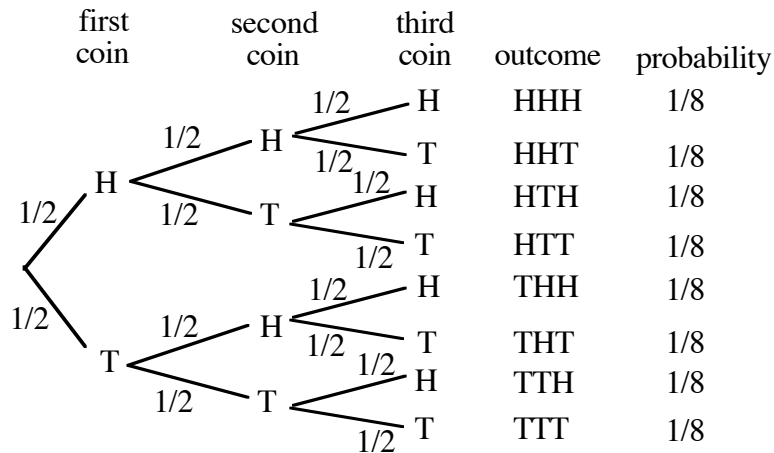
- 1.2 a.  $0.76/0.24 = 19/6$   
b.  $\frac{3/8}{5/8} = \frac{3}{5}$
- 1.3 a.  $12/48 = 1/4$   
b.  $16/48 = 1/3$   
c.  $28/48 = 7/12$
- 1.4 a.  $40 \cdot 1/2 = 20$   
b.  $60 \cdot 1/6 = 10$

### Activity 2

- 2.1 a. Using the fundamental counting principle, the number of different orders is  $4 \cdot 3 \cdot 2 \cdot 1 = 24$ .  
b.  $6/24 = 1/4$
- 2.2 a. The sample space consists of the combinations AKQ, AQK, KAQ, KQA, QAK, and QKA.  
b.  $1/3$   
c.  $2/6 = 1/3$   
d.  $1/6$

### Activity 3

3.1 a. Sample tree diagram:



b.  $1/2$

c.  $1/4$

d.  $1/8$

3.2 a.  $\frac{1}{2} \cdot 80 = 40$

b.  $\frac{1}{4} \cdot 80 = 20$

c.  $\frac{1}{8} \cdot 80 = 10$

### Activity 4

4.1 a.  $4/52 = 1/13$

b.  $13/52 = 1/4$

4.2 a. In the following list, A represents ace, K represents king, Q represents queen, b represents blue, r represents red, g represents green, and o represents orange: Ab, Ar, Ag, Ao, Kb, Kr, Kg, Ko, Qb, Qr, Qg, Qo.

b.  $1/12$

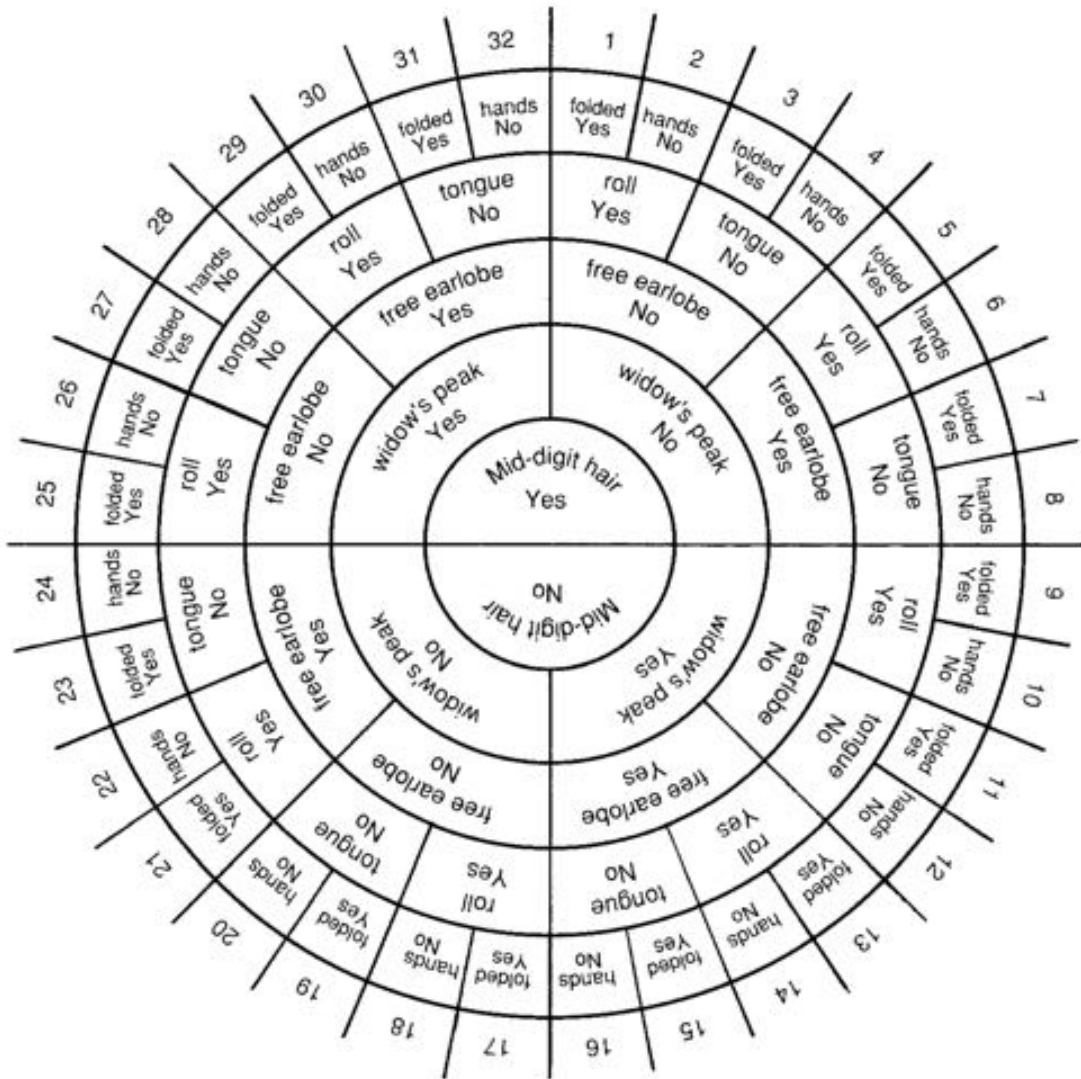
c.  $6/12 = 1/2$



## Activity 5

- 5.1**
- a.  $2^3 = 8$
  - b.  $5^{10} = 9,765,625$
  - c.  $(2/3)^{10} = 1024/59,049$
- 5.2**
- a.  $2^2 = 4$
  - b.  $2^3 = 8$
  - c.  $2^4 = 16$
  - d.  $2^5 = 32$

# Wheel of Traits Template





# There's No Place Like Home



What geometric shape does your home resemble? What do you think the houses of the future will look like? In this module, you explore some traditional American Indian housing designs—from the ground up.

*Kyle Boyce • Clay Burkett*



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## Teacher Edition

# There's No Place Like Home

### Overview

In this module, students use traditional American Indian housing designs to investigate the surface areas and volumes of some common three-dimensional figures. An intuitive understanding of limits is developed by exploring a circle as the limit of a progression of regular polygons with an increasing number of sides inscribed in the circle. Similarly, a cylinder is presented as the limit of a progression of regular polygonal right prisms inscribed in the cylinder, and a cone as the limit of a progression of regular pyramids inscribed in the cone. Students also explore the ratio of surface area to volume as a measure of structural efficiency.

### Objectives

In this module, students will:

- determine the areas of regular polygons and circles
- determine the surface areas and volumes of prisms, pyramids, cylinders, and cones
- calculate the surface areas and volumes of spheres
- identify a circle as the limiting shape for its inscribed regular polygons, a cylinder as the limiting shape for its inscribed regular prisms, and a cone as the limiting shape for its inscribed regular pyramids.

### Prerequisites

For this module, students should know:

- how to use the sine, cosine, and tangent ratios to determine unknown lengths and angle measures in right triangles
- how to calculate the perimeter of a regular polygon given the length of a side
- how to express the area of a regular polygon as the sum of the areas of congruent isosceles triangles with their vertex angle at the center of the polygon
- how to calculate the circumference and area of a circle
- the parts of a prism
- how to determine the volume of a prism
- how to make nets of three-dimensional figures
- the Pythagorean theorem.

### Time Line

Activity	1	2	3	4	5	Summary Assessment	Total
Days	1	1	2	2	2	1	9

### Materials Required

Materials	Activity					Summary Assessment
	1	2	3	4	5	
construction paper				X	X	
scissors				X	X	
tape				X	X	
cubes			X			
compass					X	
ruler					X	
rice					X	

### Technology

Materials	Activity					Summary Assessment
	1	2	3	4	5	
geometry utility	X			X	X	
graphing utility					X	
spreadsheet	X				X	

# There's No Place Like Home

## Introduction

(page 149)

This module uses traditional American Indian housing designs as a context for studying the surface areas and volumes of prisms, cylinders, pyramids, cones, and spheres.

(page 149)

## *Activity 1*

In this activity, students explore the areas of regular polygons. This provides the basis for determining surface area throughout the module. The idea of a circle as the limit of a progression of its inscribed regular polygons is introduced here.

## Materials List

- none

## Technology

- geometry utility
- spreadsheet (optional)

## Teacher Note

Since the following exploration can be time consuming, you may wish to divide the polygons in Part c among several groups, then ask students to compile their results. (If so, all students should use the same radius when constructing the inscribed polygons.)

## Exploration

(page 149)

- a–b.** Students use a geometry utility to draw an inscribed regular triangle, then determine its perimeter and area.
- c–d.** The following sample data was collected using a radius of 5 cm.

Regular Polygon	Perimeter (cm)	Area (cm <sup>2</sup> )
triangle	26.0	32.4
square	28.3	49.9
pentagon	29.4	59.3
hexagon	30.0	64.8
octagon	30.6	70.6
18-gon	31.2	76.8
circle	31.4	78.4

## Discussion

(page 150)

- a. Sample response: The number of sides in the polygon equals its number of central angles. Since there are  $360^\circ$  in a circle,  $360^\circ$  divided by the number of sides (or vertices) determines how many degrees each central angle must have. To find the vertices, locate the points on the circle where each central angle has the desired measure.
- b. The congruent central angles form congruent isosceles triangles in each polygon, with the base of each triangle representing a side of the polygon. Since the bases of these triangles are all congruent, the sides of the polygon are congruent. Since the base angles of all the isosceles triangles are congruent, the angles of the polygon are congruent.
- c. 1. One method divides the regular polygon into congruent isosceles triangles with vertex angles at the center of the polygon and bases as the sides of the polygon. The area of each of these triangles, where  $a$  is the height and  $b$  is the length of the base, is

$$\frac{1}{2}ab$$

The area of the polygon is equal to the sum of the areas of these triangles. If the polygon has  $n$  sides, then the area of the polygon is

$$n\left(\frac{1}{2}ab\right)$$

Since  $n \cdot b$  is the perimeter of the polygon, the generalized form of this method is the formula

$$A = \frac{1}{2}ap$$

where  $a$  is the apothem and  $p$  is the perimeter. **Note:** Students also may suggest using technology, as described in the exploration.

2. Sample response: Each method can be useful, depending on the information given. Using a formula is appropriate when you know the required lengths. Using technology can be faster if you have already drawn a model of the polygon.
- d. 1. The shape of the polygon approaches a circle.  
2. The area of the polygon approaches the area of the circle.  
3. The perimeter of the polygon approaches the circumference of the circle.
- e. 1. The shape of the polygon approaches a circle.  
2. The area of the polygon approaches the area of the circle.  
3. The perimeter of the polygon approaches the circumference of the circle.



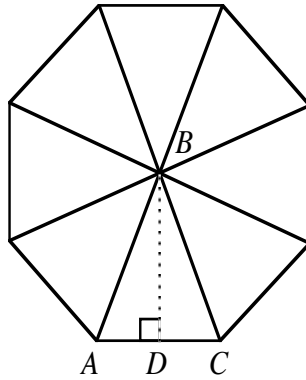
- f.
1. Sample response: As the number of sides of the polygon increases, the shape of the polygon becomes more and more circular. The apothem approaches a radius ( $r$ ) of the circle and the perimeter approaches the circumference ( $2\pi r$ ) of the circle.
  2. As the number of sides increases, the formula for the area of a regular polygon approaches the formula for the area of a circle:

$$A = \frac{1}{2}r(2\pi r) = \pi r^2$$

### Assignment

(page 153)

- 1.1
- a.  $m\angle ABC = 20^\circ$  ;  $m\angle ABD = 10^\circ$
  - b.  $AD = 5 \cdot \sin 10^\circ \approx 0.87$  m
  - c.  $AC \approx 1.74$  m
  - d. Using the Pythagorean theorem,  $BD \approx 4.9$  m .
  - e. The area of  $\triangle ABC \approx (4.9 \cdot 1.74)/2 \approx 4.3$  m<sup>2</sup> .
  - f. The area of the floor is approximately  $18(4.3) \approx 77$  m<sup>2</sup> .
- \*1.2 Student methods may vary. Using the points labeled in the diagram below,  $AB = 8$  m ,  $\sin 67.5^\circ = BD/8$ , and  $\cos 67.5^\circ = AD/8$ .



The area of the floor, therefore, can be found as follows:

$$\begin{aligned} A &= \frac{1}{2}a \cdot p \\ &= \frac{1}{2}(8 \cdot \sin 67.5^\circ) \cdot 2(8 \cdot \cos 67.5^\circ)8 \approx 181 \text{ m}^2 \end{aligned}$$

- 1.3 Using the formula for the area of a circle, the area of the floor is  $\pi(3.5/2)^2 \approx 9.6$  m<sup>2</sup>.

\* \* \* \* \*

- 1.4 Student methods may vary. Using the formula for the area of a regular polygon, the area of the floor is:

$$A = \frac{1}{2} a \cdot p$$
$$\approx \frac{1}{2} (0.87 \cdot 6) \approx 2.6 \text{ m}^2$$

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## Research Project

(page 154)

This research project gives students the opportunity to further explore the geometric and cultural aspects of traditional American Indian housing designs. You may wish to encourage students to build models or make sketches.

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(page 154)

<h2>Activity 2</h2>
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Students examine lateral surface area and total surface area using nets of right prisms and right circular cylinders. (Only right prisms and cylinders are considered in the context of this module.)

### Materials List

- none

### Technology

- geometry utility (optional)

### Discussion

(page 155)

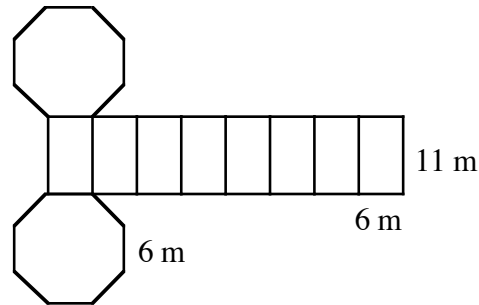
- |  |
|--|
| <p>a. 1. Sample response: The bases of a prism must be parallel. Since the rectangles are not parallel, the floor cannot be a base.</p> <p>2. Sample response: It is called a right prism because the bases are perpendicular to the lateral faces.</p> <p>b. The length of <math>\overline{AD}</math> (or <math>\overline{BC}</math>) equals the perimeter of the base.</p> |
|--|

- c. Sample response: Determine the area of a base by squaring the length of one side of the base. Find the area of the lateral surface by multiplying the perimeter of the base by the height of the prism. Add the area of the two bases to the area of the lateral surface.
- d.
1. The length of the rectangle's longer side is the circumference of the cylinder.
  2. The length of the rectangle's shorter side is the height of the cylinder.
  3. The circles represent the two bases of the cylinder.
  4. The length of the rectangle's longer side is  $2\pi r$
  5. Sample response: Multiply the circumference of the cylinder by the height to find the lateral surface area. Find the area of a base by squaring the radius and then multiplying by  $\pi$ . Add the lateral surface area to two times the area of a base.

### Assignment

(page 156)

- 2.1 a–b. Sample response:

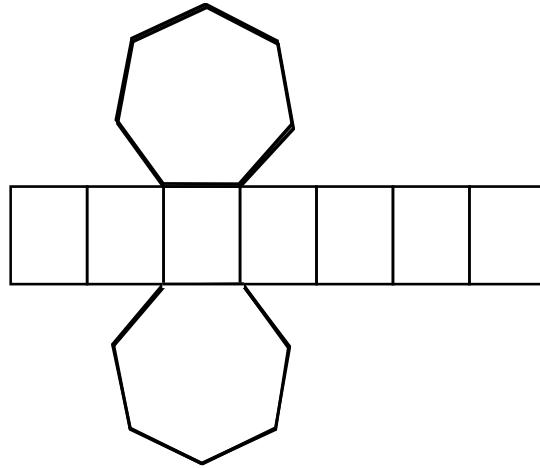


- c. The lateral surface area is  $8 \cdot 6 \cdot 11 = 528 \text{ m}^2$ .
- d. Sample response:

$$\begin{aligned}
 A &= 2 \cdot n \left( \frac{1}{2} ab \right) \\
 &= 2 \cdot 8 \left( \frac{1}{2} \cdot 6 \cdot \frac{3}{\tan 22.5^\circ} \right) \\
 &\approx 348 \text{ m}^2
 \end{aligned}$$

- e. The total surface area is  $528 + 348 \approx 876 \text{ m}^2$ .

2.2 a. Sample sketch:



b. Sample response: As the number of sides increases, the base gets closer and closer to a circle. The shape of the prism approaches a cylinder.

2.3 The lateral surface area, including the floor, is  $3 \cdot 3.2 \cdot 4.7 \approx 45 \text{ m}^2$ . The height of the triangular base is  $1.6 \cdot \tan 60^\circ \approx 2.8 \text{ m}$ . The area of each base is

$$\frac{1}{2} \cdot 2.8 \cdot 3.2 \approx 4.5 \text{ m}^2$$

The total surface area is  $45 + 2(4.5)$  or approximately  $54 \text{ m}^2$ .

2.4 The total surface area is  $2 \cdot (3.8 \cdot 3.4) + 2 \cdot (3.8 + 3.4) \cdot 2.2 = 57.5 \text{ m}^2$ .

\*2.5 a. The area of the floor is  $3.7 \cdot 4.9 \approx 18.1 \text{ m}^2$ .

b. The two walls have a total area of  $\pi(3.7/2)^2 \approx 10.8 \text{ m}^2$ .

c. The lateral surface area can be found as follows:

$$\frac{1}{2} \cdot 2\pi(3.7/2) \cdot 4.9 \approx 28.5 \text{ m}^2$$

\* \* \* \* \*

2.6 Sample response: The amount of bark required to cover the Iroquois house (not including the floor) is the area of the four walls plus the area of the half cylinder that represents the roof:

$$2(5.5 + 18.3) \cdot 2.75 + \pi(2.75)^2 + \frac{1}{2}(2\pi \cdot 2.75) \cdot 18.3 \approx 313 \text{ m}^2$$

\* \* \* \* \*

### Activity 3

Students compare the volumes of various American Indian structures. They also consider the ratio of surface area to volume as a measure of a building's heating (or cooling) efficiency.

#### Materials List

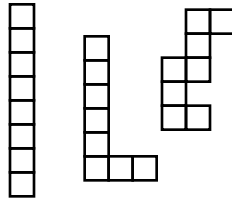
- sets of eight cubes (one or two sets per group)

#### Exploration

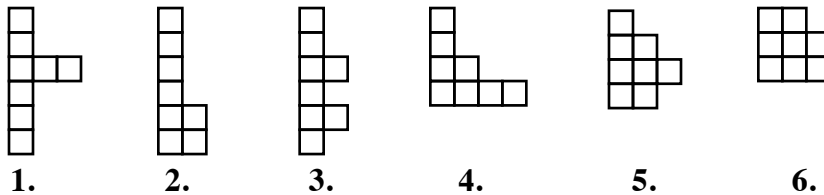
(page 159)

In this exploration, students investigate the relationship between volume and surface area.

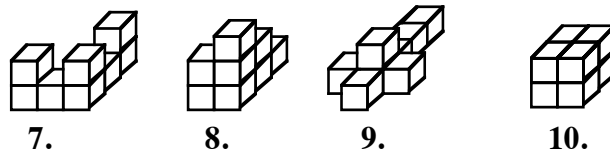
- a–b.** Students should record each arrangement of the cubes they investigate.
- c.** In general, the total surface area depends on the number of faces shared by the cubes. The diagram below shows a top view of three sample arrangements with a surface area of  $34 \text{ units}^2$ .



The following diagram shows the top view of six additional arrangements of cubes.



The diagram below shows several arrangements in which the cubes are stacked.



The total surface area and ratio of surface area to volume for each arrangement are shown in the table below.

Arrangement	Surface Area (units <sup>2</sup> )	<u>Surface Area</u> Volume
1	34	4.25
2	32	4
3	34	4.25
4	32	4
5	30	3.75
6	28	3.5
7	34	4.25
8	30	3.75
9	34	4.25
10	24	3

## Discussion

(page 159)

- a.
  1. The greatest surface area (34 units<sup>2</sup>) occurs in several different arrangements.
  2. The least surface area (24 units<sup>2</sup>) occurs in a  $2 \times 2 \times 2$  arrangement.
  3. All the arrangements have the same volume, 8 units<sup>3</sup>.
- b. Sample response: Given that two designs have the same volume, a design with more surface area (and therefore a higher ratio) would be more in contact with the outside air. While it might receive more sunlight during the day, it would also lose more heat whenever the outside air was colder than the inside air.
- c.
  1. The greatest ratio (4.25) occurs in the arrangements with the surface area of 34 units<sup>2</sup>.
  2. The least ratio (3.0) occurs in the arrangement with the surface area of 24 units<sup>2</sup>.
- d. Sample response: A cube would be a good choice because it has the smallest surface area for a given volume.
- e. Sample response: Since more surface area would provide more area for sunlight to strike, one of the shapes with a high ratio of surface area to volume would be a good choice.

## Assignment

(page 160)

- 3.1** a. The volume inside a Mohave-type house ranges from  $6^2 \cdot 2.2 \approx 79.2 \text{ m}^3$  to about  $7.5^2 \cdot 2.2 \approx 124 \text{ m}^3$ .

- b. The ratio of surface area to volume is:

$$\frac{2(6 \cdot 6) + 4(2.2 \cdot 6)}{79.2} \approx 1.58$$

- 3.2** a. The ratio of surface area to volume is:

$$\frac{2(4 \cdot 3.25 + 2.2 \cdot 3.25 + 2.2 \cdot 4)}{2.2(4)3.25} \approx 2.02$$

- b. The volume of the Pueblo dwelling is about  $13 \cdot 2.2 \approx 29 \text{ m}^3$ . This is about  $50 \text{ m}^3$  less space than the smallest Yuma dwelling.

- 3.3** a. The volume of the house is:

$$(4.1 \cdot 3.1 \cdot 1.7) + \left( \frac{3.1 \cdot 1}{2} \cdot 4.1 \right) \approx 28 \text{ m}^3$$

- b. The surface area of the house is:

$$4.1 \cdot 3.1 + 2(1.7 \cdot 3.1 + 1.7 \cdot 4.1) + 2\left( \frac{3.1 \cdot 1}{2} \right) + 2(1.84 \cdot 4.1) \approx 55 \text{ m}^2$$

The ratio of surface area to volume is  $55/28 \approx 1.96$ .

- \*3.4** a. Because the floors and side walls are the same sizes, students need only compare those portions of the structures that lie above the side walls. The surface area of this portion of the building with the curved roof is:

$$\frac{2 \cdot \pi \cdot 1.5 \cdot 3.8}{2} + \pi \cdot (1.5)^2 \approx 25 \text{ m}^2$$

The surface area of this portion of the other structure is:

$$2 \cdot \left( \frac{1.5 \cdot 3.0}{2} \right) + 2(1.5\sqrt{2} \cdot 3.8) \approx 21 \text{ m}^2$$

Therefore, the surface area of the building with the curved roof is about  $4 \text{ m}^2$  greater than that of the other building.

- b. The volume of the portion of the building with the curved roof that lies above the side walls is:

$$\frac{\pi \cdot 1.5^2 \cdot 3.8}{2} \approx 13.4 \text{ m}^3$$

The volume of the corresponding portion of the other building is:

$$\frac{3.0 \cdot 1.5}{2} \cdot 3.8 \approx 8.6 \text{ m}^3$$

Therefore, the volume of the building with the curved roof is about  $5 \text{ m}^3$  greater than that of the other building.

- c. Sample response: Because heat is lost through surface area, you might expect the building with the curved roof to lose heat more easily. But this structure also has more volume. In fact, the ratio of surface area to volume for the building with the curved roof is  $25/13.4 \approx 1.9$ , while the ratio for the other building is  $21/8.6 \approx 2.4$ . Judging from these ratios, the building with the curved roof is likely to be more efficient in retaining heat.

- 3.5**
- a. Sample response: As the number of sides increases, the volume of the prism approaches the volume of the cylinder.
- b. **1.** Sample response: Yes, the formula can also be used to find the volume of a cylinder. You just need to find the area of the circular base and multiply it by the height of the cylinder.
- 2.** Since the base of a cylinder is a circle, the formula for its area is  $A = \pi r^2$ .

\* \* \* \* \*

- 3.6**
- a. The lateral surface area of the cylindrical container is about  $1040 \text{ cm}^2$ . The total surface area is approximately  $1480 \text{ cm}^2$ . The volume is approximately  $4360 \text{ cm}^3$ .
- b. The lateral surface area of the container shaped like a rectangular prism is about  $1280 \text{ cm}^2$ . The total surface area is approximately  $1620 \text{ cm}^2$ . The volume is approximately  $4020 \text{ cm}^3$ .
- c. Sample response: Since the cylindrical container has less surface area, it should take less material to manufacture each one. Therefore, it should be cheaper to produce.

\* \* \* \* \*



## **Activity 4**

Students examine the surface area and volume of both pyramids and cones. Using nets, they also explore the notion of a cone as the limit of a progression of regular polygonal pyramids with the same heights whose bases are inscribed in the same circle.

### **Materials List**

- construction paper (one sheet per student)
- tape
- scissors

### **Technology**

- geometry utility (optional)

### **Discussion 1**

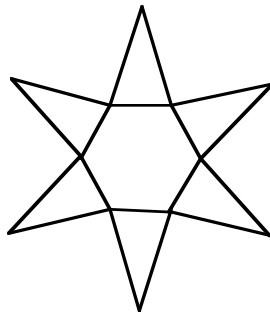
(page 163)

- a. The lateral faces are congruent isosceles triangles.
- b.
  1. The shape of the base becomes more and more circular.
  2. The lengths of the bases of the congruent isosceles triangles decrease.
  3. The pyramid looks more like a cone.
- c. Sample response: To find the surface area of a pyramid, find the area of the polygonal base and the area of all the lateral triangular faces. Add the area of the base to the area of all the triangles.

### **Exploration**

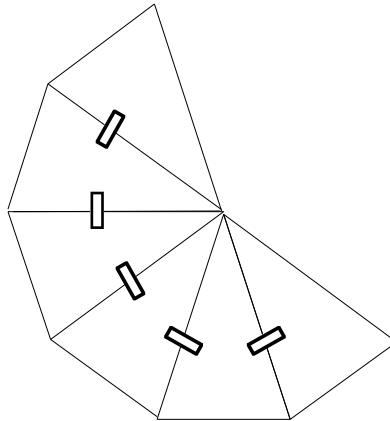
(page 163)

- a.
  1. Sample net:



- a.
  2. Students should fold the net into a pyramid.

- b. 1–2. The following diagram shows a sample net of the lateral surface (with tape):



## Discussion 2

(page 164)

- a.
1. Sample response: The shape of the net looks like a concave polygon.
  2. Sample response: As the number of faces increases, the net of the lateral surface would look more like a part of a circle.
- b.
1. The quantity  $2\pi(10)$  represents the circumference of the circle.
  2. The ratio  $40/360$  represents the fraction of the circle represented by the shaded sector in Figure 10.
- c.
- The quantity  $\pi(10)^2$  represents the area of the circle.
- d.
- As the number of faces increases, the net of the lateral surface approaches a sector of a circle.
- e.
1. The radius of the sector determines the slant height of the cone.
  2. The arc length of the sector determines the circumference of the cone's base.
  3. Sample response: Find the arc length of the sector. This is the circumference of the base of the cone. Since the circumference is  $2\pi r$ , you can then solve for the radius of the base of the cone.

## Assignment

(page 165)

- 4.1 To find the area of the base, students must determine the apothem of the hexagon,  $3\sqrt{3}$  m. To find the area of each triangular face, they must determine the slant height of the pyramid,  $\sqrt{43}$  m. The total surface area can be found as follows:

$$6\left(\frac{6 \cdot 3\sqrt{3}}{2}\right) + 6\left(\frac{\sqrt{43} \cdot 6}{2}\right) \approx 212 \text{ m}^2$$

- 4.2**
- a.**
1.  $\pi(5)^2 \approx 78.5 \text{ cm}^2$
  2.  $2\pi \cdot 5 \approx 31.4 \text{ cm}$
- b.**
1. The fraction of the circle represented by each unshaded sector is shown below:  

$$\frac{270^\circ}{360^\circ} = \frac{3}{4}, \quad \frac{240^\circ}{360^\circ} = \frac{2}{3}, \quad \frac{300^\circ}{360^\circ} = \frac{5}{6}$$
  2. The areas of these unshaded sectors are:  

$$\frac{3}{4}\pi(5)^2 \approx 58.9 \text{ cm}^2, \quad \frac{2}{3}\pi(5)^2 \approx 52.4 \text{ cm}^2, \quad \frac{5}{6}\pi(5)^2 \approx 65.4 \text{ cm}^2$$
  3. The corresponding arc lengths are:  

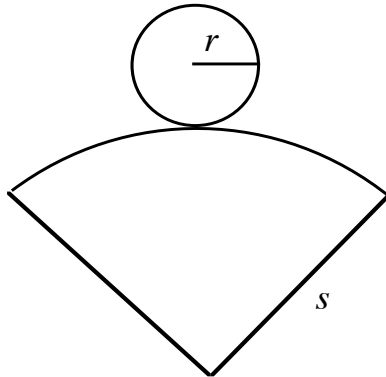
$$\frac{3}{4}(2\pi)5 \approx 23.6 \text{ cm}, \quad \frac{2}{3}(2\pi)5 \approx 20.9 \text{ cm}, \quad \frac{5}{6}(2\pi)5 \approx 26.2 \text{ cm}$$
- c.**
1. The arc length of the sector is the circumference of the base of the cone.
  2. From Part **b3**, the circumference of the base of this cone is 23.6 cm. The radius of the base, therefore, is  $r = 23.6/2\pi \approx 3.8 \text{ cm}$ . The slant height of the cone is the radius of the original circle, 5 cm. Using the Pythagorean theorem, the height of the cone can be found as follows:  

$$3.8^2 + h^2 = 5^2$$

$$h = 3.2 \text{ cm}$$
  3. The area of the base of the cone is  $\pi \cdot 3.8^2 \approx 45.4 \text{ cm}^2$ .
  4. The total surface area is:  

$$\frac{270}{360} \cdot \pi(5)^2 + \pi(3.8)^2 \approx 104.3 \text{ cm}^2$$

- \*4.3** **a.** Sample sketch:



- b. 1.  $2\pi r$   
 2.  $2\pi s$   
 3. By comparing the length of the arc of the curved part of the net with the circumference of the large circle, the fraction of the large circle represented by the cone's lateral surface is:

$$\frac{2\pi r}{2\pi s} = \frac{r}{s}$$

- c. 1.  $\pi s^2$   
 2. The area of the lateral surface of the cone is:

$$\frac{r}{s}(\pi s^2) = \pi r s$$

d. Sample response: The total surface area is the sum of the lateral surface area and the area of the base:  $A = \pi r^2 + \pi r s$ .

e. Using the Pythagorean theorem,  $h = \sqrt{s^2 - r^2}$ .

- 4.4 a. The circumference of the floor is determined by the length of the semicircular arc,  $\pi(2.5)$  m. Thus, the diameter of the floor is 2.5 m.  
 b. Since the radius of the floor is 1.25 m and the slant height of the tipi is 2.5 m,  $h = \sqrt{2.5^2 - 1.25^2} \approx 2.2$  m.  
 c. Sample response: This is a very small tipi. A person could only stand up in the center, and it could sleep no more than two people.

\* \* \* \* \*

- 4.5 a. In order to determine the radius of the template, students must first recognize that the central angle which defines each funnel, including the tab, measures  $120^\circ$ . Since the central angle for the tab measures  $10^\circ$ , the fraction of the circle that actual forms the funnel is  $110^\circ/360^\circ$ .

Since the length of the arc that forms the larger opening of the funnel is 32 cm, the radius of the circle can be found as follows:

$$\frac{110^\circ}{360^\circ} \cdot 2\pi r = 32$$

$$r = \frac{32}{2\pi} \cdot \frac{360^\circ}{110^\circ}$$

$$r \approx 16.7 \text{ cm}$$

b. Similarly, the radius of the inner circle can be determined as shown:

$$\frac{110^\circ}{360^\circ} \cdot 2\pi r = 5$$

$$r = \frac{5}{2\pi} \cdot \frac{360^\circ}{110^\circ}$$

$$r \approx 2.6 \text{ cm}$$

c. Students should recognize that the height of the funnel is the difference between the height of the cone before the tip is cut off and the height of the small cone which forms the tip.

The slant height of the larger cone is the radius of the larger circle, 16.7 cm. Since the radius of its base is approximately 5.1 cm, its height is  $\sqrt{16.7^2 - 5.1^2} \approx 15.9$  cm .

The slant height of the smaller cone (the tip) is the radius of the inner circle, 2.6 cm. Since the radius of its base is approximately 0.8 cm, its height is  $\sqrt{2.6^2 - 0.8^2} \approx 2.5$  cm .

The height of the funnel is  $15.9 - 2.5 = 13.4$  cm .

\* \* \* \* \*

(page 168)

## **Activity 5**

In this activity, students explore cones created from sectors of a circle. They also examine volume, surface area, and the ratio of surface area to volume for various cones and spheres.

### **Materials List**

- compass (one per group)
- scissors (one pair per group)
- construction paper (four sheets per group)
- ruler (one per group)
- tape
- rice (about 2 cups per group)

### **Technology**

- geometry utility
- spreadsheet

### Exploration 1

(page 168)

Students examine cones made from sectors of a circle with a fixed radius.

- a. Students draw a circle, cut it along a radius, and form a cone by overlapping the edges and taping the overlapped edge.
- b. Students measure the height and radius of their cone.
- c. Students create the lateral surface of a right circular cylinder with the same base and height as their cone. They form a cylinder by taping the edges together at the tab.
- d. The ratio of the volume of the cylinder to that of the cone should be  $3/1$ . Student results may vary slightly due to measuring and construction errors.

### Discussion 1

(page 169)

- a. The volume of the cylinder is three times the volume of a cone with the same base and height.
- b. The formula for the volume of a right circular cylinder with radius  $r$  and height  $h$  is  $V = \pi r^2 h$ . It follows that the volume of a right circular cone with radius  $r$  and height  $h$  is

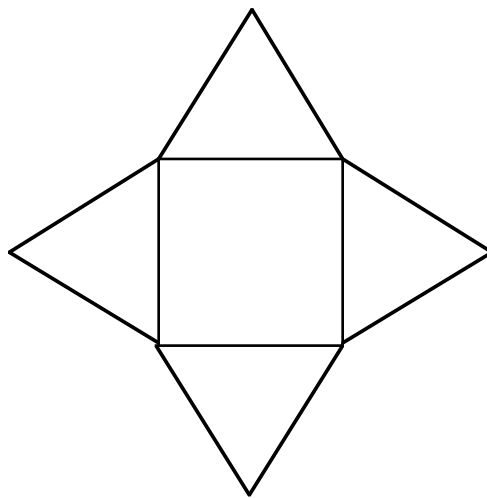
$$V = \frac{1}{3} \pi r^2 h$$

### Exploration 2

(page 169)

Students investigate the relationship between the volume of a right prism and the volume of a regular pyramid with the same base and height.

- a. Sample net:



- b–c.** Students cut the lateral faces from the net and tape them together to form the shape of a pyramid. They then measure the height of the pyramid and a side of the base.
- d.** Students create a right prism with the same base and height as the pyramid in Parts **a–c**.
- e.** The ratio of the volume of the prism to that of the pyramid should be  $3/1$ . Student results may vary slightly due to measuring and construction errors.

## Discussion 2

(page 170)

- a.** The volume of the prism is three times the volume of a pyramid with the same base and height.
- b.** The results of the explorations should agree with the formula given in the mathematics note.
- c.** Sample response: Find the area of the base by squaring  $s$ , multiply the area of the base by the height, and multiply that product by  $1/3$ .
- d.**
  - 1.** Students should rank the figures in the following order: right circular cylinder, hemisphere, right circular cone.
  - 2.** The volume of the cylinder is  $V = B \cdot h = (\pi r^2)r = \pi r^3$ . The volume of the cone is:

$$V = \frac{1}{3}B \cdot h = \frac{1}{3}\pi r^3$$

- 3.** Conjectures will vary. Based on the formulas for the volume of a cone and a cylinder, students will probably guess that the formula involves the product  $\pi r^3$ . Sample response:

$$V = \frac{2}{3}\pi r^3$$

- 4.** Since the volume of a sphere is twice that of a hemisphere with the same radius, students should suggest multiplying their response to Part **d3** of the discussion by 2.

## Assignment

(page 171)

- 5.1** a. Using the formula for the volume of a cone,

$$V = \frac{1}{3} \pi (10)^2 (15) \approx 1571 \text{ cm}^3$$

- b. The height of the cylinder must be  $1/3$  that of the cone, or 5 cm which can be determined by solving for  $h$  in  $1571 = \pi(10)^2 \cdot h$ .
- c. Students may solve for  $h$  as follows:

$$1571 = \frac{1}{3} \pi (8)^2 h$$
$$h \approx 23.4 \text{ cm}$$

- 5.2** a. The area of the regular dodecagon can be found as follows:

$$A = \frac{1}{2} ap = \frac{1}{2} \tan 75^\circ (12 \cdot 2) \approx 45 \text{ m}^2$$

Using the formula for the volume of a pyramid,

$$V \approx \frac{1}{3} \cdot 45 \cdot 5 \approx 75 \text{ m}^3$$

b.  $V = \frac{1}{3} \cdot \pi (3.75)^2 5 \approx 74 \text{ m}^3$

- c. Sample response: The radius of the dodecagon is approximately 7.4 m, which is close to the radius of the base of the cone. Therefore, the areas of the two bases are close in value. Since the pyramid and the cone have the same height, their volumes are also close in value.

- \*5.3** a. Using the Pythagorean theorem,  $h = \sqrt{36 - r^2}$ .

- b. Students should substitute for  $h$  as shown below:

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 \sqrt{36 - r^2}$$

- c. The area of the sector is a fraction of the area of the circle with radius 6 m:

$$A = \frac{2\pi r}{12\pi} \cdot \pi(6^2) = 6\pi r$$



d. Sample spreadsheet:

Radius (m)	Volume (m <sup>3</sup> )	Lateral Surface Area (m <sup>2</sup> )	<u>Surface Area</u> Volume
0.1	0.063	1.884	30.004
0.2	0.251	3.768	15.008
⋮	⋮	⋮	⋮
4.1	77.114	77.283	1.002
4.2	79.152	79.168	1.000
4.3	81.023	81.053	1.000
4.4	82.701	82.938	1.003
⋮	⋮	⋮	⋮
5.9	39.765	111.212	2.797

e. Sample response: The smallest ratio of surface area to volume occurs in a cone with a radius somewhere between 4.2 m and 4.3 m. The corresponding range for the heights is also between approximately 4.2 m and 4.3 m.

### Teacher Note

The derivation of the formula for the surface area of a sphere requires the use of calculus to find the area of a surface of revolution.

- \*5.4** a. Students should use the diameter of the pit to approximate the radius of the hemisphere. The total volume of the house can be found as follows:

$$V = (\pi \cdot 1.75^2 \cdot 1.5) + \frac{1}{2} \left( \frac{4}{3} \cdot \pi \cdot 1.75^3 \right) \approx 25.7 \text{ m}^3$$

- b. The total surface area can be found as follows:

$$A = (\pi \cdot 1.75^2) + (2 \cdot \pi \cdot 1.75 \cdot 1.5) + \frac{1}{2} (4 \cdot \pi \cdot 1.75^2) \approx 45.4 \text{ m}^2$$

- c. Sample response: Because part of a pit house is buried—and because the earth provides both insulation and heat storage—it would have less heat loss during cold months, and more moderate indoor temperatures during warm months.

\* \* \* \* \*

- 5.5** a. The radius of the base of the cone formed by the shaded sector can be determined as shown below:

$$2\pi r = \frac{120^\circ}{360^\circ}(2\pi \cdot 10)$$

$$r \approx 3.3 \text{ cm}$$

The total surface area can be found as follows:

$$A \approx \pi \cdot 3.3^2 + \frac{120^\circ}{360^\circ} \cdot \pi \cdot 10^2 \approx 139 \text{ cm}^2$$

Similarly, the radius of the base of the cone formed by the unshaded sector is:

$$2\pi r = \frac{240^\circ}{360^\circ}(2\pi \cdot 10)$$

$$r \approx 6.7 \text{ cm}$$

The total surface area of this cone is:

$$A \approx \pi \cdot 6.7^2 + \frac{240^\circ}{360^\circ} \cdot \pi \cdot 10^2 \approx 351 \text{ cm}^2$$

- b. For the cone formed by the shaded sector,  $h \approx \sqrt{10^2 - 3.3^2} \approx 9.4 \text{ cm}$ . Its volume, therefore, is:

$$V \approx \frac{1}{3} \cdot (\pi \cdot 3.3^2) \cdot 9.4 \approx 107 \text{ cm}^3$$

For the cone formed by the unshaded sector,  $h \approx \sqrt{10^2 - 6.7^2} \approx 7.4 \text{ cm}$ . Its volume is:

$$V \approx \frac{1}{3} \cdot (\pi \cdot 6.7^2) \cdot 7.4 \approx 348 \text{ cm}^3$$

- c. For the cone formed by the shaded sector, the ratio is approximately 1.3. For the cone formed by the unshaded sector, the ratio is approximately 1.0.
- 5.6** a. Since the inside radius of the hemisphere is 3 m, its volume is:

$$V = \frac{2}{3}\pi(3^3) \approx 57 \text{ m}^3$$

- b. The volume of snow is about  $5.9 \text{ m}^3$ .
- c. The ratio of surface area to volume is approximately 1. Because this ratio is relatively low, one would expect the efficiency to be fairly good—especially when considering the insulating properties of snow. Although some heat will be lost through the floor (which in the Arctic is also frozen), the temperature of the outside air during the cold months is likely to be much lower than that of the tundra.

- d. Given a volume of  $57 \text{ m}^3$ , the length of the half cylinder can be found by solving the following equation for  $h$ :

$$57 = \frac{1}{2}(\pi \cdot 3^2 \cdot h)$$

Since  $h \approx 4.1 \text{ m}$ , the surface area of the half cylinder, not including the floor, is:

$$\pi(3^2) + \pi(3)4.1 \approx 67 \text{ m}^2$$

Therefore, the ratio of surface area to volume is approximately 1.2.

- e. Answers will vary. Judging only from the calculated ratios of surface area to volume, the hemisphere should be more efficient to heat than the half cylinder of equal volume.

- 5.7 a. Using the formula for the volume of a sphere,

$$V = \frac{4}{3}\pi(75)^3 \approx 1.8 \cdot 10^6 \text{ cm}^3$$

- b. Subtracting the thickness of the walls,

$$V = \frac{4}{3}\pi(75 - 3.8)^3 \approx 1.5 \cdot 10^6 \text{ cm}^3$$

- c. The surface area of the Bathysphere is  $A = 4\pi(75)^2 \approx 71,000 \text{ cm}^2$ , while the surface area of a cube with equal volume would be approximately  $88,000 \text{ cm}^2$ . Assuming that weight is proportional to surface area, the ratio of the weight of the Bathysphere to the weight of the cube would be approximately 0.8.

\* \* \* \* \*

## Answers to Summary Assessment

(page 175)

1. a. To find the surface area of this structure, students must find the lateral surface area of a cone and the lateral surface area of a cylinder.

The slant height  $s$  of the cone equals  $\sqrt{r^2 + h^2}$ , where  $r$  is the radius of the base and  $h$  is the height of the cone. In this case,

$$s = \sqrt{5.3^2 + 2.5^2} \approx 5.9 \text{ m}$$

The lateral surface area of the cone, therefore, is:

$$\pi \cdot r \cdot s = \pi \cdot 5.3 \cdot 5.9 \approx 97 \text{ m}^2$$

The lateral surface area of the cylinder is:

$$2\pi \cdot r \cdot h = 2\pi \cdot 5.3 \cdot 2.1 \approx 69 \text{ m}^2$$

Adding the surface area of the floor ( $\pi(5.3^2) \approx 87 \text{ m}^2$ ), the total surface area is approximately  $97 + 69 + 87 = 253 \text{ m}^2$ .

- b. The total volume can be found as follows:

$$V = \frac{1}{3}\pi \cdot 5.3^2 \cdot 2.5 + \pi \cdot 5.3^2 \cdot 2.1 \approx 259 \text{ m}^3$$

2. The total inside surface area is the surface area of the hemisphere minus the area of the doorway plus the surface area of the entryway. Since the entryway is a half cylinder, its lateral surface area is:

$$\frac{1}{2}(2\pi \cdot r \cdot h) = \pi \cdot 0.5 \cdot 1 \approx 1.6 \text{ m}^2$$

The surface area of the hemisphere minus the area of the doorway can be found as follows, where  $r_i$  is the radius of the igloo and  $r_e$  is the radius of the entryway:

$$\begin{aligned} \frac{1}{2}(4\pi r_i^2 - \pi r_e^2) &= \frac{1}{2}(4\pi \cdot 2.5^2 - \pi \cdot 0.5^2) \\ &\approx 39 \text{ m}^2 \end{aligned}$$

The total inside surface area, therefore, is about  $41 \text{ m}^2$ .

The total volume is the sum of the volume of the half cylinder and the volume of the hemisphere:

$$V = \frac{1}{2}\pi(0.5^2)1 + \frac{1}{2}\left(\frac{4}{3}\pi 2.5^3\right) \approx 33 \text{ m}^3$$

3. a. For the Delaware lodge, the total surface area can be found by adding the areas of the floor, four walls, roof, and the two triangular gable ends, respectively.

$$49 + 4(7 \cdot 3) + 2(7 \cdot 3.9) + 2\left(\frac{1}{2} \cdot 7 \cdot 1.7\right) \approx 200 \text{ m}^2$$

Its volume is:

$$3(49) + \left(\frac{1}{2} \cdot 7 \cdot (1.7) \cdot 7\right) \approx 189 \text{ m}^3$$

The ratio of surface area to volume is approximately 1.

For the Miwok roundhouse, the total surface area can be found by adding the areas of the floor, round side wall, and the conical roof, respectively

$$49 + 2\pi \cdot 3.9 \cdot 3 + \pi(3.9 \cdot 4.3) \approx 175 \text{ m}^2$$

Its volume is:

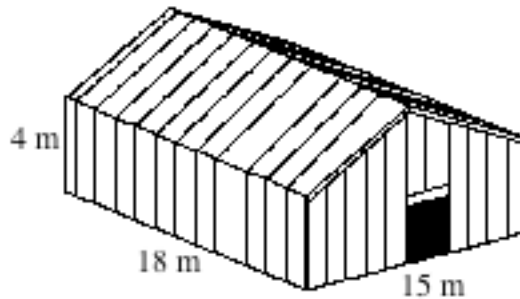
$$3(49) + \left(\frac{1}{3} \cdot 49 \cdot 1.7\right) \approx 175 \text{ m}^3$$

The ratio of surface area to volume is approximately 1, the same as that for the Delaware lodge.

- b. Answers will vary. Sample response: If materials were limited, I would choose the roundhouse. Although it has slightly less volume, it also has less surface area.
- c. Answers will vary. Sample response: Judging from ratio of surface area to volume, the two designs appear to be very close in efficiency. The roundhouse might be slightly easier to heat or cool.

## Module Assessment

1. Large plank houses were common among the American Indians of the coastal Northwest. The basic shape of this dwelling resembles that of many modern homes. For example, the floor of the plank house in the diagram below is a rectangle that measures 15 m by 18 m. The height of the side walls is 4 m; the height of the roof's peak is 7 m.



- a. Disregarding the floor, what is the surface area of this house?
  - b. What is the volume of this house?
2. A canvas shop has received an order for some material to cover a tipi. The height of the tipi is 4.5 m. The base of the tipi is an 18-sided regular polygon. Each side measures 1 m. Describe the dimensions of a piece of canvas shaped like a sector of a circle that will cover the tipi.
  3. The Anaconda Company smokestack in Anaconda, Montana, is one of the largest free-standing smokestacks in the world. Its shape approximates a cylinder 178.35 m tall, with an inside diameter of 22.96 m and an outside diameter of 26.21 m.
    - a. What volume of building material would be required to build a smokestack of this size?
    - b. How does this amount compare to the volume of concrete required to fill a rectangular classroom 7 m wide, 8 m long, and 3 m high ?
  4. Imagine that you are a member of a task force studying the potential colonization of Mars. Since humans cannot survive in the planet's atmosphere, any living area must be enclosed.
    - a. Discuss the merits of enclosing living space with a hemisphere compared to a structure of some other shape.
    - b. If the distance across the proposed colony is 1 km, how much material will be needed to enclose it with a hemispherical dome?
    - c. How much space will the dome contain?

## Answers to Module Assessment

1.
  - a. The surface area of the lower portion is about  $264 \text{ m}^2$ ; the surface area of the upper portion is about  $336 \text{ m}^2$ . The total surface area, not including the floor, is approximately  $600 \text{ m}^2$ .
  - b. The volume of the lower portion is about  $1080 \text{ m}^3$ ; the volume of the upper portion is about  $405 \text{ m}^3$ . The total volume is approximately  $1485 \text{ m}^3$ .
2. In order to determine the size of the canvas needed for the tipi, students must find the tipi's slant height. Using the Pythagorean theorem, the slant height  $s$  equals  $\sqrt{r^2 + h^2}$ , where  $r$  is the radius of the polygonal base (or the circle that circumscribes it) and  $h$  is the height of the tipi. The radius can be found using right-triangle trigonometry as follows:

$$r = 0.5/\sin 10^\circ \approx 2.9 \text{ m}$$

Therefore, the slant height is  $s = \sqrt{2.9^2 + 4.5^2} \approx 5.3 \text{ m}$ . This value is the radius of the circle from which the sector for the tipi must be cut.

The arc length that bounds this sector is the circumference of the base of the tipi, or  $2\pi(2.9) \approx 18 \text{ m}$ . The circumference of the circle is  $2\pi(5.3) \approx 33 \text{ m}$ . Therefore, the measure of the central angle that defines the sector can be found as follows:

$$\begin{aligned}x/360^\circ &= 18/33 \\x &\approx 196^\circ\end{aligned}$$

Therefore, the piece of canvas needed for this tipi is a sector of a circle with a radius of  $5.3 \text{ m}$  whose central angle measures  $196^\circ$ .

3.
  - a. The volume of building material is approximately  $96,227 - 73,843 = 22,384 \text{ m}^3$ .
  - b. Since a rectangular classroom with these dimensions has a volume of  $168 \text{ m}^3$ , the material for the smokestack would completely fill 133 classrooms.
4.
  - a. Sample response: Compared to other building shapes, a hemisphere provides a reasonable living space for a given floor area, as well as a low ratio of surface area to volume. One possibility would be a geodesic dome with many sides, thus approaching the advantages of a hemisphere, yet maintaining angles which are easier to build.

- b. The total surface area for this hemisphere (including the floor) is:

$$\frac{1}{2}(4\pi(0.5)^2) + \pi(0.5)^2 \approx 2.4 \text{ km}^2$$

- c. The volume of a hemisphere with radius  $0.5 \text{ km}$  is about  $0.26 \text{ km}^3$

## **Selected References**

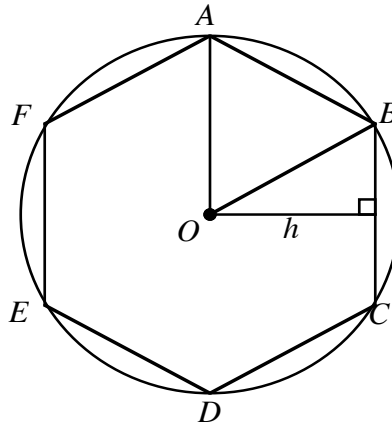
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# Flashbacks

## Activity 1

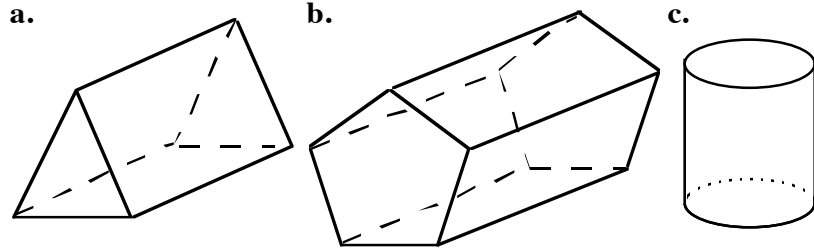
- 1.1 Write the definition of a regular polygon.
- 1.2 The following diagram shows a regular hexagon inscribed in a circle with a radius of 6 cm. Use this diagram to complete Parts **a–d** below.



- a. Find the measure of  $\angle AOB$ .
- b. Identify what type of triangle is  $\triangle AOB$ .
- c. Find  $h$ .
- d. Find the area of the hexagon.
- e. Find the area of circle  $O$ .

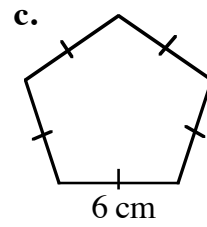
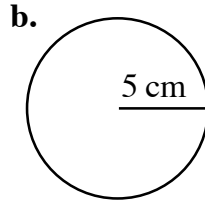
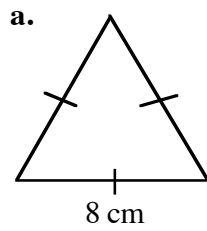
## Activity 2

2.1 Sketch a net for each of the following figures.



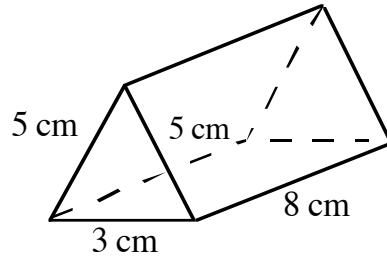
2.2 Describe the bases of each figure in Flashback 2.1.

2.3 Find the area of each of the following figures.

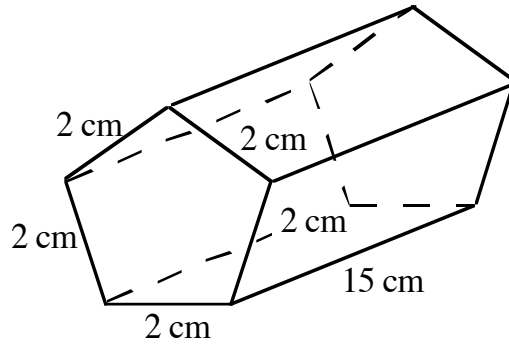


### Activity 3

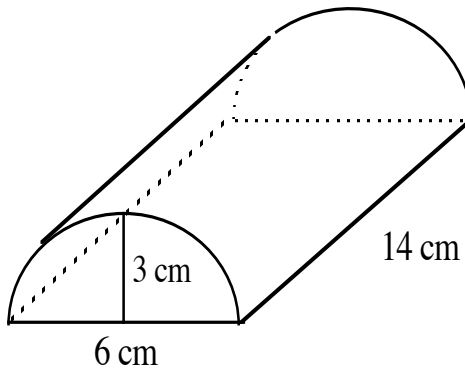
- 3.1 Determine the surface area of the right triangular prism shown below.



- 3.2 Find the surface area of the following right pentagonal prism.



- 3.3 The diagram below shows half of a right circular cylinder. What is the surface area of this figure?

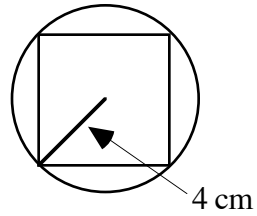


### Activity 4

4.1 Sketch a net for each of the following:

- a. a pyramid with a square base
- b. a pyramid with a pentagonal base.

4.2 The diagram below shows a square inscribed in a circle with a radius of 4 cm.



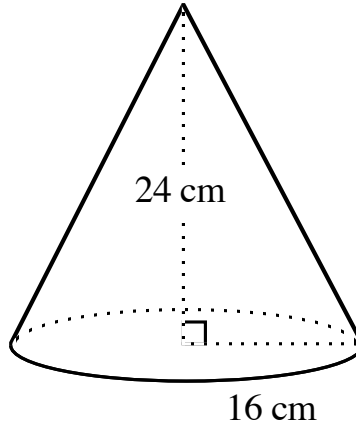
- a. What is the diameter of the circle?
- b. What is the area of the circle?
- c. What is the circumference of the circle?
- d. What is the perimeter of the square?
- e. What is the area of the square?

### Activity 5

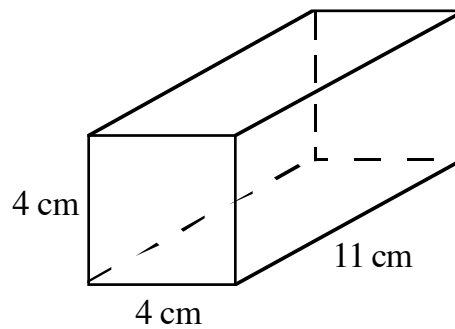
- 5.1 Solve the following equation for  $a$ .

$$4^2 = a^2 + b^2$$

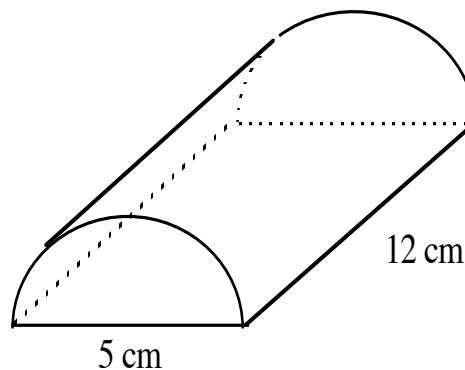
- 5.2 The diagram below shows a right circular cone with a height of 24 cm. The radius of the base is 16 cm.



- a. What is the lateral surface area of this cone?  
b. What is the total surface area of this cone?
- 5.3 a. Find the surface area, volume, and ratio of surface area to volume for the right square prism shown below.



- b. Repeat Part a for the half of the right circular cylinder in the following diagram.



# Answers to Flashbacks

## Activity 1

1.1 Sample response: A regular polygon is a polygon in which all angles are congruent and all sides are congruent.

1.2 a.  $m\angle AOB = 60^\circ$   
b.  $\triangle AOB$  is an equilateral triangle.

c.  $h = 3\sqrt{3} \approx 5.2$  cm

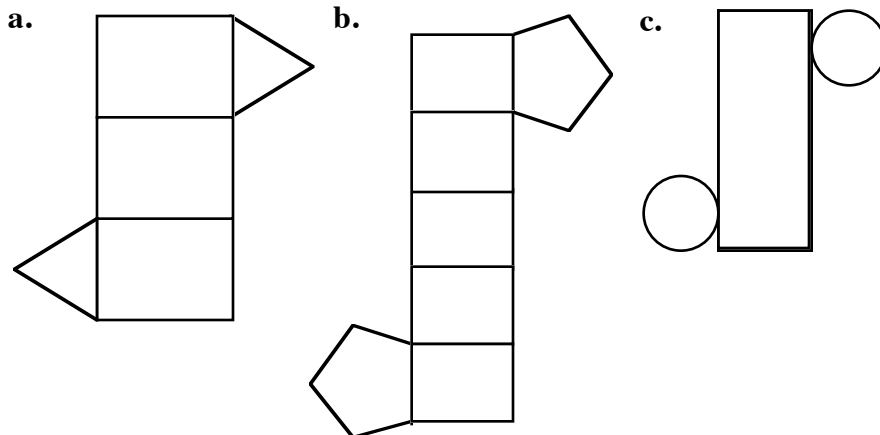
d. The area of the hexagon is six times the area of  $\triangle AOB$ :

$$\frac{1}{2}(6 \cdot 3\sqrt{3}) \cdot 6 \approx 94 \text{ cm}^2$$

e. The area of the circle is  $\pi(6^2) \approx 113 \text{ cm}^2$ .

## Activity 2

2.1 Sample sketches:



2.2 a. The bases are triangles.  
b. The bases are pentagons.  
c. The bases are circles.

- 2.3 a. Using the formula for the area of a triangle,

$$\frac{1}{2} \cdot 8 \cdot 4\sqrt{3} \approx 27.7 \text{ cm}^2$$

b.  $\pi(5^2) \approx 78.5 \text{ cm}^2$

- c. Using the formula for the area of a regular polygon,

$$\frac{1}{2} \left( \frac{3}{\tan 36^\circ} \right) (6 \cdot 5) \approx 62 \text{ cm}^2$$

### Activity 3

- 3.1 The surface area can be found as follows:

$$\frac{4.8 \cdot 3}{2} \cdot 2 + (5 + 5 + 3)8 \approx 118 \text{ cm}^2$$

- 3.2 The surface area can be found as follows:

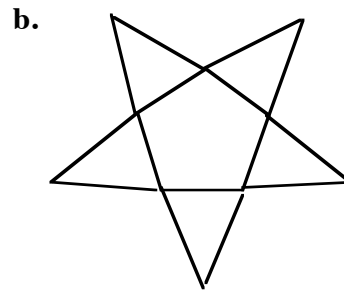
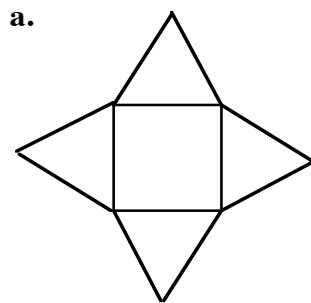
$$\left( \frac{(1/\tan 36^\circ) \cdot 2}{2} \cdot 5 \right) \cdot 2 + 15 \cdot 2 \cdot 5 \approx 164 \text{ cm}^2$$

- 3.3 The surface area can be found as follows:

$$\pi \cdot 3^2 + \frac{1}{2} (2\pi \cdot 3 \cdot 14) \approx 160 \text{ cm}^2.$$

### Activity 4

- 4.1 Sample sketches:



- 4.2**
- a. The diameter of the circle is 8 cm.
  - b. The area of the circle is approximately  $\pi(4^2) \approx 50 \text{ cm}^2$ .
  - c. The circumference of the circle is approximately  $2\pi(4) \approx 25 \text{ cm}$ .
  - d. The perimeter of the square is:

$$\frac{8}{\sqrt{2}} \cdot 4 \approx 23 \text{ cm}$$

- e. The area of the square is:

$$\frac{8}{\sqrt{2}} \cdot \frac{8}{\sqrt{2}} \approx 32 \text{ cm}^2$$

### Activity 5

**5.1**  $a = \sqrt{4^2 - b^2}$  or  $a = -\sqrt{4^2 - b^2}$

- 5.2**
- a. The slant height of the cone is  $\sqrt{24^2 + 16^2} \approx 29 \text{ cm}$ . The lateral surface area of the cone is  $\pi \cdot 16 \cdot 29 \approx 1460 \text{ cm}^2$ .

- b. The total surface area of the cone is approximately  $\pi \cdot 16 \cdot 29 + \pi \cdot 16^2 \approx 2260 \text{ cm}^2$ .

- 5.3**
- a. The surface area is  $2 \cdot 4 \cdot 4 + 4 \cdot 11 \cdot 4 = 208 \text{ cm}^2$  and the volume is  $4 \cdot 11 \cdot 4 = 176 \text{ cm}^3$ . The ratio of surface area to volume is  $208/176 \approx 1.2$ .

- b. The surface area is

$$\pi(2.5)^2 + \frac{1}{2}(2\pi \cdot 2.5 \cdot 12) \approx 114 \text{ cm}^2$$

and the volume is

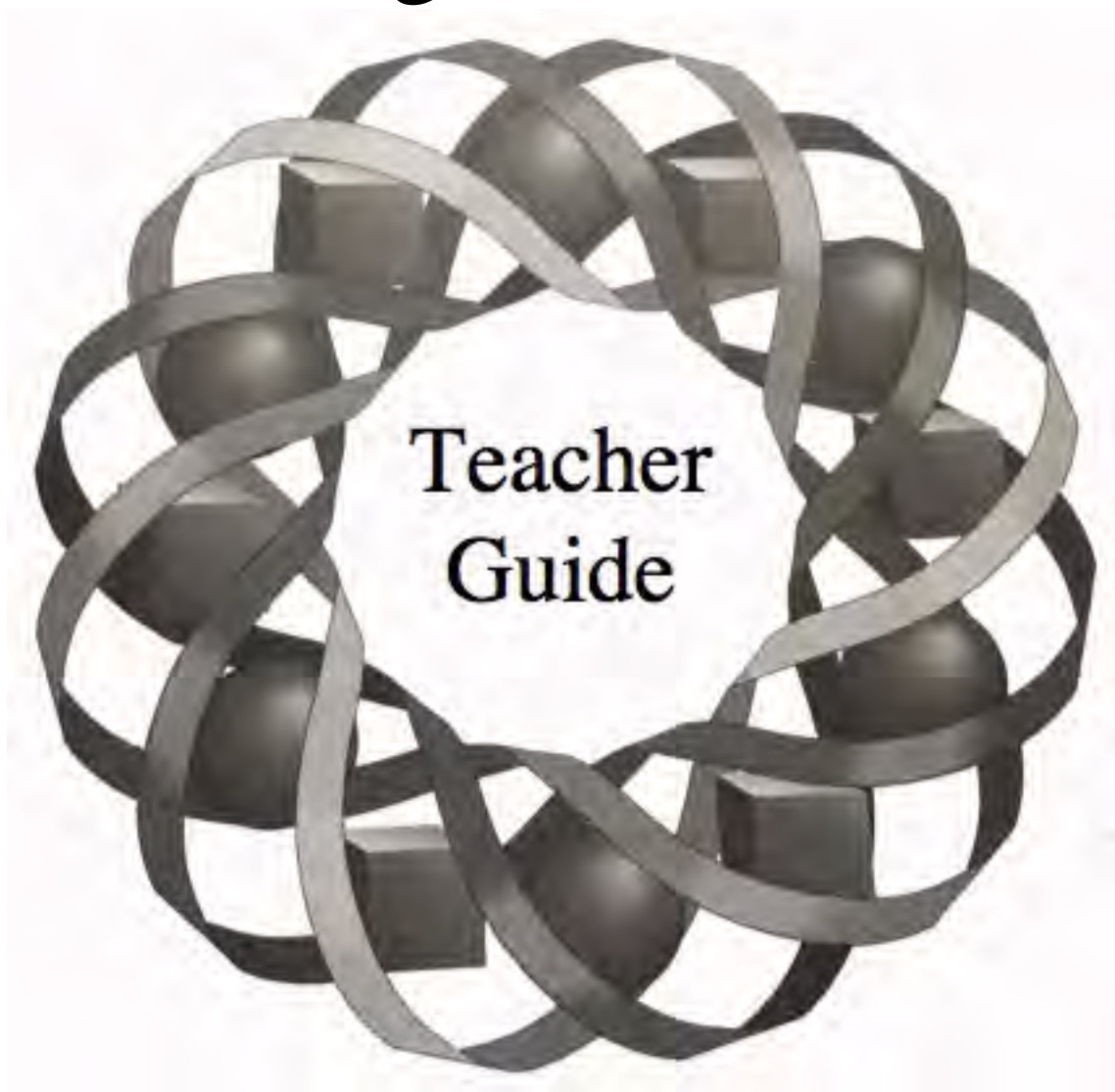
$$\frac{1}{2}\pi(2.5)^2 \cdot 12 \approx 118 \text{ cm}^3$$

The ratio of surface area to volume is  $114/118 \approx 0.97$ .





# Making Concessions



If there's going to be a prom this spring, the class needs to raise some cash. What to sell? How much to buy? How much to charge? Unless the class makes good choices, the big spring dance could be doomed.

*Bonnie Eichenberger • Paul Swenson*



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## Teacher Edition

# Making Concessions

### Overview

Students examine linear programming in two dimensions and in three dimensions. To find corner points of figures that enclose feasible solutions, they solve systems of linear equations using substitution and matrices.

### Objectives

In this module, students will:

- determine constraints for linear programming problems
- write objective functions
- interpret the meaning of points in a feasible region
- find the corner points of a feasible region
- develop the corner principle for optimization
- find solutions to systems of inequalities in two variables
- solve systems of equations in two and three variables graphically, algebraically (by substitution), and by using matrices
- use linear programming to make decisions involving the buying and selling of goods.

### Prerequisites

For this module, students should know:

- how to graph inequalities on a number line
- how to graph inequalities on a coordinate plane
- how to write the equation of a line
- how to solve a system of linear equations involving two variables using substitution
- how to multiply matrices.

### Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	2	3	2	3	1	11

### Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
graph paper	X	X	X		
cardboard box				X	
scissors				X	
tape				X	

### Technology

Software	Activity				
	1	2	3	4	Summary Assessment
graphing utility	X	X	X		
3-D graphing utility				X	
matrix manipulator			X	X	

### Teacher Note

In Activity 4, students model feasible regions in three dimensions. This may be treated as an optional activity.

# Making Concessions

## Introduction

(page 181)

In this module, students use feasibility studies and linear programming to help plan concession sales.

(page 181)

## *Activity 1*

This activity focuses on a typical decision-making process in a business setting. Students represent ranges of values on number-line graphs. They then use these ranges to establish feasible regions for linear programming problems.

## Materials List

- graph paper (several sheets per student; optional)

## Technology

- graphing utility (optional)

## Teacher Note

The sample data given in the following exploration will be used to generate sample responses throughout the module.

## Exploration

(page 181)

Students consider some basic questions concerning the planning and start-up of a concession stand. This works well as a group activity.

- a–e.** Students make decisions based on their own experiences and wishes. The only restriction is the \$100 spending limit. The profits shown in the following sample table assume that all items are sold:

Item	How Sold	Selling Price	Cost per Item	Quantity Purchased	Amount Spent	Potential Profit
pizza	by piece	\$2.00	\$5.00	10 pizzas	\$50.00	\$110.00
pop	by can	\$1.00	\$2.00	15 six-packs	\$30.00	\$60.00
nachos	by weight	\$1.50	\$10.00	2 kits	\$20.00	\$28.00
<b>Total</b>					\$100.00	\$198.00

- f.** Students write summaries of their recommendations. **Note:** Students use their data on quantity and profit in the assignment.

## Discussion

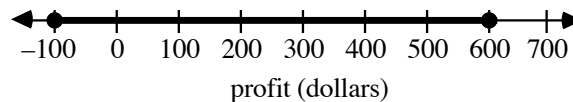
(page 182)

- a. Since students have no indication of the potential market or acceptable selling prices, they may have some difficulty deciding how much of each item to buy and how much to charge.
- b. Sample response: It would have helped to know the number of people who would buy concessions, the number interested in each item, the cost to the class for each item, the number of home games left, and the availability of storage for unused products.
- c. Questions should reflect student concerns in Parts **a** and **b**.
- d. Sample response: We assumed that the items could be obtained for a certain price, that we would not have to rent selling space, and that all the food would be sold.
- e. **1–2.** Answers will vary. Some students may recommend that none of a particular item be bought. The greatest recommended number must keep the total cost below \$100.  
**3.** Sample response: The profit ranges from \$40 to \$600.
- f. Since it is possible to spend \$100 and not sell anything, the least possible profit is  $-\$100$ .

## Assignment

(page 182)

- 1.1**
- a. Sample response: The profit could range from  $-\$100$  to  $\$600$ .
  - b. Sample graph:

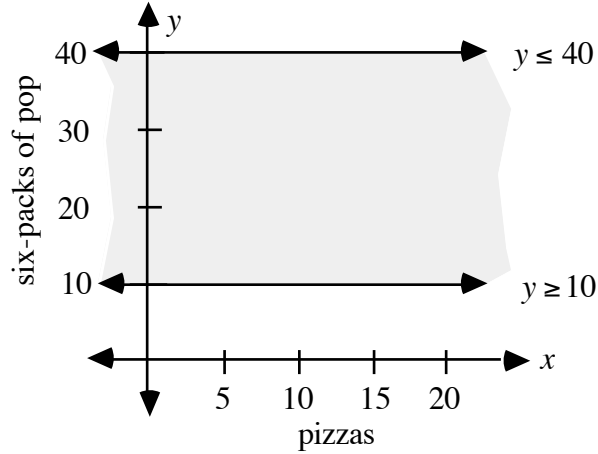


- c. Answers will vary. The sample response given in Part **a** can be written as  $-\$100 \leq \text{profit} \leq \$600$ .
  - d. Sample response: I used the symbols  $\leq$  and  $\geq$  because the endpoints of the interval were part of the range of values.
- 1.2**
- a. **1.** Point *A* represents the purchase of 20 pizzas and 30 six-packs.  
**2.** Point *B* represents the purchase of 20 pizzas and 0 six-packs.
  - b. Sample response: The *x*-coordinate of point *C* is always between 5 and 20. The *y*-coordinate can take on any real-number value.

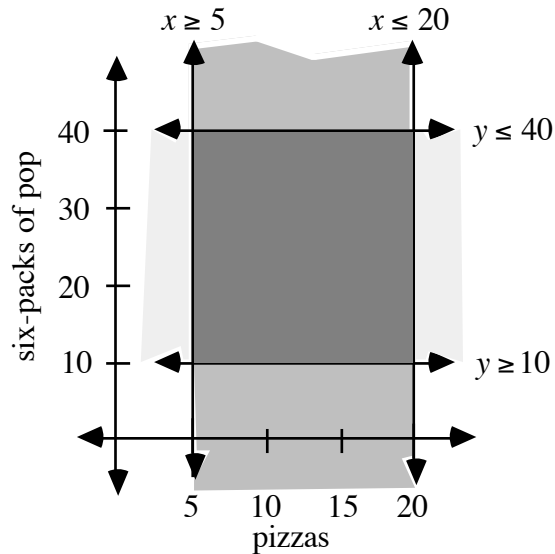
- c. Sample response: The shaded region below the  $x$ -axis has no meaning because it is impossible to buy a negative number of six-packs. Since you can't buy fractions of pizzas or six-packs, any points with non-integer coordinates also have no meaning.

Because the class has only \$100 to spend, the shaded region above the number of six-packs that can be purchased for \$100 is not relevant in this setting.

- d. Sample graph:

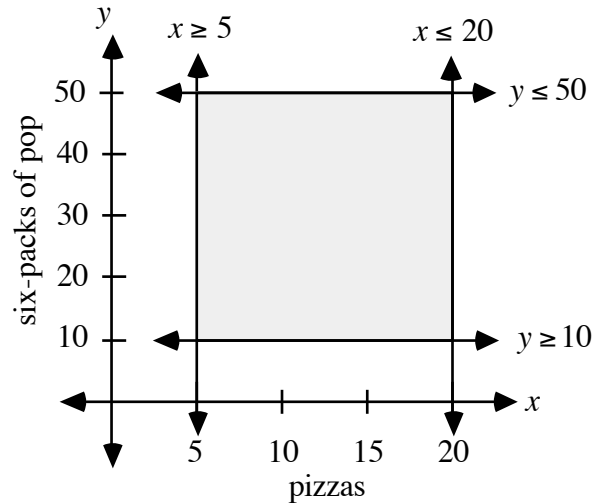


- e. Sample graph:



- 1.3 Answers will vary, depending on the class data. The following sample responses are based on ranges of 5–20 pizzas and 10–50 six-packs.

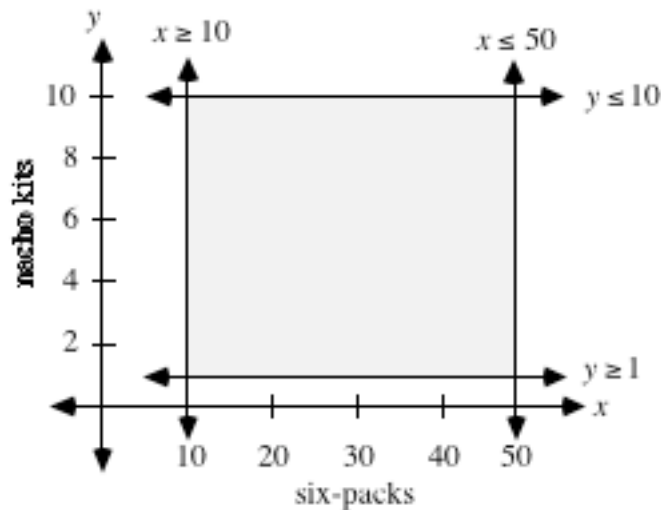
a. Sample graph:



- b. Sample response: The rectangular region formed by the intersection of  $5 \leq x \leq 20$  and  $10 \leq y \leq 50$  shows all the possible combinations of pizzas and six-packs where no more than \$100 is spent on either pizza or pop.

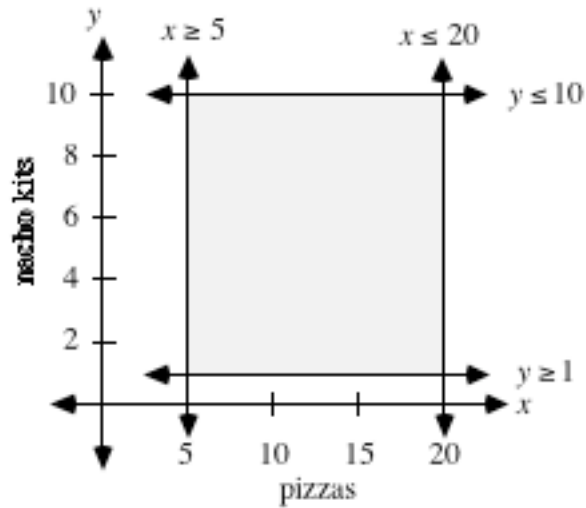
**Note:** A point such as (20,50) would not meet class recommendations given a total spending limit of \$100. (See Problem 1.6.)

1.4 a. Sample graph:



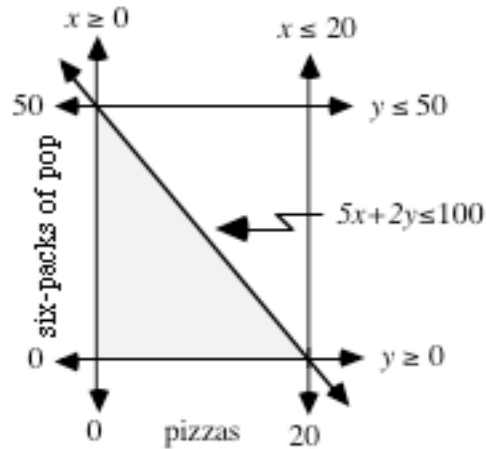


b. Sample graph:



- 1.5**
- a. For the sample graph given in Problem **1.2e**, the intersection is the set of all ordered pairs such that  $5 \leq x \leq 20$  and  $10 \leq y \leq 40$ .
  - b. Since the boundary lines represent values that satisfy the constraints, the symbols  $\leq$  and  $\geq$  should be used.
  - c. Sample response: No. Pizzas, six-packs, and nacho kits can only be bought in whole numbers, so only points whose coordinates are integers [lattice points] should be considered.
- \*1.6**
- a. Sample response: Medium pizzas cost \$5.00 each and pop costs \$2.00 per six-pack. **Note:** The sample responses given below are based on these costs.
  - b.
    1.  $0 \leq x \leq 20$
    2.  $0 \leq y \leq 50$
    3. See sample graph given in Part **d** below.
  - c.
    1. Sample response:  $5x + 2y \leq 100$ .
    2. See sample graph given in Part **d** below.

- d. The shaded area in the following sample graph indicates where the combined cost of pizzas and six-packs is less than or equal to \$100.



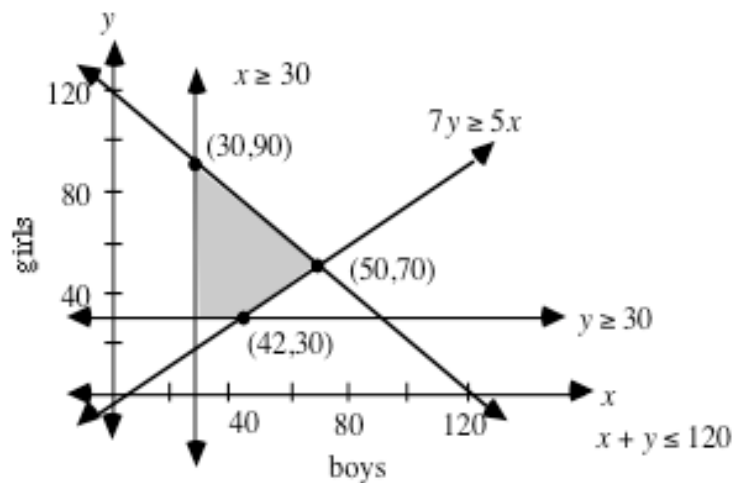
- e. 1. The coordinates of the corner points in the sample graph above are  $(0,50)$ ,  $(20,0)$ , and  $(0,0)$ .
2. Student responses may include descriptions of graphing and tracing, solving a system of linear equations by substitution, or the use of a symbolic manipulator.

\*\*\*\*\*

- 1.7 a. In the following sample responses,  $x$  represents the number of boys and  $y$  represents the number of girls.

1.  $x \geq 30$
2.  $y \geq 30$
3.  $x + y \leq 120$
4.  $7y \geq 5x$

- b-c. As shown in the sample graph below, all of the corner points are in the feasible region.



- 1.8**
- a.  $x > 0$ ,  $y \geq 30$ ,  $x + y \leq 70$ , and  $y \geq x$ .
  - b. The corner points are (30,30), (35,35), (0,70), and (0,30). Corner points (30,30) and (35,35) lie within the feasible region.
  - c. Sample response: The constraints indicate that no more than 70 students will be surveyed. Of those surveyed, the number of girls must be greater than or equal to the number of boys. There must be at least 30 girls interviewed.

\* \* \* \* \*

(page 186)

## *Activity 2*

Students continue to graph feasible regions and identify the corner points. They then explore the use of the corner principle for solving linear programming problems.

### **Materials List**

- graph paper (several sheets per student; optional)

### **Technology**

- graphing utility

### **Exploration 1**

(page 187)

- a. Students explore how the graph of an objective function is affected as the value of the objective (in this case, profit) changes.
- b.
  1. Students should observe that all the lines are parallel.
  2. Sample response: As the value of  $P$  increases, so does the  $y$ -intercept.

### **Discussion 1**

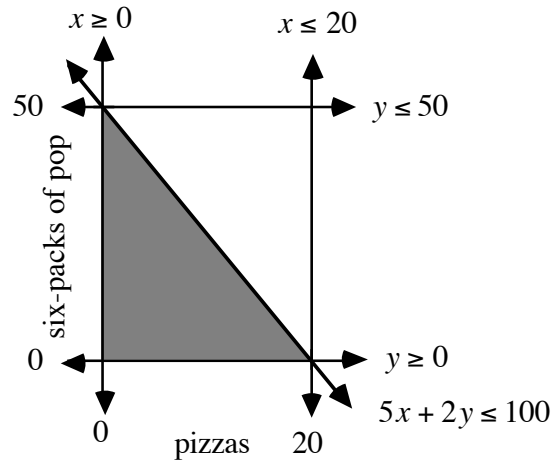
(page 187)

- a. Sample response: The lines are parallel (they all have the same slope).
- b. Sample response: All the lines are parallel because the  $x$ -coefficient and  $y$ -coefficient, which deteriorate the slope, remained the same.
- c. Sample response: For a given profit function, the value of  $P$  remains constant. As the value of  $x$  increases, the value of  $y$  decreases.
- d. Sample response: The graph representing the greatest profit is located above the others. This happens because the value of  $P$  affects the  $y$ -intercept. The greater the value of  $P$  in the objective function, the greater the  $y$ -intercept.
- e. Sample response: The graph that represents the least profit is the one with the lowest  $y$ -intercept in the feasible region. This line is located below the others.

## Exploration 2

(page 187)

a. Sample graph:



- b. 1. Sample response: The point (10,25) lies in the feasible region. Substituting these values in the equation  $P = 3x + 2y$  yields  $P = 80$ .
2. Substituting 80 for  $P$  results in the equation  $80 = 3x + 2y$ .
- c. 1. The line that indicates the maximum profit intersects the feasible region at the corner point (0,50).
2. Since the equation of this line is  $100 = 3x + 2y$ , the maximum profit is \$100.
- d. Students compare values for maximum profit with others in the class.

## Discussion 2

(page 188)

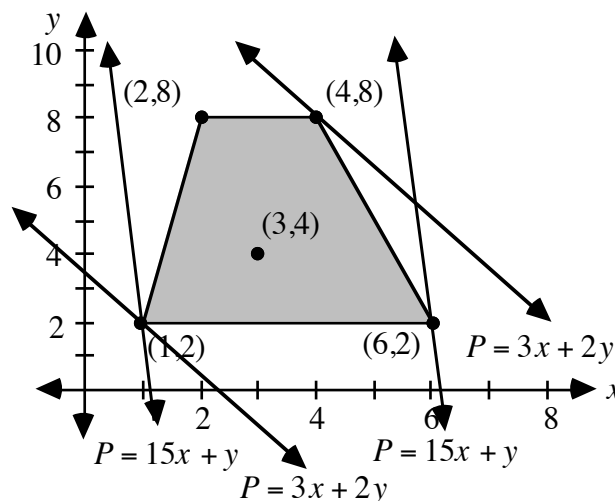
- a. 1. Two lines must intersect to determine a vertex.
2. Two equations are needed to find the coordinates of the vertex.
- b. Many graphing utilities will only graph functions. Constraints in the form  $x = c$ , where  $c$  is a constant, are not functions.
- c. The line that indicates the maximum profit passes through the corner point (0,50). All other lines that pass through the feasible region are below that line.

- d. Sample response: To find the coordinates of the vertex in the lower right-hand corner of the feasible region, for example, you can use the lines defined by the equations  $5x + 2y = 100$  and  $x = 20$  (the two intersecting boundary lines). Substituting 20 for the value of  $x$  in the equation  $5x + 2y = 100$  results in  $5(20) + 2y = 100$ . Solving this equation for  $y$  yields  $y = 0$ . Since the point with coordinates  $(20,0)$  satisfies both equations, it is a point of intersection and thus a corner point of the feasible region.
- e. Sample response: You could find the coordinates for all the corner points of the feasible region and substitute the values of the coordinates into the objective function until you found the greatest value for  $P$ .
- f. Sample response: The graph of the objective function that maximizes the objective would coincide with the edge of the feasible region. The maximum value would occur at any point along that border, so there might be more than one way to maximize the objective function.
- g. The maximum profit can be found by substituting the coordinates into the objective function.
- h. Sample response: Since  $5/8$  of a six-pack is equivalent to  $3\frac{3}{4}$  cans, this is not a possible amount to buy.
- i. Sample response: This means that the maximum value occurs at any point on the segment joining  $(0,4)$  and  $(3,3)$ .

### Assignment

(page 190)

- 2.1 Since lines  $b$  and  $c$  appear to be parallel, they could represent the same profit function for different values of  $P$ .
- 2.2 a. Sample graph for Parts 1–4:

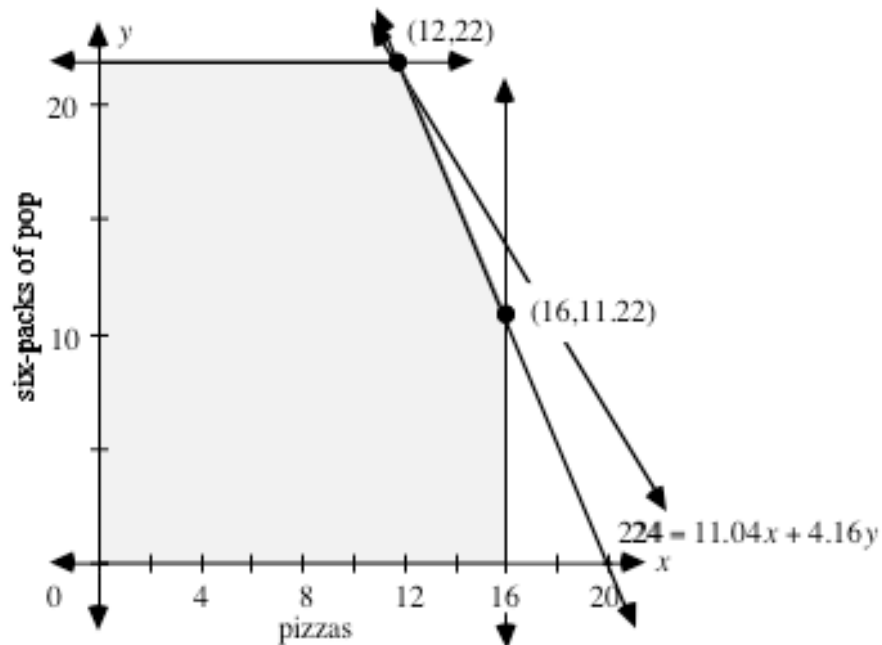


1.  $28 = 3(4) + 2(8)$
2.  $92 = 15(6) + 2$
3.  $7 = 3(1) + 2(2)$
4.  $17 = 15(1) + 2$

b. Sample response: The slope of the objective function determines which corner point of the feasible region represents a maximum or minimum value of the function.

**\*2.3**

- a. The constraints on the problem can be represented by the inequalities  $0 \leq x \leq 16$ ,  $0 \leq y \leq 22$ , and  $4.96x + 1.84y \leq 100$  where  $x$  represents the number of pizzas and  $y$  represents the number of six-packs of pop.
- b. See sample graph given in Part e below.
- c. The profit is \$11.04 on each pizza and \$4.16 on each six-pack.
- d. Sample response:  $P = 11.04x + 4.16y$ , where  $P$  is profit,  $x$  is the number of pizzas, and  $y$  is the number of six-packs.
- e. The maximum value of  $P = 11.04x + 4.16y$  occurs at the corner point  $(12,22)$ . If all items are sold, the maximum profit is  $12(11.04) + 22(4.16)$  or \$224.00. The graphs of the two oblique lines nearly coincide. The sample graph distorts that difference. Sample graph:  $224 = 11.04x + 4.16y$



f. Sample response: The class should buy 12 pizzas and 22 six-packs of pop. Since these are whole number quantities, this seems to be a reasonable answer.

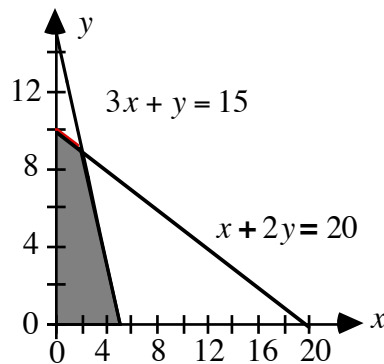
- 2.4 a. Substituting 5.5 for  $x$  in each equation yields the following:

$$\begin{aligned} y_1 &= 0.8(5.5) + 4 \\ &= 4.4 + 4 \\ &= 8.4 \end{aligned} \qquad \begin{aligned} y_2 &= -(5.5) + 13.9 \\ &= 8.4 \end{aligned}$$

Therefore,  $(5.5, 8.4)$  is a solution to both equations.

- b. The coordinates of the corner points are  $A(0,0)$ ,  $B(0,4)$ ,  $D(8.4,5.5)$ , and  $E(8.95,0)$ . Students should verify these coordinates by substituting them in the equations defining the corresponding boundaries of the feasible region.

- 2.5 a. Sample graph:



- b. The coordinates of the corner points are  $(0,0)$ ,  $(5,0)$ ,  $(2,9)$ , and  $(0,10)$ .  
 c. 1. As shown in the table below, the minimum of  $-5$  occurs at  $(5,0)$ .

$(x,y)$	$P = 2y - x$	$P = 3y + 2x$
$(0,0)$	0	0
$(5,0)$	-5	10
$(2,9)$	16	31
$(0,10)$	20	30

2. As shown in the table above, the minimum of 0 occurs at  $(0,0)$ .

- 2.6 a. The maximum value of the objective function occurs at the corner point  $(9\frac{1}{3}, 22)$ , where  $x$  represents number of pizzas and  $y$  represents number of six-packs. Since the value for  $x$  is not a whole number, it does represent a reasonable amount of pizza to buy.  
 b. To find the maximum profit which corresponds to a point with integer coordinates, students may suggest rounding solutions to the nearest integer or substituting the coordinates of points near the corner point into the objective function.

In this case, the point (9,22) results in a profit of \$244. Although the points (10,22) and (10,21) produce greater profits, they are not in the feasible region. The point (10,20) is in the feasible region but results in a profit of only \$240.

**Note:** Neither of these methods guarantee that the optimum value will be found.

- c. Sample response: A businessperson might buy less if the profit is not greatly affected. They may buy slightly more if they believe they could sell the extra or easily absorb the loss.

**2.7** Answers will vary. Students should record the quantities of cheese and chips needed to make an appropriate number of nacho servings, the quantity and cost of each item to be purchased, and the selling price of each item. If the corner principle does not give a solution with integer coordinates, they should use an appropriate strategy to determine the quantities to purchase.

\* \* \* \* \*

**2.8**  $A(0,0), B(0,5), C(4.2,7.1), D(9.3,3.7), E(7.45,0)$

- 2.9**
- a. 1.  $D(10,8)$
  - 2.  $A(0,0)$

- b. 1.  $m = -8/5$
- 2. At both  $D$  and  $E$ , the value of the objective function is 120.
- 3. The equation of this line in slope-intercept form is:

$$y = -\frac{8}{5}x + \frac{P}{5}$$

The slope of the objective function is  $-8/5$ .

- 4. Since the slope of the objective function is the same as the slope of the line that passes through  $D$  and  $E$ , the value of the function is the same at both  $D$  and  $E$  (and at any other point on  $\overline{DE}$ ).
- 5. All three lines would be parallel to  $\overline{DE}$ .
- c. 1. Sample response:  $P = 2x + 5y$ .
- 2. Sample response:  $P = x + 5y$ .
- 3. Sample response:  $P = 10x + 2y$ .

\* \* \* \* \*



## Research Project

(page 194)

Students plan a fund-raiser that involves selling two different items. To help determine the feasibility of their plan, they conduct a survey. They then prepare a report designed to persuade their classmates to invest in the venture. This works well as a small-group activity.

You may wish to allow students to implement a plan selected by the class. Students then can evaluate the accuracy of their projections.

(page 194)

### Activity 3

Students use matrices to solve systems of two equations in two unknowns. Matrix equations are then used to find corner points of feasible regions.

#### Materials

- graph paper (several sheets per student)

#### Technology

- graphing utility
- matrix manipulator

#### Exploration

(page 194)

Students examine the relationship between a system of two equations and two unknowns and the corresponding matrix equation. The concept of inverse matrices is introduced. Students then use technology to solve matrix equations.

- a. Students should obtain the following equation:

$$\begin{bmatrix} -8x & 10y \\ 10x & 10y \end{bmatrix} = \begin{bmatrix} 40 \\ 139 \end{bmatrix}$$

- b. Sample response: The entries in the matrices are the terms of the equations. The first equation's terms are in the top rows of the matrices; the second equation's terms are in the bottom rows of the matrices.
- c. One way for students to verify the value of  $\mathbf{M}^{-1}$  is to enter matrix  $\mathbf{M}$  in their technology, then use the multiplicative inverse command. A second approach involves entering both  $\mathbf{M}$  and  $\mathbf{M}^{-1}$ , then finding their product—which should be the identity matrix.

d. Matrices will vary. Sample response:

$$\begin{bmatrix} 6 & 4 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -2 & 5 \end{bmatrix}$$
$$\begin{bmatrix} 8 & 3 \\ 10 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 & 3 \\ 10 & -5 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 10 & -5 \end{bmatrix}$$

e. 1. The matrix equation is shown below:

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 9 \end{bmatrix}$$

2. The inverse of the coefficient matrix is:

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

3. The resulting equation is:

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 22 \\ 9 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

4. The matrix equation verifies the solution:

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 9 \end{bmatrix}$$

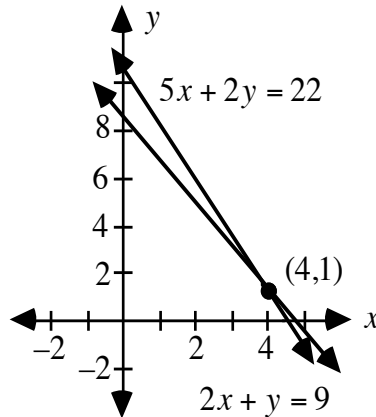
5. The solution is (4,1).

6. Students should substitute as follows:

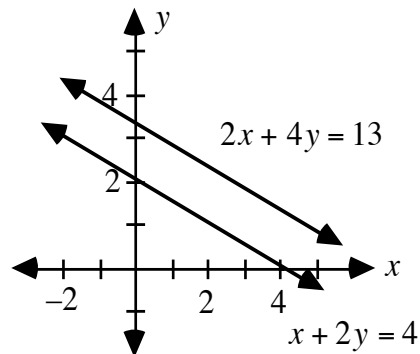
$$\begin{array}{rcl} 5(4) + 2(1) & = & 22 \\ 20 + 2 & = & 22 \\ 22 & = & 22 \end{array} \qquad \begin{array}{rcl} 2(4) + 1 & = & 9 \\ 8 + 1 & = & 9 \\ 9 & = & 9 \end{array}$$

Therefore, (4,1) is a solution to both equations.

7. Since the graphs of the two lines intersect at  $(4,1)$ , the graph verifies the solution.



- f. 1. The system is inconsistent because the graphs are parallel and do not have a common point.



2. The process fails because the coefficient matrix does not have an inverse.

## Discussion

(page 197)

- a. Sample response: You can enter the coefficient matrix  $\mathbf{M}$  and use the multiplicative inverse button to find  $\mathbf{M}^{-1}$ . Or you can enter  $\mathbf{M}$  and  $\mathbf{M}^{-1}$ , then find their product to see if you get the identity matrix.
- b. The coefficient matrix does not have an inverse.
- c. The coefficient matrix is a  $2 \times 2$  matrix with a  $y$  coefficient equal to 0:

$$\begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix}$$

- d. Sample response: Matrix multiplication is not commutative. In matrix multiplication, the number of columns in the first matrix must equal the number of rows in the second matrix. Both the coefficient and the constant matrix must be multiplied by the inverse matrix on the left in order for the dimensions to match.

- e. Using a process similar to that described for systems of two equations and two unknowns, the system can be represented as a matrix equation in the form  $\mathbf{M} \cdot \mathbf{X} = \mathbf{C}$  as shown below:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ -7 \end{bmatrix}$$

## Assignment

(page 198)

- 3.1 a. The multiplicative inverse of the matrix is shown below:

$$\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$$

- b. Sample response:

$$\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$$

- c. Answers will vary. Sample response: The following matrix does not have an inverse.

$$\begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$$

When the elements of the matrix are used as coefficients, the graphs of the equations (such as  $2x + 8y = 1$  and  $x + 4y = 3$ ) are parallel. When I try to find the multiplicative inverse using technology, I get an error message.

- 3.2 Students can verify their solutions by substituting the values obtained into the original equations.

a.  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

b.  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

c.  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

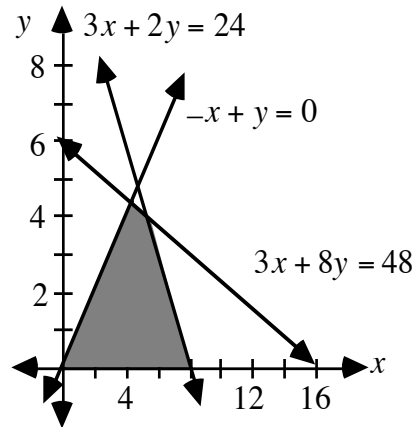
3.3

$$\begin{bmatrix} 80 & 33 \\ -20 & 73 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 466 \\ 46 \end{bmatrix}$$

$$\begin{bmatrix} 80 & 33 \\ -20 & 73 \end{bmatrix}^{-1} \cdot \left( \begin{bmatrix} 80 & 33 \\ -20 & 73 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 80 & 33 \\ -20 & 73 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 466 \\ 46 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

3.4 a. Sample graph:



b.  $(0,0), (8,0), (5\frac{1}{3}, 4), (4\frac{4}{11}, 4\frac{4}{11})$

c. As shown in the table below, the maximum of 24 occurs at two corner points. Therefore, the maximum occurs at any point on the segment connecting these two points.

$(x,y)$	$P = 2y + 3x$
$(0,0)$	0
$(8,0)$	24
$(5\frac{1}{3}, 4)$	24
$(4\frac{4}{11}, 4\frac{4}{11})$	$21\frac{9}{11}$

\*3.5 a. In the following inequalities,  $x$  represents number of pizzas and  $y$  represents number of six-packs:

$$5x + 2y \leq 100$$

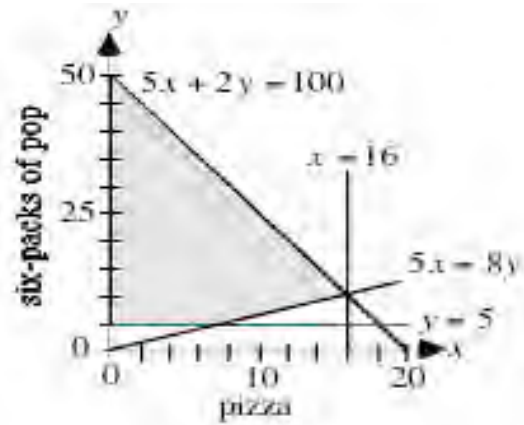
$$2y \geq 10$$

$$5x \leq 80$$

$$5x \leq 8y$$

$$x \geq 0$$

b. Sample graph:



c. As shown below, the profit function  $P = 11x + 2y$  has its maximum value at the corner point (16,10).

$(x,y)$	$P = 11x + 2y$
(0,5)	10
(8,5)	98
(16,10)	196
(0,50)	100

The maximum potential profit of \$196 occurs if the class buys (and sells) 16 pizzas and 10 six-packs.

\*\*\*\*\*

3.6 a. In the following inequalities,  $x$  represents millions of liters from the reservoir and  $y$  represents millions of liters from the aquifer.

$$x + y \geq 20$$

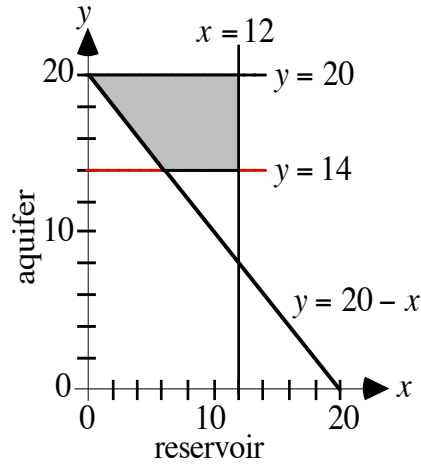
$$x \leq 12$$

$$y \geq 14$$

$$y \leq 20$$

$$x \geq 0$$

- b. In the following sample graph, units on the axes represent millions of liters. The feasible region is shaded.



- c. As shown below, the minimum value of the objective function  $C = 80x + 100y$  occurs at the corner point (6,14). The maximum occurs at the point (12,20).

$(x,y)$	$C = 80x + 100y$
(0,20)	2000
(6,14)	1880
(12,14)	2360
(12,20)	2960

1. \$1880
2. \$2960

\* \* \* \* \*

(page 200)

### Activity 4

Students extend linear programming concepts into three dimensions by considering the sale of three items instead of two. **Note:** This may be treated as an optional activity.

#### Materials List

- cardboard boxes (one per group)
- scissors (one pair per group)
- tape

## Technology

- 3-D graphing utility (optional)
- matrix manipulator

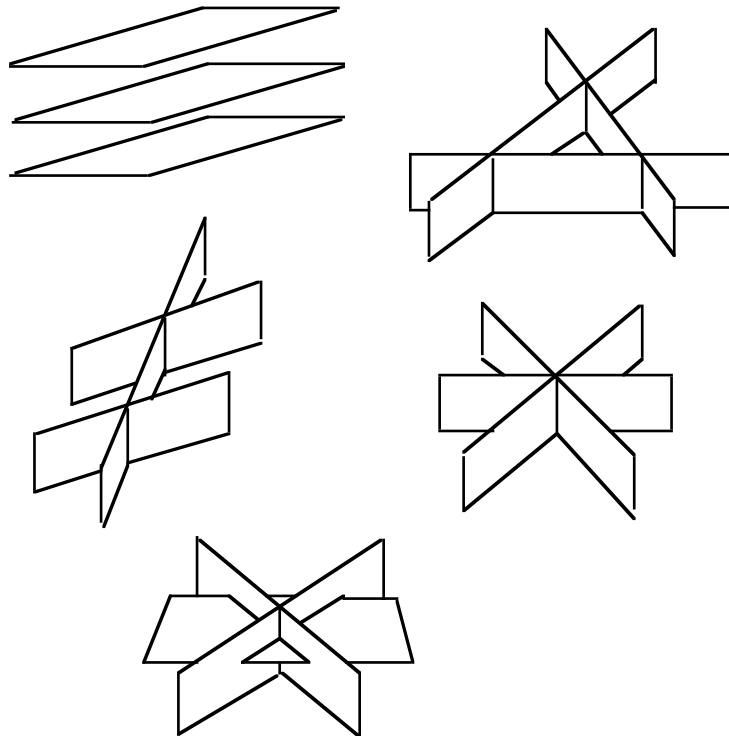
## Discussion 1

(page 200)

- Sample response: Three planes would be needed: one to graph pizza versus pop, one for pizza versus nachos, and one for pop versus nachos.
- Sample response: Three variables are necessary: one each for pizza, pop, and nachos.
- Sample response: No. One plane can only accommodate two different variables.
- Sample response: The constraints for three items could be displayed on a three-dimensional coordinate system where planes would represent the boundaries.

## Teacher Note

You may wish to discuss the various ways in which three planes can intersect one another. These are illustrated below.





## Exploration

(page 200)

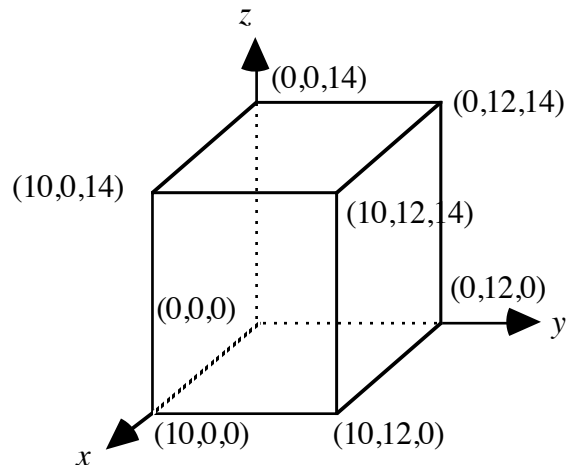
Students build a model of the first octant of a three-dimensional coordinate system. This activity works well in small groups.

- a–b.** Students cut cardboard boxes and label the axes of their coordinate systems. **Note:** The leftover cardboard should be saved for use in Part **d**.
  - c–d.** On each plane surface of the first octant, students shade one of three regions defined by inequalities. They then cut a model of each region from the leftover pieces of cardboard.
  - e.** Students move the model of region A through the various values of the constraint placed on  $z$ .
  - f.** Students should tape the model regions to form a rectangular prism in the first octant. This prism defines the set of feasible solutions for the constraints  $0 \leq x \leq 6$ ,  $0 \leq y \leq 4$ , and  $0 \leq z \leq 5$ .
  - g.** The coordinates of the eight corner points are:  $(0,0,0)$ ,  $(0,0,5)$ ,  $(6,0,0)$ ,  $(6,0,5)$ ,  $(0,4,0)$ ,  $(0,4,5)$ ,  $(6,4,0)$ , and  $(6,4,5)$ .
- a.** The set of points describes five planes which are parallel to Region A and each measures 4 units by 6 units. The planes are located at  $z = 1, 2, 3, 4,$  and  $5$ .
  - b.**
    - 1.** Three planes may intersect in a single point. Two planes may intersect in a line.
    - 2.** Each corner point is formed from the intersection of three planes. There are four ways in which three planes can be selected from four planes:  $ABC$ ,  $ABD$ ,  $ACD$ , and  $BCD$ .
    - 3.** The maximum number of corner points is 10. Given five planes  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ , the 10 possible sets of three planes are  $ABC$ ,  $ABD$ ,  $ABE$ ,  $ACD$ ,  $ACE$ ,  $ADE$ ,  $BCD$ ,  $BCE$ ,  $BDE$ , and  $CDE$ .
  - c.**
    - 1.** The shape is a rectangular prism. A rectangular prism has eight vertices.
    - 2.** Sample response: No. Not all sets of constraints that involve six planes would have three pairs of parallel planes. The parallel planes reduce the number of possible corner points.
  - d.**
    - 1.** Sample response: By the corner principle, the maximum will occur at one of the eight corner points. The specific point where the maximum occurs would depend on the objective function.
    - 2.** Sample response: The coordinates of each corner point could be substituted into the objective function to determine which represents the maximum value.
  - e.** Sample response: No, this would require four dimensions.

## Assignment

(page 202)

- 4.1**
- This graph shows the constraints on pizza and pop. The shaded region is defined by  $0 \leq x \leq 10$ ,  $0 \leq y \leq 12$ , and  $z = 0$ .
  - This graph shows the constraints on pop and nachos. The shaded region is defined by  $0 \leq y \leq 12$ ,  $0 \leq z \leq 14$ , and  $x = 0$ .
  - This graph shows the constraints on pizza and nachos. The shaded region is defined by  $0 \leq x \leq 10$ ,  $0 \leq z \leq 14$ , and  $y = 0$ .
- 4.2** Sample response: The class should purchase from 0 to 10 pizzas, from 0 to 12 six-packs of pop, and from 0 to 14 nacho kits.
- 4.3**
- $y = 12$ ,  $y = 0$ ,  $z = 14$ ,  $z = 0$
  - $x = 10$ ,  $x = 0$ ,  $z = 14$ ,  $z = 0$
  - Sample response: The points of intersection consist of all the possible combinations of ordered triples formed by the  $x$ -values,  $y$ -values, and  $z$ -values of the equations.
- 4.4** Sample sketch:



- 4.5** As shown in the table below, the maximum value of  $P = x + y + z$  occurs at the point  $(10,12,14)$ .

$(x,y,z)$	$P = x + y + z$
$(0,0,0)$	0
$(10,0,0)$	10
$(0,12,0)$	12
$(0,0,14)$	14
$(10,12,0)$	22
$(10,0,14)$	24
$(0,12,14)$	26
$(10,12,14)$	36

- 4.6 a.  $(-11, -3, 9)$   
 b. This system is inconsistent.  
 c.  $\left(\frac{23}{5}, -\frac{8}{5}, -\frac{1}{5}\right)$  or  $(4.6, -1.6, -0.2)$
- 4.7 a. The profit is \$11.00 on each pizza, \$4.00 on each six-pack of pop, and \$14.00 on each nacho kit.  
 b.  $P = 11x + 4y + 14z$
- \*4.8 a. The constraints are  $0 \leq x \leq 10$ ,  $0 \leq y \leq 25$ , and  $0 \leq z \leq 5$ .  
 b. Six planes enclose the feasible solutions.  
 c. Sample response: Eight points must be examined to find the maximum profit. There are eight corner points because the figure formed by the planes is a rectangular prism. **Note:** Students who do not visualize the prism may respond that 20 points must be examined (the maximum number of ways in which six planes can be chosen three at a time).
- 4.9 The maximum value of  $P = 11x + 4y + 14z$  occurs at the corner point  $(10, 25, 5)$ . The profit at this point is  $11(10) + 4(25) + 14(5) = \$280$ .

\*\*\*\*\*

- 4.10 Answers will vary. Students should record the cost and number of each item to be purchased, the number of servings that can be produced from each one, the selling prices, and the profit.
- 4.11 The solution may be found as follows:

$$\begin{array}{l} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 2 & 1 & 2 & 26 \\ 5 & 2 & 3 & 43 \end{array} \right] \begin{array}{l} [x] \\ [y] \\ [z] \end{array} \\ \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 2 \\ 5 & 2 & 3 & 5 & 2 & 3 \end{array} \right] \begin{array}{l} [x] \\ [y] \\ [z] \end{array} \end{array}$$

\*\*\*\*\*

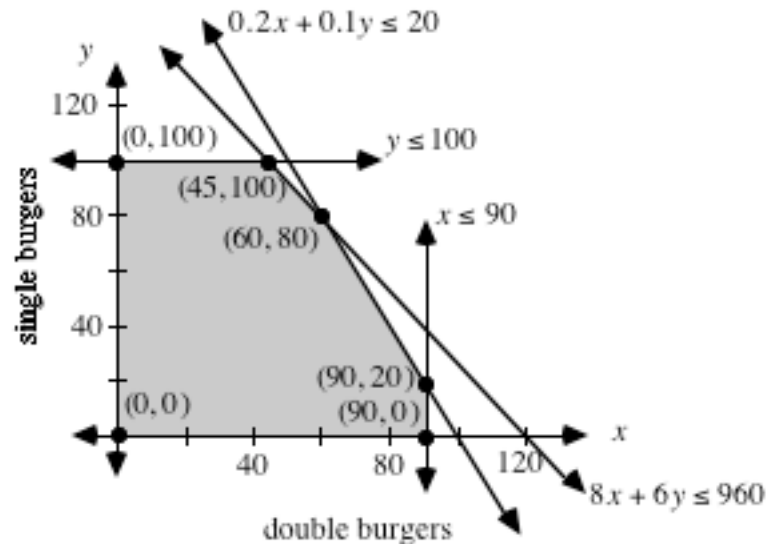
## Answers to Summary Assessment

(page 204)

1. Sample response: Let  $x$  represent the number of double hamburgers and  $y$  represent the number of single hamburgers. The following inequalities are the constraints that define the feasible region:  
 $0.2x + 0.1y \leq 20$ ,  $x \geq 0$ ,  $y \geq 0$ ,  $x \leq 90$ ,  $y \leq 100$ , and  $8x + 6y \leq 960$ .

The profit function is the equation  $p = 0.90x + 0.70y$ .

The graph below shows the feasible region.



Since the maximum profit of \$110.50 occurs at the corner point  $(45,100)$ , the restaurant should prepare 45 double burgers and 100 singles for the lunch-time rush.

2. Sample response: Using  $h$  for the number of hot dogs,  $c$  for the number of bags of chips, and  $p$  for the number of cans of pop, the three constraints can be expressed as:

$$0.15h + 0.2c + 0.35p \leq 108$$

$$c \leq 2h$$

$$c + h \leq 3p$$

The corresponding matrix equation for the boundaries of these constraints is:

$$\begin{bmatrix} 0.15 & 0.2 & 0.35 \\ -2 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} h \\ c \\ p \end{bmatrix} = \begin{bmatrix} 108 \\ 0 \\ 0 \end{bmatrix}$$

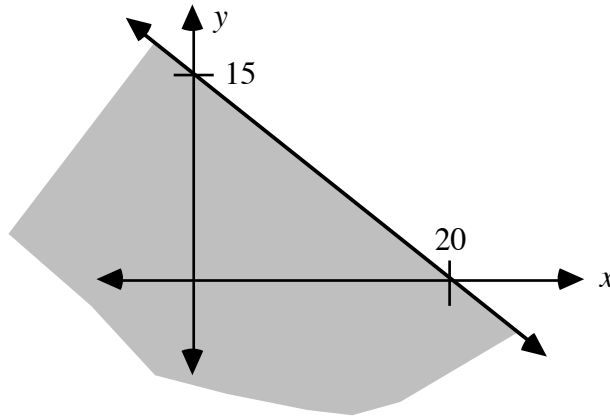
The solution to this system is  $(120,240,120)$ . The profit equation is  $P = 0.85h + 0.8c + 0.65p$ . At this corner point, the profit is \$372.

## Module Assessment

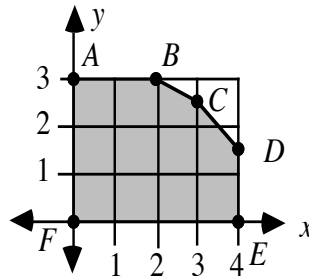
1. Use inequalities to describe the points graphed on the following number line.



2. Use inequalities to describe the points graphed in the shaded region below.



3. Which point in the graph below maximizes the value of the objective function  $P = 3x + y$ ? Defend your response.



4. Consider the system of linear equations below:

$$2x - 3y + z = 2.7$$

$$4x + 2.4y - 1.5z = 11.7$$

$$\frac{1}{2}x + 9y + 3.8z = 39.96$$

- a. Write a matrix equation for this system.
- b. Solve the system and verify your solution.
- c. Determine the value of the objective function  $P = 3x + 2y - 5z$  at the point described in Part b.

## Answers to Module Assessment

- $3 < x \leq 21$
- $15x + 20y \leq 300$
- Sample response: According to the corner principle, the maximum value of an objective function occurs at a corner point of the feasible region. As shown in the table below, the maximum of 13.5 occurs at point  $D$ .

Point	Coordinates	$P = 3x + y$
$A$	(0,3)	3
$B$	(2,3)	9
$C$	(3,2.5)	11.5
$D$	(4,1.5)	13.5
$E$	(4,0)	12
$F$	(0,0)	0

- The corresponding matrix equation is shown below:

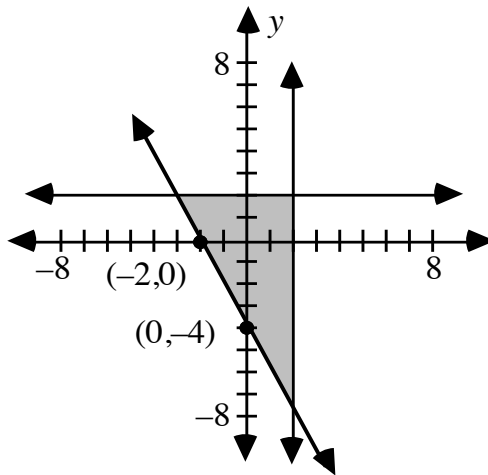
$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 2.4 & -1.5 \\ 0.5 & 9 & 3.8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2.7 \\ 11.7 \\ 39.96 \end{bmatrix}$$

- The solution is (3,2.5,4.2). Students should verify their solutions by substituting into the original equations.
- $P = -7$

# Flashbacks

## Activity 1

- 1.1** Graph each of the following inequalities on a number line:
- $x \leq 25$
  - $-16 < x$
  - $-2 \leq x < 4$
- 1.2** Graph  $x \geq -4$  and  $y < 7$  on a two-dimensional coordinate system and name three points in the intersecting region.
- 1.3.** Graph  $y \geq -4$ ,  $x \leq 0$ , and  $2x \geq y - 4$  on a two-dimensional coordinate system and name three points in the intersecting region.
- 1.4** Identify the inequalities whose intersection is the shaded region on the graph below.



## Activity 2

- 2.1** Identify the slope and y-intercept of the line represented by each of the following equations.
- $2y - 4x = 12$
  - $-y + 16 = 3x$
- 2.2** Given the equation  $w = 3.56a - 4.1b$ , find  $w$  for each of the following values of  $a$  and  $b$ :
- $a = 2$  and  $b = 6$
  - $a = -5$  and  $b = 0$
- 2.3** Determine the equation, in slope-intercept form, of the line that passes through  $(2,4)$  and  $(-5,1)$ .

## Activity 3

- 3.1** Complete each of the following matrix multiplications.

- $$\begin{bmatrix} 2 & 7 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$
- $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 2 & -6 \end{bmatrix}$$
- $$\begin{bmatrix} 2 & 11 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

## Activity 4

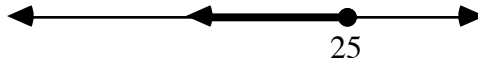
- 4.1** Each of the following ordered triples represents the coordinates of a point in the form  $(x,y,z)$ . Plot and label each point on a three-dimensional coordinate system.
- $(0,3,0)$
  - $(5,3,2)$
  - $(8,7,3)$
- 4.2** Find the points of intersection, if any, for each of the following pairs of lines.
- $x + y = 6$  and  $y = 4$
  - $2x - y = 4$  and  $x + 3y = -3$
  - $x = 4 - 2y$  and  $y = -3$



# Answers to Flashbacks

## Activity 1

1.1 a. Sample graph:



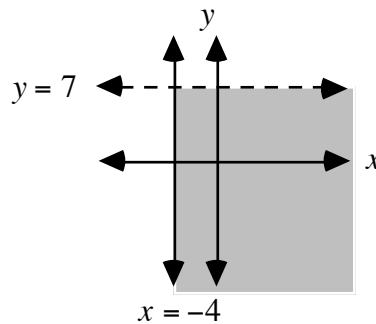
b. Sample graph:



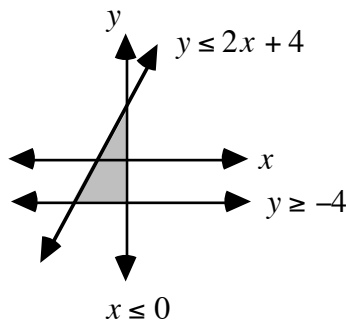
c. Sample graph:



1.2 Sample response: Three points in the intersecting region are  $(0,0)$ ,  $(-4,6.9)$ , and  $(40,-50)$ .



1.3 Sample response: Three points in the intersecting region are:  $(0,0)$ ,  $(-4,-4)$ , and  $(-1,-1)$ .



1.4  $y \geq -2x - 4$ ,  $y \leq 2$ ,  $x \leq 2$

## Activity 2

- 2.1 a.  $m = 2, b = 6$   
b.  $m = -3, b = 16$
- 2.2 a.  $w = -17.48$   
b.  $w = -17.8$
- 2.3 Sample response:

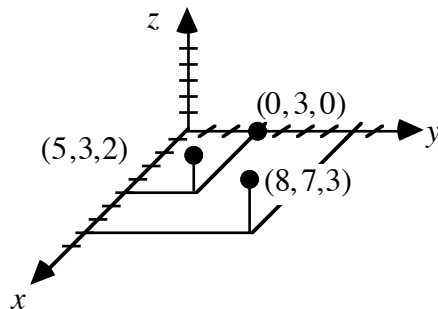
$$(y - 4) = \left( \frac{4 - 1}{2 - (-5)} \right) (x - 2)$$
$$y = \frac{3}{7}x + \frac{22}{7}$$

## Activity 3

- 3.1 a.  $\begin{bmatrix} 3 \\ -21 \end{bmatrix}$
- b.  $\begin{bmatrix} 3 & 4 \\ 2 & -6 \end{bmatrix}$
- c.  $\begin{bmatrix} 24 & 11 \\ 3 & 5 \end{bmatrix}$

## Activity 4

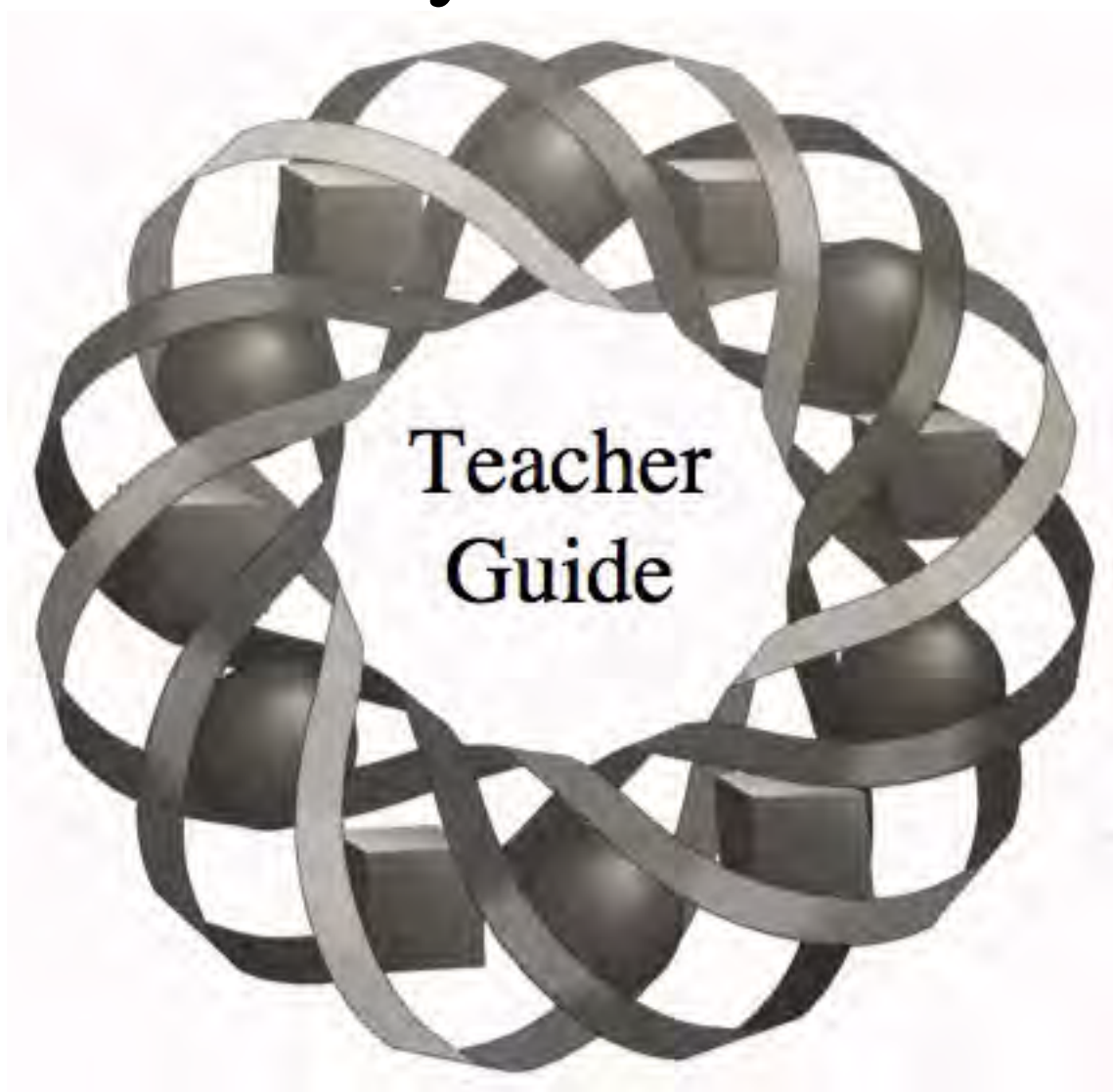
- 4.1 a-c. Sample graph:



- 4.2 a.  $(2, 4)$
- b.  $\left( \frac{9}{7}, \frac{-10}{7} \right)$
- c.  $(10, -3)$



# Crazy Cartoons



How are cartoon characters made? How do they change shape? How can you make an inanimate object appear to move? A cartoonist—or a math student—can help answer these questions.

*Monty Brekke • John Carter • Janet Kuchenbrod*



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# Teacher Edition

## Crazy Cartoons

### Overview

Students explore transformations in a plane, including dilations, translations, rotations, reflections, and composite transformations.

### Objectives

In this module, students will:

- use the properties of similar figures
- calculate distances between points in the Cartesian plane
- explore the geometric relationships in perspective drawings, dilations centered at the origin, translations, rotations about the origin, reflections, and composite transformations
- write matrices for dilations centered at the origin, translations, rotations about the origin, and reflections
- use matrix equations to transform geometric figures on the Cartesian plane.

### Prerequisites

For this module, students should know:

- how to graph points in the Cartesian plane
- the relationships between similar figures
- how to solve systems of linear equations
- how to perform operations on matrices, including scalar multiplication, matrix addition, and matrix multiplication
- how to find the equation of a line given two points on the line
- how to use trigonometric ratios to determine angle measures
- how to reflect a set of points in a line.

### Time Line

Activity	1	2	3	4	5	Summary Assessment	Total
Days	3	2	2	2	2	1	12

## Teacher Note

Students may use graph paper, tracing paper, MIRAs™, transparent grids, graphing calculators, spreadsheets, or other appropriate graphing utilities to perform transformations. In addition, you may want to use technology to solve matrix equations or simultaneous equations.

## Materials Required

Materials	Activity					Summary Assessment
	1	2	3	4	5	
rulers	X	X	X	X	X	X
graph paper	X	X	X	X	X	X
protractors	X			X		X
template for Problem 2.7		X				
tracing paper		X	X	X	X	X
decks of playing cards				X		

## Teacher Note

A blackline master of the template appears at the end of the teacher edition FOR THIS MODULE.

## Technology

Software	Activity					Summary Assessment
	1	2	3	4	5	
geometry utility	X		X		X	X
graphing utility	X	X	X	X		X
spreadsheet		X	X	X	X	X
matrix manipulator		X	X	X	X	X

# Crazy Cartoons

## Introduction

(page 209)

Students describe transformations in general terms, without the use of coordinates.

## Discussion

(page 209)

- a. Sample response: The bug in frame B is a reflection of the bug in frame A with respect to a vertical line. The bug in frame C is a  $60^\circ$  clockwise rotation of the bug in frame A. The bug in frame D is a  $90^\circ$  counterclockwise rotation of the bug in frame A. The bug in frame E is a scaled-down version of the bug in frame A. The bug in frame F is a scaled-up version of the bug in frame A.
- b. Sample response: The bug in frame B can be described as the combination of a  $180^\circ$  rotation of the bug in frame A, followed by a reflection with respect to a horizontal line.

(page 210)

## *Activity 1*

In this activity, students explore dilations by graphing shapes in the coordinate plane. They create and measure similar shapes using a geometry utility.

### Materials List

- graph paper (one sheet per student)
- rulers (one per student)
- protractors (one per student)

### Technology

- geometry utility

## Teacher Note

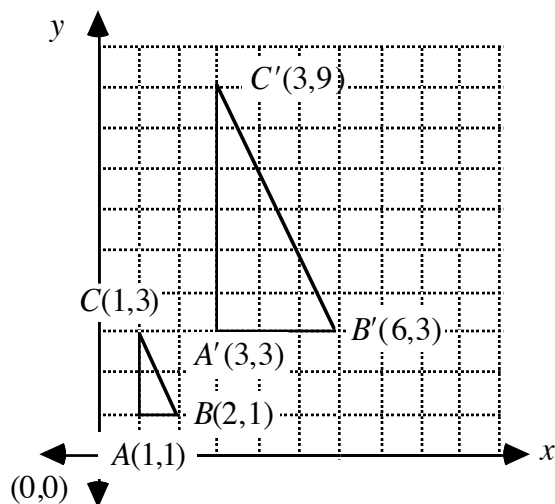
Point of perspective and dilations are important in both drawing and painting. You may wish to use one or both of the following quotations to initiate a class discussion.

- The optical model, or the visual pyramid, can secure the correctness of the painter's representation of reality. The painting . . . should be conceived as a "planar representation" of the visual pyramid. All the figures, objects, and spatial distances painted on the artist's panel are fully proportional to those found in the actual world that the painting represents. Perspective construction, then, makes it provable that a picture is "correct" in its rendering of space. (Barasch, *Theories of Art*)
- . . . perspective construction begins with a series of arbitrary acts, as the painter selects a visual field of "whatever size I want," picks a size for the human figures, and places his centric point "wherever I wish." (Hulse, *The Rule of Art*)

## Exploration

(page 211)

a–b. Sample graph:

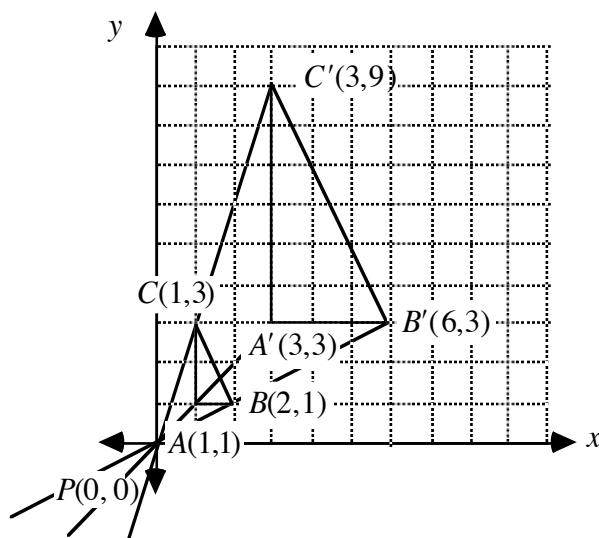


- c. In the following sample table, lengths are measured in centimeters and angles are measured in degrees.

Preimage		Image	
$AB$	1	$A'B'$	3
$BC$	$\sqrt{5} \approx 2.2$	$B'C'$	$3\sqrt{5} \approx 6.6$
$CA$	2	$C'A'$	6
$m\angle ABC$	$63^\circ$	$m\angle A'B'C'$	$63^\circ$
$m\angle BCA$	$27^\circ$	$m\angle B'C'A'$	$27^\circ$
$m\angle CAB$	$90^\circ$	$m\angle C'A'B'$	$90^\circ$



- d. The ratios of corresponding lengths are  $3/1$ .
- e. Sample graph:



- f.  $PA = \sqrt{2} \approx 1.4$ ,  $PB = \sqrt{5} \approx 2.2$ ,  $PC = \sqrt{10} \approx 3.2$ ,  $PA' = 3\sqrt{2} \approx 4.2$ ,  
 $PB' = 3\sqrt{5} \approx 6.6$ ,  $PC' = 3\sqrt{10} \approx 9.5$

- g–h. As in Part d, the ratios of corresponding lengths are  $3/1$ .

## Discussion

(page 212)

- a. Sample response: The segment connecting two points can be thought of as the hypotenuse of a right triangle. The length of the horizontal side is  $|x_2 - x_1|$  and the length of the vertical side is  $|y_2 - y_1|$ . Using the Pythagorean theorem, the distance  $d$  between the two points is:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- b. The ratios of corresponding lengths are equal.
- c. Sample response: The point of perspective is the center of dilation. The resulting figures are similar.
- d. The corresponding angles are congruent.
- e. In a dilation, the corresponding lengths are proportional and the corresponding angles are congruent.
- f. 1.  $\triangle ABC \sim \triangle A'B'C'$   
 2.  $\triangle PAC \sim \triangle P'A'C'$ ,  $\triangle PAB \sim \triangle P'A'B'$ , and  $\triangle PBC \sim \triangle P'B'C'$

- g.
  1. The scale factor equals the ratio of corresponding lengths.
  2. The scale factor equals the ratio of perimeters.
  3. The ratio of the areas is the square of the scale factor.
- h. Sample response: Yes, because the two triangles would still be similar.
- i. A scale factor  $x$ , where  $-1 < x < 1$ , reduces the size of the original shape.
- j. The ratio is the scale factor.

### Assignment

(page 214)

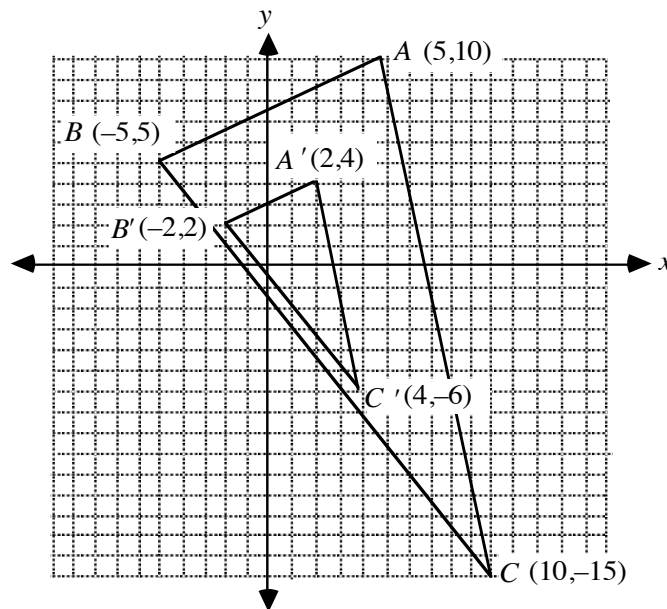
- 1.1 a. Sample response: The center of the dilation can be found by drawing a line through  $A$  and  $A'$  and another line through  $C$  and  $C'$  (or through any two pairs of corresponding points in the image and the preimage). The point of intersection of the two lines is the center of dilation.

- b. The scale factor is the ratio of corresponding lengths:

$$\frac{55}{22} = \frac{5}{2} = \frac{2.5}{1}$$

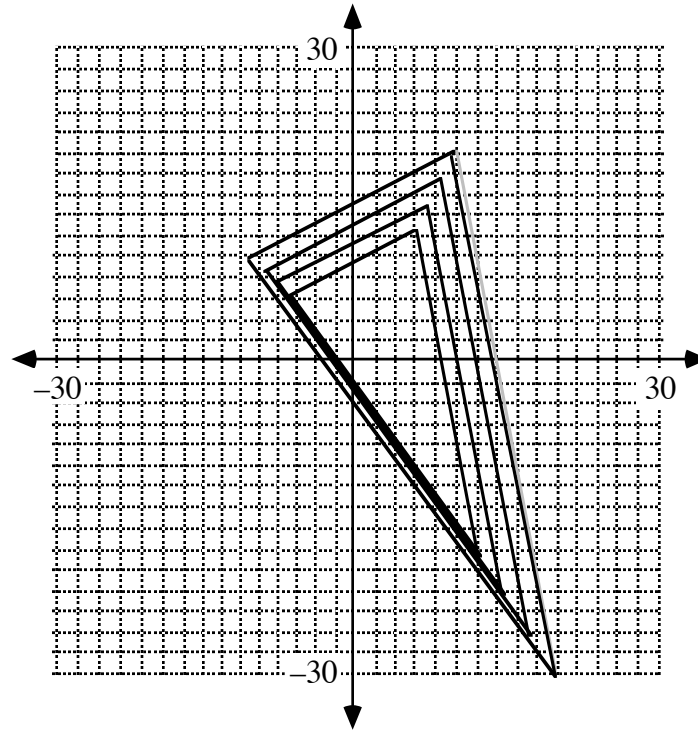
- 1.2 Sample response: No. The figures are not similar; therefore, they could not have been produced by a dilation. There is no point of perspective.

- 1.3 a–c. Sample graph:



- d.
  1. 2
  2.  $-1/5$
  3. 2.5

- 1.4 a. Sample graph:



- b. Sample response: The triangle would appear to double in size, growing like an animated cartoon.
- c. Sample response: Assign one figure to represent the preimage and the other to represent the image. Using the origin as the center of dilation, dilate the preimage by a scale factor equivalent to the ratio of the corresponding lengths of the image to the preimage.

- \*1.5 a.  $3/1$   
 b.  $1/2$   
 c.  $3/2$

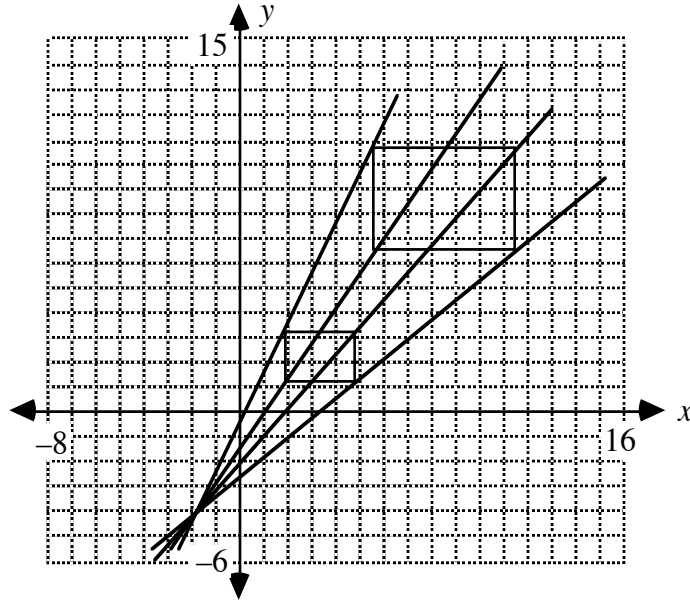
- d. Sample response: They could not have the same center. When you draw lines through corresponding points in the images and preimages, they do not intersect at the same point.

\* \* \* \* \*

- 1.6 The center of dilation is the origin and the scale factor is  $2/1$ .
- 1.7 Sketches will vary. Students should record the coordinates of the point of perspective (which is also the center of dilation). They should also describe how the coordinates of an image point  $X'$  and its preimage  $X$  can be used to find the scale factor  $r$  as follows:

$$r = \frac{PX'}{PX}$$

- 1.8
- See sample graph given in Part **d** below.
  - Sample response: The ratio of the sides of the image to the sides of the preimage is  $2/1$  and all the angles are right angles. Since the corresponding sides are proportional and all the angles are congruent, the quadrilaterals are similar.
  - The scale factor is 2.
  1. Sample graph:



- Sample response: The coordinates of the point of perspective are approximately  $(-2, -4)$ .
- Sample response: The equation of the line containing  $A'$  and  $A$  is:

$$y - 1.2 = \left( \frac{6.7 - 1.2}{5.3 - 1.9} \right) \cdot (x - 1.9)$$

$$y = \frac{55}{34}x - \frac{637}{340}$$

The equation of the line containing  $B'$  and  $B$  is:

$$y - 1.2 = \left( \frac{6.7 - 1.2}{11.3 - 4.9} \right) \cdot (x - 4.9)$$

$$y = \frac{55}{64}x - \frac{1927}{640}$$

Their point of intersection is  $(-1.5, -4.3)$ .

\*\*\*\*\*

## **Activity 2**

In this activity, students use matrices to store the coordinates of points that define a figure. They discover that a dilation with center at the origin can be represented by scalar multiplication of the matrix or by a multiplication on the left by a  $2 \times 2$  matrix.

### **Materials List**

- graph paper (one sheet per student)
- ruler (one per student)
- tracing paper (one sheet per student)
- template for Problem 2.7 (one per student; a blackline master appears at the end of the teacher edition for this module)

### **Technology**

- matrix manipulator
- spreadsheet
- graphing utility

### **Teacher Note**

When using some types of technology (such as spreadsheets, for example) to draw closed figures, it may be necessary to repeat the coordinates of some points in a matrix. For example, a triangle can be defined on a coordinate grid by listing the coordinates of three points (the three vertices) in a  $2 \times 3$  matrix. However, in order to use a spreadsheet to draw the same triangle from a matrix, it may be necessary to list the coordinates of both a starting point and a stopping point (or four points in all).

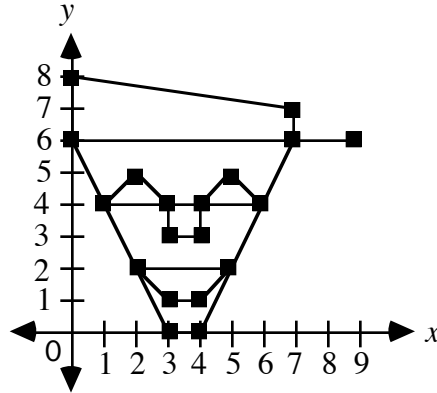
Alternately, students may simply use the spreadsheet utility to print an unconnected scatterplot, then connect the points by hand.

## Exploration

(page 217)

In this exploration, students draw and transform Skip on a coordinate plane.

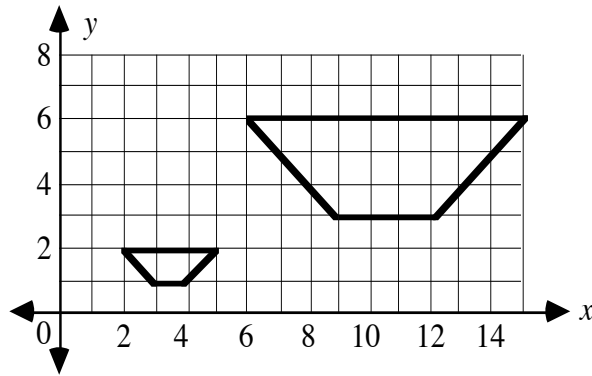
- a. Sample graph:



- b. Sample matrix:

$$\mathbf{M} = \begin{bmatrix} 2 & 5 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

- c. Sample graph:



- d. Sample matrix:

$$\mathbf{M}' = \begin{bmatrix} 6 & 15 & 12 & 9 \\ 6 & 6 & 3 & 3 \end{bmatrix}$$

- e. Students may use the following matrix equation to create eight equations with four unknowns.

$$\mathbf{D} \cdot \mathbf{M} = \mathbf{M}'$$

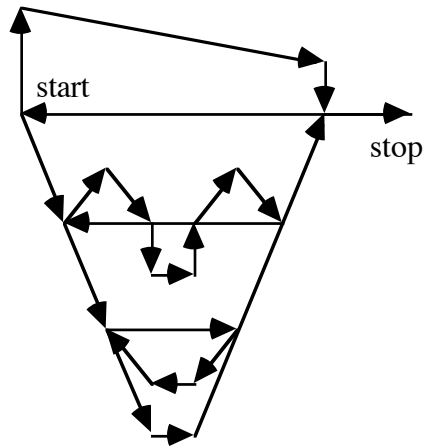
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 12 & 9 \\ 6 & 6 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2a+2b & 5a+2b & 4a+1b & 3a+1b \\ 2c+2d & 5c+2d & 4c+1d & 3c+1d \end{bmatrix} = \begin{bmatrix} 6 & 15 & 12 & 9 \\ 6 & 6 & 3 & 3 \end{bmatrix}$$

By solving this system of equations, students can determine the elements of  $\mathbf{D}$ .

$$\begin{array}{rcl} 2a + 2b = 6 & 2c + 2d = 6 & \\ 5a + 2b = 15 & 5c + 2d = 6 & \\ 4a + 1b = 12 & 4c + 1d = 3 & \Rightarrow \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ 3a + 1b = 9 & 3c + 1d = 3 & \end{array}$$

- f. Skip can be drawn without lifting a pencil or retracing a segment using the circuit shown below. Because Skip can be drawn with a circuit, students can use a spreadsheet to graph the cartoon without adding additional, unwanted segments. **Note:** There are several other ways to trace through the figure. Only one possibility is shown here.



In the following sample matrix, the coordinates of some points have been repeated in order to obtain the graph of a closed figure.

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 7 & 7 & 0 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 6 & 1 & 2 & 5 & 4 & 3 & 2 & 3 & 4 & 7 & 9 \\ 6 & 8 & 7 & 6 & 6 & 4 & 5 & 4 & 3 & 3 & 4 & 5 & 4 & 4 & 2 & 2 & 1 & 1 & 2 & 0 & 0 & 6 & 6 \end{bmatrix}$$

- g. The elements in matrix  $\mathbf{S}'$  are three times greater than the corresponding elements in matrix  $\mathbf{S}$ .

## Discussion

(page 218)

- a. Sample response: A dilation by a scale factor of 3, with center at the origin, can be accomplished by multiplying the preimage matrix by the scalar 3 to obtain the image matrix.
- b. In order to multiply matrices, the number of columns in the matrix on the left must equal the number of rows in the matrix on the right.

c. 1.  $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

2.  $\begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$

3.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- d. As shown in the following equation, when an identity matrix is multiplied by a scalar  $k$ , the product is the transformation matrix for a dilation of scale factor  $k$ , with center at the origin.

$$k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

- e. Sample response: To determine the matrix that represents the image of Skip, you could multiply each entry by  $n$ , multiply the matrix that represents Skip's face by a scalar of  $n$ , or multiply on the left by the following matrix:

$$\begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$$

- e. Sample response: In a dilation, the image and preimage are similar. They have the same shape and their corresponding angles are congruent. The lengths of the sides of the image are proportional to those of the preimage.

## Teacher Note

To complete Problem 2.7, each student will require a copy of the template. A blackline master appears at the end of the teacher edition for this module.



## Assignment

(page 219)

- 2.1 a. Answers may vary. Sample matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 2 & 5 & 6 & 3 & 0 \\ 3 & 7 & 5 & 2 & 0 & 3 \end{bmatrix}$$

- b. 1. Sample response using scalar multiplication:

$$\mathbf{P}' = 3.5 \cdot \begin{bmatrix} 0 & 2 & 5 & 6 & 3 & 0 \\ 3 & 7 & 5 & 2 & 0 & 3 \end{bmatrix}$$

2. Sample response using multiplication by a  $2 \times 2$  matrix:

$$\mathbf{P}' = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 5 & 6 & 3 & 0 \\ 3 & 7 & 5 & 2 & 0 & 3 \end{bmatrix}$$

- c. Sample matrix:

$$\mathbf{P}' = \begin{bmatrix} 0 & 7 & 17.5 & 21 & 10.5 & 0 \\ 10.5 & 24.5 & 17.5 & 7 & 0 & 10.5 \end{bmatrix}$$

- 2.2 a. Sample response:  $5 \cdot \mathbf{S} = \mathbf{S}'$  or

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \cdot \mathbf{S} = \mathbf{S}'$$

- b. Sample response:

$$\mathbf{C}' = 0.75 \cdot \begin{bmatrix} 9 & 0 & 0 & 7 & 7 \\ 6 & 6 & 8 & 7 & 6 \end{bmatrix}$$

- c. Using matrix  $\mathbf{M}$  from Part **b** of the exploration,

$$\mathbf{M} = \begin{bmatrix} 2 & 5 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

then the coordinates for the image of Skip's mouth under the given dilation are represented in the following matrix:

$$\mathbf{M}' = \begin{bmatrix} 1.5 & 3.75 & 3 & 2.25 \\ 1.5 & 1.5 & 0.75 & 0.75 \end{bmatrix}$$

- \*2.3 a. Sample matrix:

$$\mathbf{B} = \begin{bmatrix} a & i & g & e & c \\ b & j & h & f & d \end{bmatrix}$$

- b. The scale factor from  $B$  to  $A$  is  $1/2$  or  $0.5$ . The scale factor from  $B$  to  $C$  is  $3/1$ .

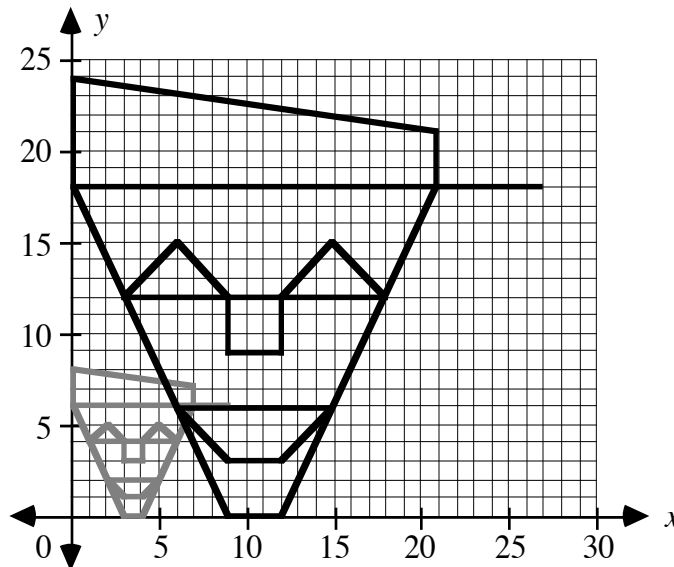
c. Sample response:

$$\begin{aligned} \mathbf{A} &= 0.5 \cdot \mathbf{B} \\ &= \begin{bmatrix} 0.5a & 0.5i & 0.5g & 0.5e & 0.5c \\ 0.5b & 0.5j & 0.5h & 0.5f & 0.5d \end{bmatrix} \end{aligned}$$

d. Sample response:

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0.5a & 0.5i & 0.5g & 0.5e & 0.5c \\ 0.5b & 0.5j & 0.5h & 0.5f & 0.5d \end{bmatrix} \\ &= \begin{bmatrix} 3a & 3i & 3g & 3e & 3c \\ 3b & 3j & 3h & 3f & 3d \end{bmatrix} \end{aligned}$$

**\*2.4** The dilation of Skip by a scale factor of 3 with center at the origin can be accomplished using the matrix operation  $3 \cdot \mathbf{S} = \mathbf{S}'$ , where  $\mathbf{S}$  is the matrix representation for Skip and  $\mathbf{S}'$  is the matrix representation for Skip's image. Sample graph:



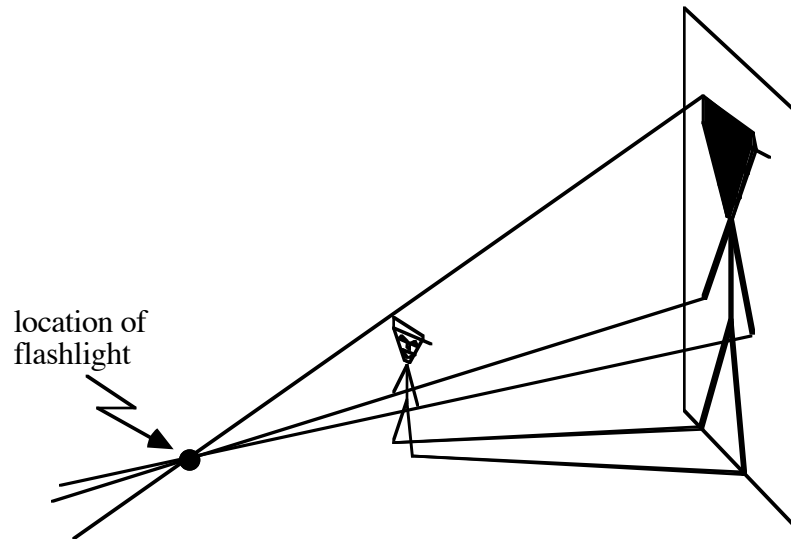
\*\*\*\*\*

**2.5** Sample response:

	small	medium	large	x-large
blue	120	280	120	160
red	200	320	80	40

- 2.6**
- a. Sample response: At 6:00 p.m. on Tuesday, the diameter of the earth's image appears to be about 0.8 cm. At 1:00 a.m. on Wednesday, the diameter of the earth's image appears to be about 2.3 cm. The scale factor is about  $2.3/0.8 \approx 2.9$ .
- b. Sample response: After another 7 hr, the diameter of the earth's image should appear to be about 6.7 cm, which is 2.9 times larger than 2.3 cm.

- 2.7 a. As shown in the diagram below, the position of the flashlight can be determined by drawing the lines that connect corresponding vertices.



- b. Sample response: Chip should hold the flashlight so that its distance from the wall is five times its distance from Skip. Then the ratio  $PX'/PX$ , or the scale factor, will equal 5.
- c. Sample response: As Chip moves the flashlight toward Skip, his shadow will grow larger. This is because the ratio of the distance from the flashlight to the wall to the distance from the flashlight to Skip becomes larger.

\* \* \* \* \*

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### **Activity 3**

In this activity, students explore translations in the coordinate plane.

#### **Materials List**

- graph paper
- tracing paper
- ruler (one per student)

## Technology

- geometry utility
- matrix manipulator
- graphing utility
- spreadsheet

## Teacher Note

As an extension to Activity 3, you may wish to explore dilations with center at  $C(a,b)$ . Given a scale factor  $s$ , a dilation with center at  $C(a,b)$  maps the point  $P(x,y)$  onto the point  $P'(s(x-a)+a, s(y-b)+b)$ . For example, a dilation by a scale factor of 0.5 with center at  $C(5,6)$  of the triangle defined by matrix  $\mathbf{S}$  yields matrix  $\mathbf{S}'$  as follows:

$$\mathbf{S} = \begin{bmatrix} 3 & 5 & -2 \\ 6 & -1 & -3 \end{bmatrix}$$

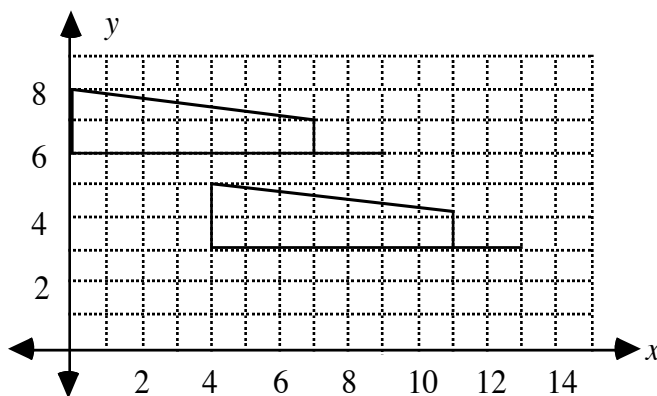
$$\mathbf{S}' = \begin{bmatrix} 0.5(3-5)+5 & 0.5(5-5)+5 & 0.5(-2-5)+5 \\ 0.5(6-6)+6 & 0.5(-1-6)+6 & 0.5(-3-6)+6 \end{bmatrix}$$

$$\mathbf{S}' = \begin{bmatrix} 4 & 5 & 1.5 \\ 6 & 2.5 & 1.5 \end{bmatrix}$$

## Exploration

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- a. 1–2. Students trace Skip’s movements through the translation vector, then through this vector’s vertical and horizontal components. In both cases, Skip’s image moves to the same position at the bottom of the slide.
- b. Sample graph:



- c. Students should observe that every  $x$ -value in matrix  $\mathbf{C}$  is increased by 4 and every  $y$ -value is decreased by 3.

**d–e.** Students should write and verify the following equation:

$$\mathbf{C} + \mathbf{T} = \mathbf{C}'$$

$$\begin{bmatrix} 9 & 0 & 0 & 7 & 7 \\ 6 & 6 & 8 & 7 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ -3 & -3 & -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 13 & 4 & 4 & 11 & 11 \\ 3 & 3 & 5 & 4 & 3 \end{bmatrix}$$

## Discussion

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- a. Sample response: The two images coincided.
- b. Skip's preimage and image are congruent.
- c. Sample response: The image has been moved 4 units to the right and 3 units down from the preimage.
- d. The following matrix translates the seven points in matrix **C** 5 units down and 4 units to the left.

$$T_{C,C'} = \begin{bmatrix} -4 & -4 & -4 & -4 & -4 \\ -5 & -5 & -5 & -5 & -5 \end{bmatrix}$$

- e. Sample response: Since the translation vector and its vertical and horizontal components form a right triangle, you could find the length of the translation vector using the Pythagorean theorem.
- f. Sample response: To find the measure of the angle between the vertical component and the translation vector, you could use the inverse tangent.
- g. Sample response: The distance between a point in the preimage and the corresponding point in the image is the length of the translation vector.
- h.
  1. Using the distance formula, the length of the translation vector is  $\sqrt{(x+h-x)^2 + (y+k-y)^2} = \sqrt{h^2 + k^2}$ .
  2. Sample response: The angle between the horizontal component and the translation vector is  $\tan^{-1}(k/h)$ .

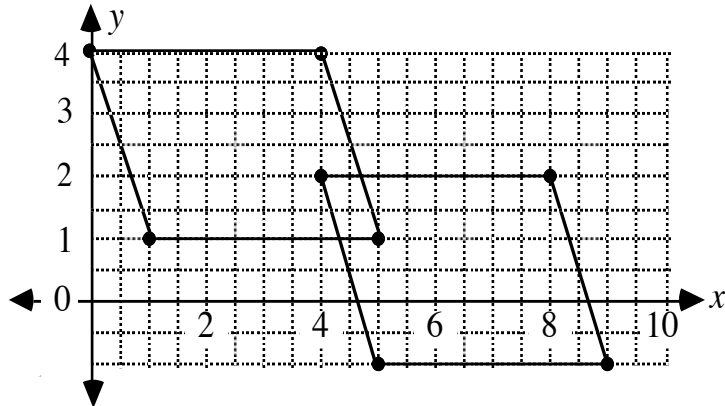
## Assignment

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- 3.1 Sample response: Skip was translated 5 units in a direction approximately  $37^\circ$  measured counterclockwise from the horizontal.
- 3.2 a. Sample matrix equation:

$$\begin{bmatrix} 1 & 5 & 4 & 0 \\ 1 & 1 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 & 4 \\ -2 & -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 8 & 4 \\ -1 & -1 & 2 & 2 \end{bmatrix}$$

b. Sample graph:



- 3.3 a. Sample response: The matrix consists of five columns and two rows. Each entry in the first row is 7, to represent the horizontal movement. Each entry in the second row is  $-6$ , to represent the vertical movement.

b. Sample matrix equation:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 6 & 4 & 6 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 7 & 7 & 7 & 7 \\ -6 & -6 & -6 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 10 & 11 & 12 \\ -5 & 0 & -2 & 0 & -5 \end{bmatrix}$$

- c. The measure of the angle is approximately  $49.4^\circ$ .
- d. The length of the translation vector can be found as follows:  
 $\sqrt{(8-1)^2 + (-5-1)^2} = \sqrt{49+36} = \sqrt{85} \approx 9.22$ .

3.4 a. See diagram given in Problem 3.3 (student edition).

b. Sample response: As you flipped through the cards, the M would appear to move down the page. **Note:** You may wish to allow students to actually create the flip cards. If so, the origin must be in the same position on each card, and each drawing must be taped in the appropriate location relative to the origin.

c. Sample response: Create a transformation matrix by subtracting the row and column values in the preimage matrix from the corresponding values in the image matrix. Then multiply this matrix by  $1/n$ , where  $n$  is the total number of images desired.

- \*3.5 a. Since students choose their own translations, answers will vary. In each translation matrix, both  $h$  and  $k$  should be less than 0.
- b. To obtain the image matrix  $S'$ ,  $h$  is added to each element in row 1 and  $k$  is added to each element in row 2.

- \*3.6 a. 1.** Sample equation for the translation:

$$\begin{bmatrix} 3 & 2 & -1 & -2 & -1 & 2 \\ 1 & 7 & 7 & 1 & -4 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ -8 & -8 & -8 & -8 & -8 & -8 \end{bmatrix} =$$

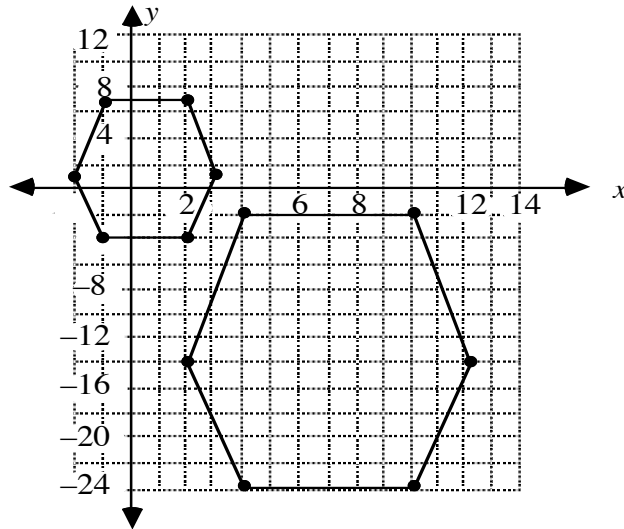
$$\begin{bmatrix} 6 & 5 & 2 & 1 & 2 & 5 \\ -7 & -1 & -1 & -7 & -12 & -12 \end{bmatrix}$$

Sample equation for the dilation:

$$2 \cdot \begin{bmatrix} 6 & 5 & 2 & 1 & 2 & 5 \\ -7 & -1 & -1 & -7 & -12 & -12 \end{bmatrix} =$$

$$\begin{bmatrix} 12 & 10 & 4 & 2 & 4 & 10 \\ -14 & -2 & -2 & -14 & -24 & -24 \end{bmatrix}$$

- 2.** The final image has vertices at (12,-14), (10,-2), (4,-2), (2,-14), (4,-24), and (10,-24). Sample graph:

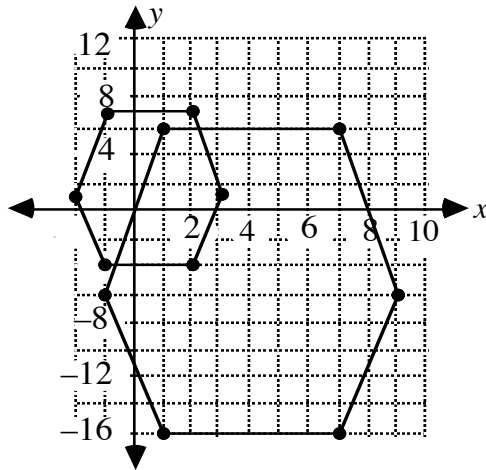


- 3.** In the sample equation below, **P** is the matrix representation of the original preimage:

$$2 \cdot \left[ \mathbf{P} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ -8 & -8 & -8 & -8 & -8 & -8 \end{bmatrix} \right] =$$

$$\begin{bmatrix} 12 & 10 & 4 & 2 & 4 & 10 \\ -14 & -2 & -2 & -14 & -24 & -24 \end{bmatrix}$$

- b. The final image has vertices at  $(9,-6)$ ,  $(7,6)$ ,  $(1,6)$ ,  $(-1,-6)$ ,  $(1,-16)$ , and  $(7,-16)$ . Sample graph:



In the sample equation below,  $\mathbf{P}$  is the matrix representation of the original preimage:

$$2 \cdot \mathbf{P} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ -8 & -8 & -8 & -8 & -8 & -8 \end{bmatrix} \\ = \begin{bmatrix} 9 & 7 & 1 & -1 & 1 & 7 \\ -6 & 6 & 6 & -6 & -16 & -16 \end{bmatrix}$$

- c. Sample response: The transformations are not equivalent. The two figures are congruent but positioned differently. The image in Part **a** is 3 units to the right and 8 units down from the image in Part **b**. In other words, it seems to have been translated one additional time. **Note:** It can be shown algebraically that these two composite transformations are not equivalent using the distributive property of multiplication over addition. The transformation in Part **a** is  $2(P + T) = 2P + 2T$ ; the transformation in Part **b** is  $2P + T$ .

\* \* \* \* \*

- 3.7 a. Sample response: In this translation, the image of quadrilateral  $ABCD$  moves 8 units to the right and 2 units up.

b. 
$$\begin{bmatrix} 8 & 8 & 8 & 8 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

- c. The distance between each point  $X$  and its image  $X'$  is  $\sqrt{8^2 + 2^2} = \sqrt{68} \approx 8.25$ .



- 3.8**
- a. The coordinates are  $A''(0, 2)$ ,  $B''(-6, 4)$ , and  $C''(-1, -4)$ .
- b. 1. In this translation, the image of triangle  $ABC$  moves 2 units to the left and 2 units down.
2. 
$$\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}$$
- c. The distance between each point  $X$  and its image  $X'$  is  $\sqrt{(-2)^2 + (-2)^2} = \sqrt{8} \approx 2.83$ .

\* \* \* \* \*

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### Activity 4

Students investigate rotations about the origin for specific degree measures.

#### Materials List

- graph paper (one sheet per student)
- ruler (one per student)
- protractor (one per student)
- tracing paper (one sheet per student)
- standard deck of playing cards (for Problem **4.10**)

#### Technology

- matrix manipulator
- graphing utility
- spreadsheet

#### Exploration

(page 227)

Students develop the transformation matrices for  $180^\circ$ ,  $360^\circ$ , and  $90^\circ$  rotations.

- a–b.** By rotating the tracing paper  $180^\circ$  counterclockwise about the origin, students should be able to estimate the coordinates for the image matrix of Chip’s mouth. Sample matrix:

$$\mathbf{P}' = \begin{bmatrix} -2 & -4 & -6 \\ -2 & 0 & -2 \end{bmatrix}$$

- c. Sample response: The matrix of the image formed by a  $180^\circ$  counterclockwise rotation can be found by multiplying the matrix of the preimage by the scalar  $-1$ .
- d. When multiplied on the left of a matrix of ordered pairs, the matrix below produces a  $180^\circ$  rotation about the origin:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- e. 1. By rotating the tracing paper  $360^\circ$  counterclockwise about the origin, students should be able to estimate the coordinates for the following image matrix:

$$\mathbf{P}' = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \end{bmatrix}$$

Students may conjecture that multiplication by the scalar 1 is required for a  $360^\circ$  counterclockwise rotation. When multiplied on the left of a matrix of ordered pairs, the matrix below produces a  $360^\circ$  rotation about the origin:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. By rotating the tracing paper  $90^\circ$  counterclockwise about the origin, students should be able to estimate the coordinates for the following image matrix:

$$\mathbf{P}' = \begin{bmatrix} -2 & 0 & -2 \\ 2 & 4 & 6 \end{bmatrix}$$

When multiplied on the left of a matrix of ordered pairs, the matrix below produces a  $90^\circ$  counterclockwise rotation about the origin:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

## Discussion

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- a. Sample response: The image would be the same as the preimage.
- b. Sample response: Since multiplication by the identity matrix would not change the preimage matrix, it could be thought of as producing a  $0^\circ$  rotation about the origin.
- c. In general, a rotation about the origin of any multiple of  $360^\circ$  will produce the same result as a rotation of  $360^\circ$ .

- d. A clockwise rotation of  $270^\circ$  produces the same image as a counterclockwise rotation of  $90^\circ$ . In general, a clockwise rotation of  $k^\circ$  is equivalent to a counterclockwise rotation of  $(n \cdot 360^\circ - k)$  for any integer  $n$ .
- e. Sample response: Yes. Since the end result is the same, the rotations are equivalent.
- f. In a rotation, the image and the preimage are congruent.

**Assignment**

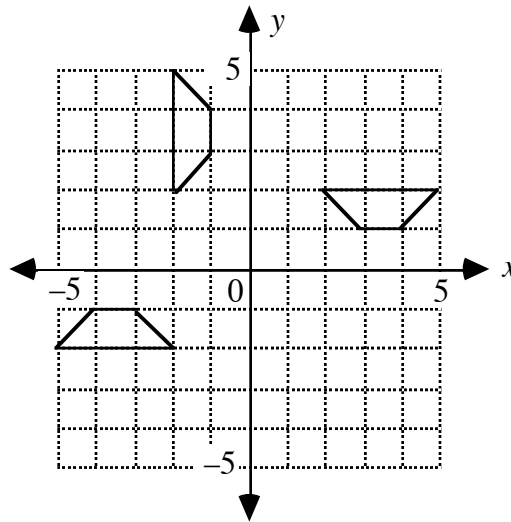
(page 229)

- 4.1 a. 1. The image matrix is the same as matrix **M**.

2. 
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \mathbf{M} = \mathbf{M}' = \begin{bmatrix} -2 & -5 & -4 & -3 \\ -2 & -2 & -1 & -1 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \mathbf{M} = \mathbf{M}' = \begin{bmatrix} -2 & -2 & -1 & -1 \\ 2 & 5 & 4 & 3 \end{bmatrix}$$

- b. Sample graph:



- \*4.2 a. The image matrix for Skip's mouth under a  $270^\circ$  counterclockwise rotation about the origin is:

$$\mathbf{M}' = \begin{bmatrix} -2 & -2 & -1 & -1 \\ 2 & 5 & 4 & 3 \end{bmatrix}$$

- b. When multiplied on the left of a matrix of ordered pairs, the following matrix produces a  $270^\circ$  rotation about the origin:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- 4.3**
- a. 1.  $a = \cos 30^\circ \approx 0.87$   
 2.  $b = \sin 30^\circ = 0.5$   
 3.  $\mathbf{P}' = \begin{bmatrix} \cos 30^\circ \\ \sin 30^\circ \end{bmatrix} \approx \begin{bmatrix} 0.87 \\ 0.5 \end{bmatrix}$
- b. 1.  $\mathbf{P}' = \begin{bmatrix} 2 \cos 30^\circ \\ 2 \sin 30^\circ \end{bmatrix} \approx \begin{bmatrix} 1.73 \\ 1 \end{bmatrix}$   
 2.  $\mathbf{P}' = \begin{bmatrix} c(\cos 30^\circ) \\ c(\sin 30^\circ) \end{bmatrix} \approx \begin{bmatrix} 0.87c \\ 0.5c \end{bmatrix}$
- 4.4**
- a. 1.  $a = \sin 30^\circ = 0.5$   
 2.  $b = \cos 30^\circ \approx 0.87$   
 3.  $\mathbf{Q}' = \begin{bmatrix} -\sin 30^\circ \\ \cos 30^\circ \end{bmatrix} \approx \begin{bmatrix} -0.5 \\ 0.87 \end{bmatrix}$
- b. 1.  $\mathbf{Q}' = \begin{bmatrix} -2 \sin 30^\circ \\ 2 \cos 30^\circ \end{bmatrix} \approx \begin{bmatrix} -1 \\ 1.73 \end{bmatrix}$   
 2.  $\mathbf{Q}' = \begin{bmatrix} -d(\sin 30^\circ) \\ d(\cos 30^\circ) \end{bmatrix} \approx \begin{bmatrix} -0.5d \\ 0.87d \end{bmatrix}$
- 4.5**
- a.  $\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \approx \begin{bmatrix} 0.87 & -0.5 \\ 0.5 & 0.87 \end{bmatrix}$
- b.  $\begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix} \approx \begin{bmatrix} 0.94 & -0.34 \\ 0.34 & 0.94 \end{bmatrix}$
- c.  $\begin{bmatrix} \cos n^\circ & -\sin n^\circ \\ \sin n^\circ & \cos n^\circ \end{bmatrix}$
- 4.6**
- a.  $\begin{bmatrix} 2 \cos 30^\circ & -2 \sin 30^\circ \\ 2 \sin 30^\circ & 2 \cos 30^\circ \end{bmatrix} \approx \begin{bmatrix} 1.73 & -1 \\ 1 & 1.73 \end{bmatrix}$
- b. 1.  $\begin{bmatrix} c(\cos 30^\circ) & -c(\sin 30^\circ) \\ c(\sin 30^\circ) & c(\cos 30^\circ) \end{bmatrix} \approx \begin{bmatrix} 0.87c & -0.5c \\ 0.5c & 0.87c \end{bmatrix}$   
 2.  $\begin{bmatrix} c(\cos n^\circ) & -c(\sin n^\circ) \\ c(\sin n^\circ) & c(\cos n^\circ) \end{bmatrix}$

- 4.7** Answers will vary, since students choose their own angles of rotation. The following sample equation summarizes a  $90^\circ$  rotation about the origin of Skip's cap:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 9 & 7 & 0 & 0 & 7 & 7 \\ 6 & 6 & 6 & 8 & 7 & 6 \end{bmatrix} = \begin{bmatrix} -6 & -6 & -6 & -8 & -7 & -6 \\ 9 & 7 & 0 & 0 & 7 & 7 \end{bmatrix}$$

**4.8** a.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -8 & -2 \end{bmatrix}$

b.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -8 & -2 \\ 5 & 0 & 0 \end{bmatrix}$

- 4.9** Sample response: For each of the following rotations about  $O$ , each point in the image coincides with the preimage:  $240^\circ$ ,  $360^\circ$ ,  $0^\circ$ ,  $-120^\circ$ , or  $-240^\circ$ .

- 4.10** Answers will vary, depending on the particular design of the cards.

- 4.11** Sample response: Skip's face does not have rotational symmetry. There is no point about which the face can be rotated to produce an image that coincides with the preimage.

- \*4.12** a. Sample response: A square has rotational symmetry of  $90^\circ$  about its center.



- b. a regular octagon

- c. 1.  $60^\circ$   
 2.  $72^\circ$   
 3.  $36^\circ$   
 4.  $360^\circ/n$

\* \* \* \* \*

4.13 a. 1.  $S' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \bullet S = \begin{bmatrix} 2 & 4 & 0 & 2 & 0 & -4 & 0 & -2 \\ 0 & -3.4 & -3.4 & 0 & 3.4 & -3.4 & -3.4 & 0 \end{bmatrix}$

2.  $S' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bullet S = \begin{bmatrix} -2 & -4 & 0 & -2 & 0 & 4 & 0 & 2 \\ 0 & -3.4 & -3.4 & 0 & 3.4 & -3.4 & -3.4 & 0 \end{bmatrix}$

3.  $S' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bullet S = \begin{bmatrix} -2 & -4 & 0 & -2 & 0 & 4 & 0 & 2 \\ 0 & 3.4 & 3.4 & 0 & -3.4 & 3.4 & 3.4 & 0 \end{bmatrix}$

4.  $S' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bullet S = \begin{bmatrix} -2 & -4 & 0 & -2 & 0 & 4 & 0 & 2 \\ 0 & 3.4 & 3.4 & 0 & -3.4 & 3.4 & 3.4 & 0 \end{bmatrix}$

5.  $S' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bullet S = \begin{bmatrix} 0 & -3.4 & -3.4 & 0 & 3.4 & -3.4 & -3.4 & 0 \\ -2 & -4 & 0 & -2 & 0 & 4 & 0 & 2 \end{bmatrix}$

b. Sample response: The angles which give rotational symmetry about the center are  $120^\circ$  and  $240^\circ$ .

4.14 Sample response: A  $90^\circ$  rotation about the origin takes all points in the first quadrant to points in the second quadrant.

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## Activity 5

In this activity, students examine reflections in the coordinate plane. The exploration builds on students' previous experiences with reflections. (See the Level 1 module, "Reflect on This.")

### Materials List

- graph paper (one sheet per student)
- ruler (one per student)
- MIRA™ (optional)

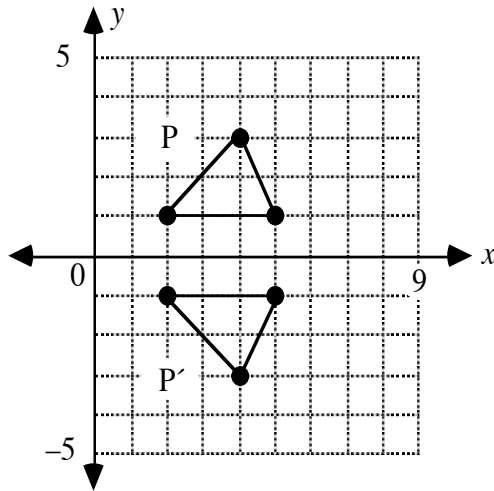
### Technology

- matrix manipulator
- spreadsheet
- geometry utility (optional)

## Exploration

(page 234)

- a. 1–2. Sample graph:

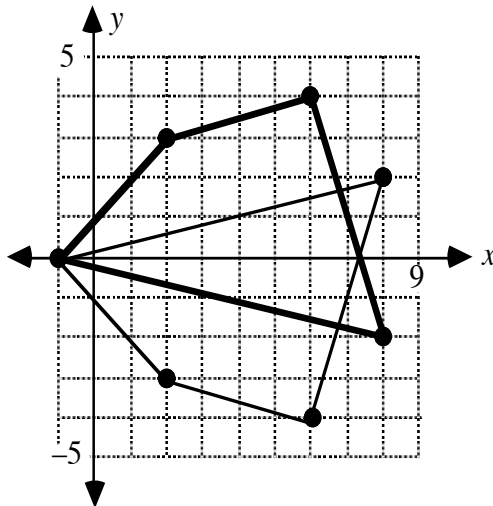


3. Sample matrix:

$$\mathbf{P}' = \begin{bmatrix} 2 & 4 & 5 \\ -1 & -3 & -1 \end{bmatrix}$$

- 4–5. The  $x$ -axis is the perpendicular bisector of the segments that connect the corresponding points of the preimage and image.

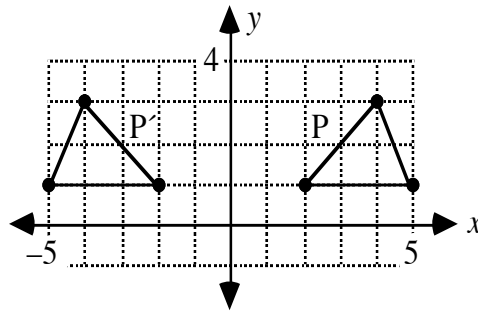
- b. 1–2. Sample graph:



3. Sample matrix:

$$\mathbf{H}' = \begin{bmatrix} -1 & 2 & 6 & 8 \\ 0 & -3 & -4 & 2 \end{bmatrix}$$

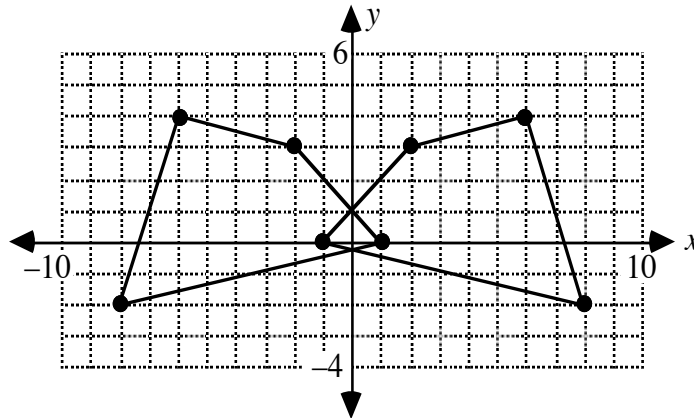
- c. The following sample graph shows the triangle represented by matrix  $\mathbf{P}$  reflected in the  $y$ -axis:



The coordinates of the image can be represented in the matrix below:

$$\mathbf{P}' = \begin{bmatrix} -2 & -4 & -5 \\ 1 & 3 & 1 \end{bmatrix}$$

The following sample graph shows the quadrilateral represented by matrix  $\mathbf{H}$  reflected in the  $y$ -axis:

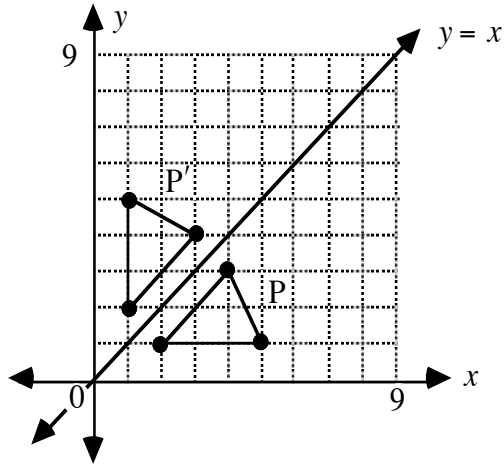


The coordinates of the image can be represented in the matrix below:

$$\mathbf{H}' = \begin{bmatrix} 1 & -2 & -6 & -8 \\ 0 & 3 & 4 & -2 \end{bmatrix}$$



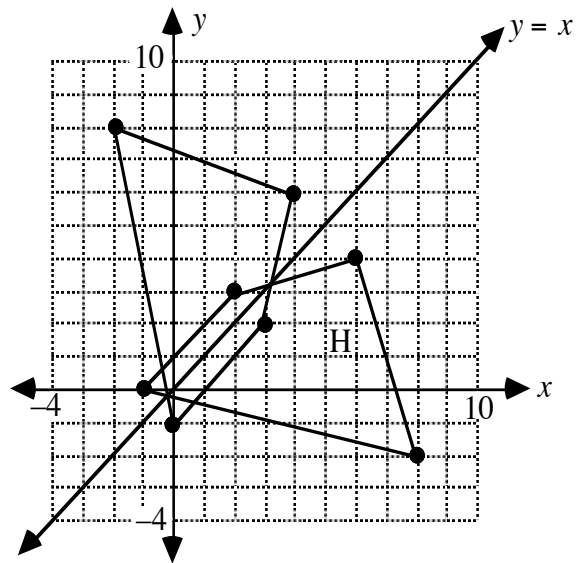
- d. 1. Sample graph:



2. Sample image matrix:

$$\mathbf{P}' = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \end{bmatrix}$$

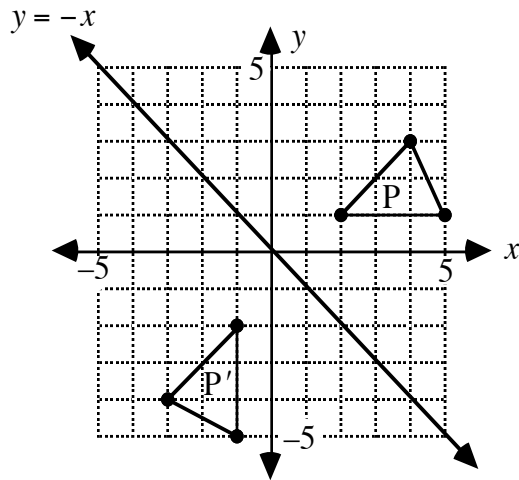
3. Sample graph:



The coordinates of the image can be represented in the matrix below:

$$\mathbf{H}' = \begin{bmatrix} 0 & 3 & 4 & -2 \\ -1 & 2 & 6 & 8 \end{bmatrix}$$

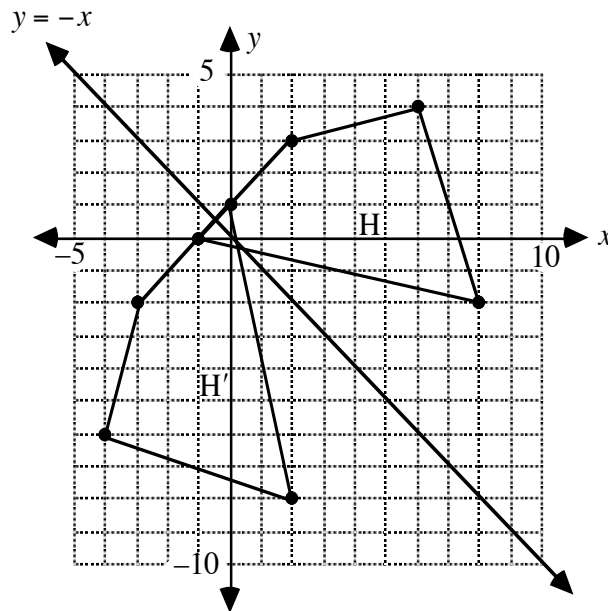
- e. Sample graph for the triangle represented by matrix **P**:



The coordinates of the image can be represented in the matrix below:

$$\mathbf{P}' = \begin{bmatrix} -1 & -3 & -1 \\ -2 & -4 & -5 \end{bmatrix}$$

Sample graph for the quadrilateral represented by matrix **H**:



The coordinates of the image can be represented in the matrix below:

$$H' = \begin{bmatrix} 0 & -3 & -4 & 2 \\ -1 & -2 & -6 & -8 \end{bmatrix}$$

- f. Matrix **A** reflects a set of points with respect to the line  $y = x$ . Matrix **B** reflects a set of points with respect to the  $y$ -axis. Matrix **C** reflects a set of points with respect to the line  $y = -x$ . Matrix **D** reflects a set of points with respect to the  $x$ -axis.

## Discussion

(page 236)

- a.
1. Sample response: When reflecting in the  $x$ -axis, the  $x$ -values remain unchanged but each  $y$ -value is the additive inverse of the corresponding  $y$ -value in the preimage.
  2. Sample response: When reflecting in the  $y$ -axis, the  $y$ -values remain unchanged but each  $x$ -value is the additive inverse of the corresponding  $x$ -value in the preimage.
- b.
1. Sample response: When reflecting in the line  $y = x$ , the  $x$  and  $y$ -values are switched.
  2. Sample response: When reflecting in the line  $y = -x$ , the  $x$  and  $y$ -values are switched and the signs are changed.
- c.
- Sample responses: Both rotations and reflections use multiplication on the left by  $2 \times 2$  matrices. Dilations use scalar multiplication or matrix multiplication. Translations use matrix addition. The number of columns in a translation matrix depends on the number of points being transformed.
- d.
- Sample response: Dilations produce images that are similar to the preimage. Translations, rotations, and reflections produce images that are congruent to the preimage.

## Assignment

(page 236)

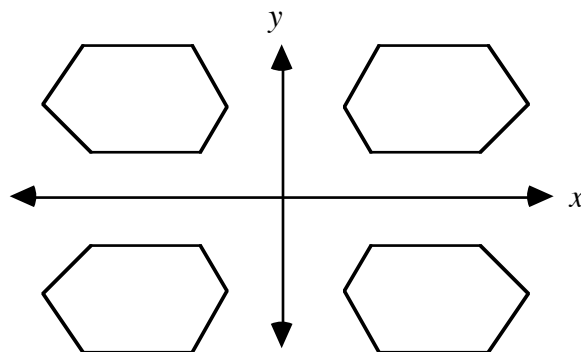
- 5.1 a. Sample response:

$$\mathbf{F}' = \mathbf{B} \cdot \mathbf{F}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3.5 & 6.5 & 7.5 & 6 & 3 \\ 4 & 6 & 6 & 4 & 3 & 3 \end{bmatrix}$$

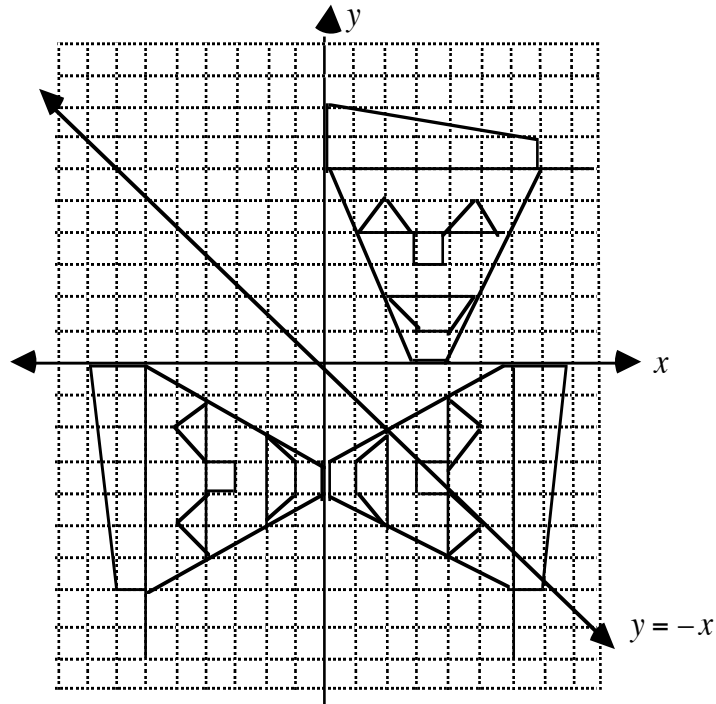
b.  $\mathbf{F}' = \begin{bmatrix} -2 & -3.5 & -6.5 & -7.5 & -6 & -3 \\ 4 & 6 & 6 & 4 & 3 & 3 \end{bmatrix}$

- c. Sample graph:

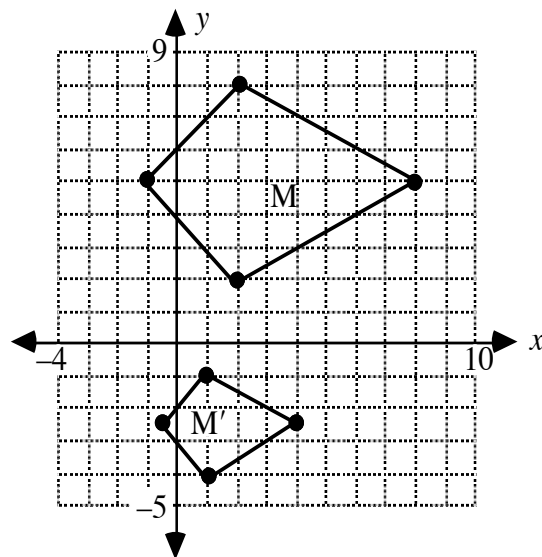


- d. Sample response: No, you can't. For any preimage, one of the three possible images is the result of two reflections: one in the  $x$ -axis and one in the  $y$ -axis. Therefore, there is always one image that cannot be created by a single reflection.

\*5.2 Answers will vary. The following sample graph shows Skip reflected in the line  $y = -x$ , then in the  $y$ -axis.



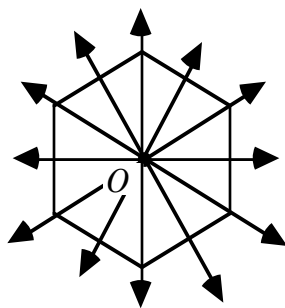
5.3 a–b. The coordinates of the vertices are  $(-0.5, -2.5)$ ,  $(1, -4)$ ,  $(4, -2.5)$ , and  $(1, -1)$ . Sample graph:



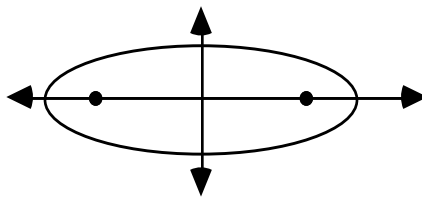
- c. The composite transformation can be described by the matrix equation below:

$$\begin{aligned} M'' &= 0.5 \cdot D \cdot M \\ &= 0.5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 8 & 2 \\ 5 & 8 & 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 & 1 & 4 & 1 \\ -2.5 & -4 & -2.5 & -1 \end{bmatrix} \end{aligned}$$

- 5.4 a–c.** Students should observe that a reflection in the  $x$ -axis, followed by a reflection in the  $y$ -axis, is equivalent to a  $180^\circ$  rotation about the origin. Multiplication of the corresponding matrices confirms this result.
- \*5.5 a.** A regular hexagon has six lines of symmetry, as shown below.



- b. A line has an infinite number of lines of symmetry, including line  $l$  itself and all lines perpendicular to line  $l$ .
- c. An ellipse has two lines of symmetry, as shown below.



- d. A circle has an infinite number of lines of symmetry, which all intersect at the center  $C$ .
- 5.6** Sample response: Skip does not have a line of symmetry, unless we exclude his hat.

\* \* \* \* \*

- 5.7 a.** Matrix  $\mathbf{Z}$  is a reflection of  $\mathbf{D}$  in the line  $y = -x$ .
- b. The matrix that describes this transformation is:

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

c. The following matrix equation verifies the responses given above:

$$\mathbf{R} \cdot \mathbf{D} = \mathbf{Z}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 & -6 & -4 & -2 \\ 1 & 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 & -3 \\ 3 & 6 & 4 & 2 \end{bmatrix}$$

**5.8** a. These do not create equivalent transformations. An example for  $P(3,4)$  is shown below.

$$4 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix} \neq 4 \cdot \left( \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right)$$

$$\begin{bmatrix} 10 \\ 21 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 36 \end{bmatrix}$$

b. These create equivalent transformations, as shown below for  $P(a,b)$ .

$$7 \cdot \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \left( 7 \cdot \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

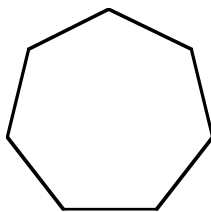
$$\begin{bmatrix} 7b \\ 7a \end{bmatrix} = \begin{bmatrix} 7b \\ 7a \end{bmatrix}$$

c. These create equivalent transformations, as shown below for  $P(a,b)$ .

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \left( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

**5.9** Answers will vary. Sample response: The following regular heptagon has seven lines of symmetry.



\*\*\*\*\*

The flip-card cartoons should demonstrate students' knowledge of the various transformations and their matrix representations. The following program--plotspt--works on the TI-92 Graphing Calculator. It allows students to "plot points" using the calculator. If they combine that with the program "CyclePic" (found under "catalog" on the TI-92), they can also animate the cartoon.

**plotpts(list)**

**Prgm**

©Plots a matrix

©The matrix is stored in *list*

**Local siz,pts,i**

©Defines temporary program variables

©Deleted at end of program execution

**dim(list) ) siz**

©Stores dimensions of matrix in *siz*

**siz[2]-1 pts**

©Stores 2nd dimension value minus 1 in *pts*

©This is the number of points minus 1

**For i,1,pts**

**Line list[1,i], list[2,i], list[1,i+1], list[2,i+1]**

**EndFor**

©Draws lines between points of matrix

**EndPrgm**

---

**Answers to Summary Assessment**

(page 240)

1. **Note:** This problem was designed to be completed on a calculator capable of storing three matrices. It is adaptable to other types of technology and may be done in stages on a spreadsheet.
  - a. The graph of the triangle should include the vertices (0,0), (16,0), and (16,16).
  - b. 1. Sample translation matrix:

$$\mathbf{T} = \begin{bmatrix} -2 & -2 & -2 \\ 6 & 6 & 6 \end{bmatrix}$$

2. The rotation matrix is shown below.

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3. This dilation may be accomplished through scalar multiplication or through multiplication on the left by the following matrix:

$$\mathbf{D} = \begin{bmatrix} 3/4 & 0 \\ 0 & 3/4 \end{bmatrix}$$

- c. Repeating the composite transformation produces a succession of triangles defined by the following matrices:

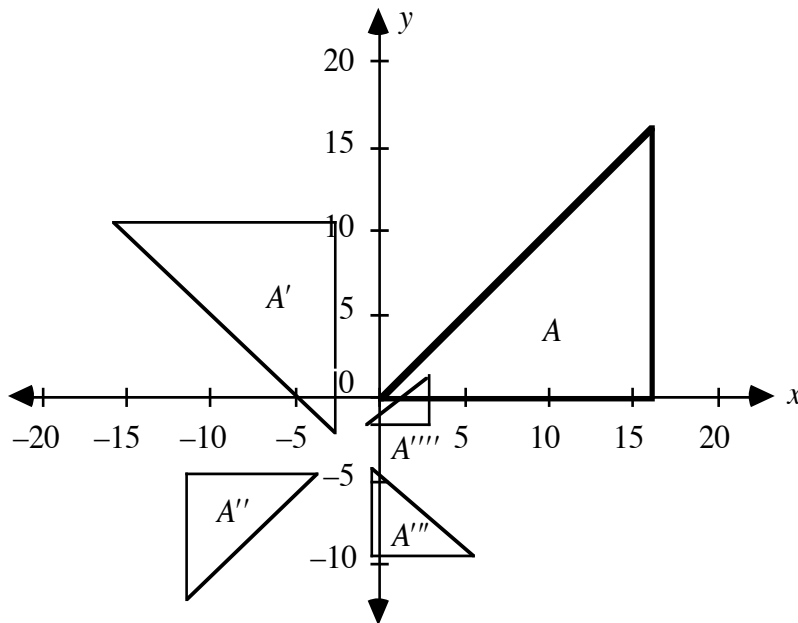
$$\mathbf{A}' = \begin{bmatrix} -9/2 & -9/2 & -33/2 \\ -3/2 & 21/2 & 21/2 \end{bmatrix}$$

$$\mathbf{A}'' = \begin{bmatrix} -27/8 & -99/8 & -99/8 \\ -39/8 & -39/8 & -111/8 \end{bmatrix}$$

$$\mathbf{A}''' = \begin{bmatrix} -27/32 & -27/32 & 182/32 \\ -129/32 & -345/32 & -345/32 \end{bmatrix}$$

$$\mathbf{A}'''' = \begin{bmatrix} -189/128 & 459/128 & 459/128 \\ -273/128 & -273/128 & 395/128 \end{bmatrix}$$

Sample graph:



2. Responses will vary. Students should describe their results mathematically.



## *Module Assessment*

1. a. Write a matrix equation that describes a dilation, with center at the origin and a scale factor of  $3/4$ , of the polygon defined by matrix  $\mathbf{M}$  below.

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & -4 & -8 & -3 \\ -1 & 4 & 5 & 1 & -3 \end{bmatrix}$$

- b. Determine the scale factor of the dilation, with center at the origin that, transforms  $\mathbf{M}$  into  $\mathbf{M}'$ .

$$\mathbf{M}' = \begin{bmatrix} 0.5 & 0.25 & -1 & -2 & -0.75 \\ -0.25 & 1 & 1.25 & 0.25 & -0.75 \end{bmatrix}$$

2. a. Graph the triangles described by matrices  $\mathbf{H}$  and  $\mathbf{H}'$ .

$$\mathbf{H} = \begin{bmatrix} 7 & -6 & 2 \\ 2 & -1 & 8 \end{bmatrix} \quad \mathbf{H}' = \begin{bmatrix} 17 & 4 & 12 \\ 1 & -2 & 7 \end{bmatrix}$$

- b. Identify the transformation of the preimage that occurred in Part a.  
c. Use a matrix equation to verify your response to Part b.
3. a. Graph the triangle described by matrix  $\mathbf{N}$  below and rotate it  $180^\circ$  about the origin.

$$\mathbf{N} = \begin{bmatrix} -5 & 5 & 1 \\ 4 & 4 & 0 \end{bmatrix}$$

- b. Write a matrix equation that describes this rotation.
4. a. Graph the line segments described by matrices  $\mathbf{S}$  and  $\mathbf{S}'$ .

$$\mathbf{S} = \begin{bmatrix} -3 & 2 \\ 7 & 1 \end{bmatrix} \quad \mathbf{S}' = \begin{bmatrix} -10 & 0 \\ 6 & -6 \end{bmatrix}$$

- b. Describe a composite transformation that could have generated  $\mathbf{S}'$  from  $\mathbf{S}$ . Use a matrix equation to support your response.

- 5.**
- a.** Graph a quadrilateral on a coordinate plane.
  - b.** Transform the quadrilateral using three of the four transformations listed below. Graph each image in your composite transformation and write the matrix equation that describes each one.
    - a dilation with center at the origin and a scale factor of  $\frac{2}{3}$
    - a translation 8 units to the left and 3 units up
    - a  $270^\circ$  rotation about the origin
    - a reflection in the line  $y = -x$
  - c.** Perform the three transformations you selected in Part **b** in a different order. Graph each image in the composite transformation and write the matrix equation that describes each one.
  - d.** Write a paragraph comparing the final image in Part **b** with the final image in Part **c**.

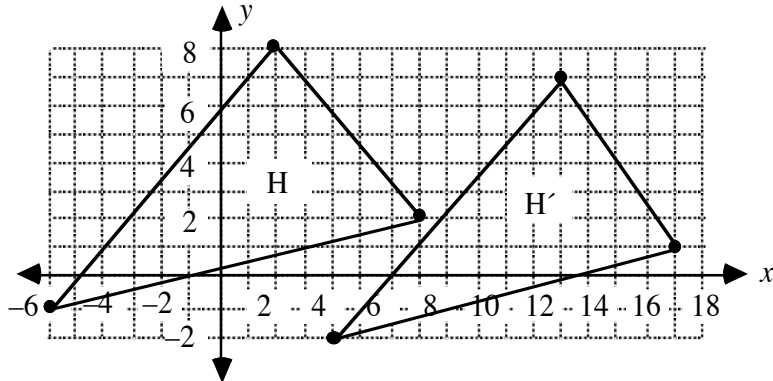
## Answers to Module Assessment

1. a. Sample matrix equation:

$$0.75 \cdot \begin{bmatrix} 2 & 1 & -4 & -8 & -3 \\ -1 & 4 & 5 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.75 & -3 & -6 & -2.25 \\ -0.75 & 3 & 3.75 & 0.75 & -2.25 \end{bmatrix}$$

- b. The scale factor that dilates matrix  $\mathbf{M}$  to matrix  $\mathbf{M}'$  is  $1/4$ .

2. a. Sample graph:

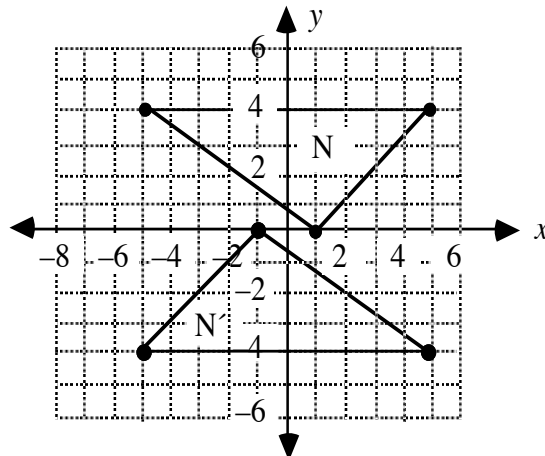


- b. The transformation is a translation 10 units to the right and 1 unit down.

- c. Sample matrix equation:

$$\begin{bmatrix} 7 & -6 & 2 \\ 2 & -1 & 8 \end{bmatrix} + \begin{bmatrix} 10 & 10 & 10 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 17 & 4 & 12 \\ 1 & -2 & 7 \end{bmatrix}$$

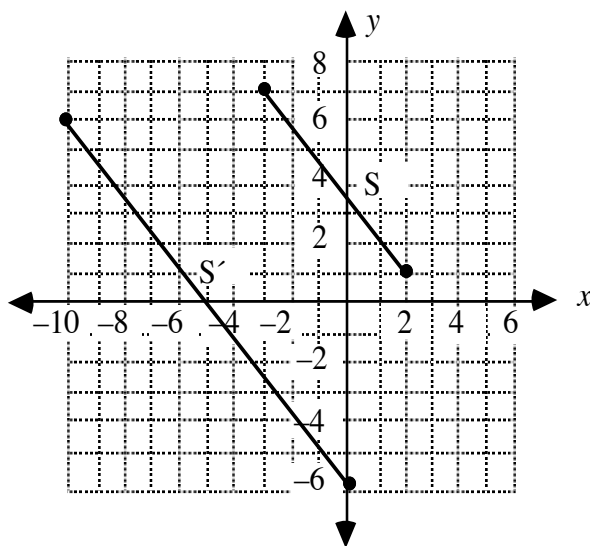
3. a. Sample graph:



- b. Sample matrix equation:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 5 & 1 \\ 4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -5 & -1 \\ -4 & -4 & 0 \end{bmatrix}$$

4. a. Sample graph:



- b. Answers may vary. Sample response: The segment represented by matrix  $\mathbf{S}$  was dilated by a scale factor of 2 with center at the origin. The image was then translated 4 units left and 8 units down. A matrix equation describing this composite transformation is:

$$2\mathbf{S} + \mathbf{T} = \mathbf{S}'$$

$$2\begin{bmatrix} -3 & 2 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} -4 & -4 \\ -8 & -8 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 6 & -6 \end{bmatrix}$$

It is also possible that the segment represented by matrix  $\mathbf{S}$  was translated 2 units left and 4 units down. This image was then dilated by a scale factor of 2 with center at the origin. In this case, the corresponding matrix equation is:

$$2(\mathbf{S} + \mathbf{T}) = \mathbf{S}'$$

$$2\left(\begin{bmatrix} -3 & 2 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\right) = \begin{bmatrix} -10 & 0 \\ 6 & -6 \end{bmatrix}$$

5. Answers will vary. There are 24 possible orders in which the specified transformations could be performed. Some of the orders produce the same result. **Note:** For assistance in scoring, you may wish to store the following matrices in a calculator or other matrix manipulator. When added to the matrix of ordered pairs, matrix  $\mathbf{T}$  produces a translation. When multiplied on the left of a matrix of ordered pairs, matrix  $\mathbf{R}$  produces a  $270^\circ$  rotation about the origin, while matrix  $\mathbf{X}$  produces a reflection in the line  $y = -x$ .

$$\mathbf{T} = \begin{bmatrix} -8 & -8 & -8 & -8 \\ 3 & 3 & 3 & 3 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

### **Selected References**

Barasch, M. *Theories of Art: From Plato to Winkelman*. New York: New York University Press, 1985.

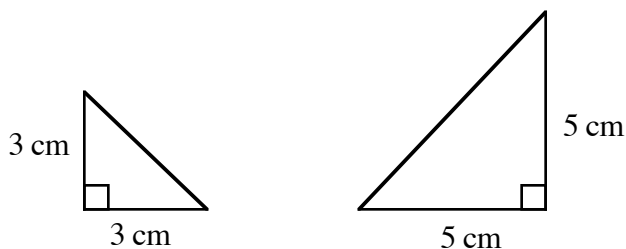
Hulse, C. *The Rule of Art: Literature and Painting in the Renaissance*. Chicago: University of Chicago Press, 1990.

Papy, G. *Modern Mathematics*. London: Collier-Macmillan, 1968.

# Flashbacks

## Activity 1

- 1.1 a. Use the Pythagorean theorem to find the length of the hypotenuse of each right triangle in the following diagram.



- b. Are these two triangles similar? Explain your response.
- 1.2. Consider the following points:  $A(1,2)$ ,  $B(-3,4)$ ,  $C(0,5)$ , and  $D(2,0)$ .
- a. 1. Find the equation of the line that contains  $A$  and  $B$ .  
2. Find the equation of the line that contains  $C$  and  $D$ .
- b. Identify the point of intersection of the two lines in Part a.

## Activity 2

- 2.1. Complete the following operations given matrices  $\mathbf{A}$  and  $\mathbf{B}$  below.

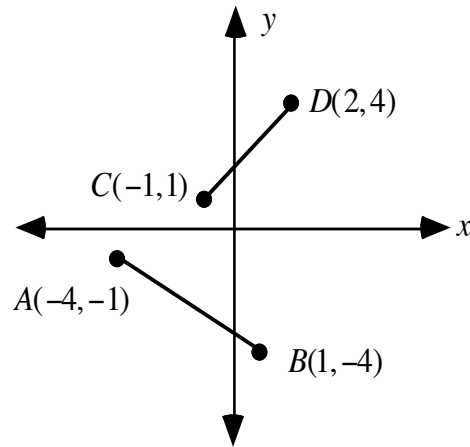
$$\mathbf{A} = \begin{bmatrix} 2 & 6 & 8 \\ -9 & 6 & -2 \\ 6 & -1 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

- a.  $\mathbf{A} + \mathbf{B}$
- b.  $3\mathbf{A}$
- c.  $\mathbf{A} \cdot \mathbf{B}$
- 2.2. Is it possible to perform the matrix multiplication shown below? Explain your response:

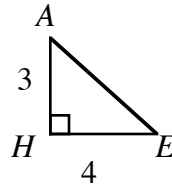
$$\begin{bmatrix} 2 & 3 & 4 \\ 9 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 & -9 \\ 4 & 5 & 0 \end{bmatrix}$$

### Activity 3

- 3.1 Find the lengths of  $\overline{AB}$  and  $\overline{CD}$  in the diagram below.



- 3.2 What angle measure corresponds with a tangent of 0.25?  
3.3 Determine the measure of  $\angle A$  in the following right triangle.



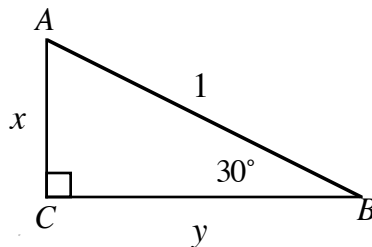
### Activity 4

- 4.1 Graph the figure described by each of the following matrices on a coordinate plane.

a.  $\begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix}$

b.  $\begin{bmatrix} 2 & 5 & 6 & 1 \\ 4 & -1 & -5 & -3 \end{bmatrix}$

- 4.2 Determine the unknown lengths in the triangle below.



### Activity 5

**5.1** Describe the graphs of  $y = x$  and  $y = -x$ .

**5.2** Complete the following operations given matrices **A**, **B**, and **C** below.

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 7 & 4 & -6 \\ 7 & -1 & 0 & 0 & -4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 7 & 7 & 7 & 7 & 7 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a.  $\mathbf{C} \cdot \mathbf{A}$

b.  $\mathbf{C} \cdot (\mathbf{A} + \mathbf{B})$



## Answers to Flashbacks

### Activity 1

- 1.1 a.  $\sqrt{3^2 + 3^2} = \sqrt{18} \approx 4.24$  cm;  $\sqrt{5^2 + 5^2} = \sqrt{50} \approx 7.07$   
b. Sample response: Yes, the two triangles are similar. The corresponding angles are congruent and the corresponding sides are proportional. The ratio of corresponding sides is  $3/5$ .

- 1.2 a. 1. Sample response:

$$y - 4 = \frac{4 - 2}{-3 - 1}(x - (-3))$$
$$y = -0.5x + 2.5$$

2. Sample response:

$$y - 0 = \frac{5 - 0}{0 - 2}(x - 2)$$
$$y = -2.5x + 5$$

- b. (1.25, 1.875)

### Activity 2

2.1 a.  $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 6 & 7 \\ -9 & 7 & 1 \\ 8 & -2 & 13 \end{bmatrix}$

b.  $3\mathbf{A} = \begin{bmatrix} 6 & 18 & 24 \\ -27 & 18 & -6 \\ 18 & -3 & 27 \end{bmatrix}$

c.  $\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 24 & -2 & 48 \\ -40 & 8 & 19 \\ 42 & -10 & 27 \end{bmatrix}$

- 2.2 Sample response: It is not possible to multiply a  $2 \times 3$  matrix by another  $2 \times 3$  matrix. The number of columns in the left-hand matrix must equal the number of rows in the right-hand matrix.

### Activity 3

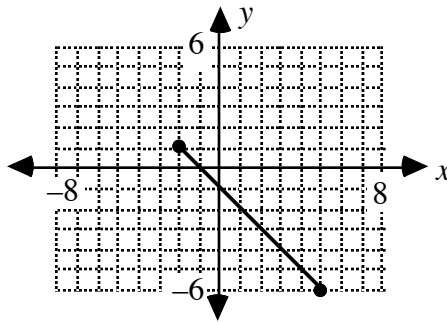
3.1  $AB = \sqrt{3^2 + 5^2} = \sqrt{34} \approx 5.83$ ;  $CD = \sqrt{3^2 + 3^2} = \sqrt{18} \approx 4.24$

3.2  $\tan^{-1}(0.25) \approx 14^\circ$

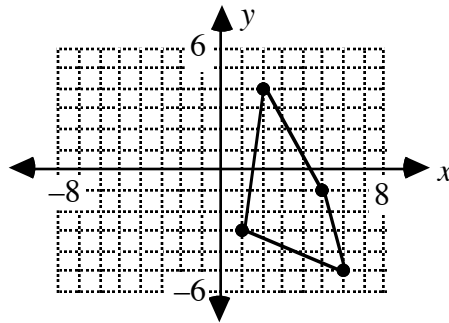
3.3  $m\angle A = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ$

## Activity 4

4.1 a. Sample graph:



b. Sample graph:



4.2  $y = \sqrt{3}/2 \approx 0.87$ ;  $x = 1/2$

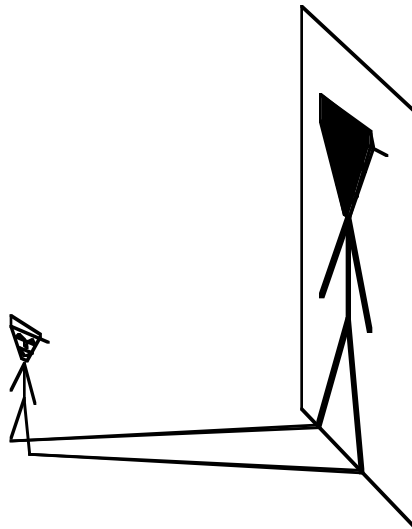
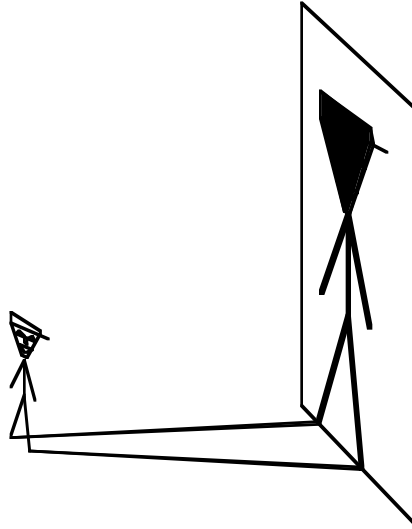
## Activity 5

5.1 Sample response: Both graphs are lines that pass through the origin. The slope of  $y = x$  is 1; the slope of  $y = -x$  is -1.

5.2 a.  $\begin{bmatrix} 7 & -1 & 0 & 0 & -4 \\ 2 & 4 & 7 & 4 & -6 \end{bmatrix}$

b.  $\begin{bmatrix} 14 & 6 & 7 & 7 & 3 \\ 4 & 6 & 9 & 6 & -4 \end{bmatrix}$

## Template for Problem 2.7







# Hurry! Hurry! Hurry!

## Step Right Up!



Have you ever played a carnival game? Better yet, have you ever won a big prize? In this module, you use probability to explore the carnival midway.

*Bill Chalgren • Mike Trudnowski • Anne Watkins*



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**Teacher Edition**  
**Hurry! Hurry! Hurry! Step Right Up!**

**Overview**

This module uses gaming situations to introduce geometric probability and conditional probability. Students also determine multistage probabilities involving independent events and use expected value to evaluate fair games.

**Objectives**

In this module, students will:

- use geometric models for determining probability
- use expected value to determine fair games
- explore probabilities of multistage experiments
- examine the difference between independent and dependent events
- explore conditional probability.

**Prerequisites**

For this module, students should know:

- how to determine the area of regular polygons and circles
- how to graph inequalities on a coordinate plane
- how to create tree diagrams and use them to determine probabilities
- how to identify the sample space for an experiment
- how to determine theoretical and experimental probabilities
- how to calculate expected value.

**Time Line**

<b>Activity</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>Summary Assessment</b>	<b>Total</b>
<b>Days</b>	3	2	4	1	10

## Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
dimes	X			
quarters	X			
centimeter graph paper	X			
cardboard	X			
tape	X			
rulers	X			
dice		X		
ping-pong balls			X	
markers			X	
paper bags			X	

## Technology

Software	Activity			
	1	2	3	Summary Assessment
random number generator			X	



# Hurry! Hurry! Hurry! Step Right Up!

## Introduction

(page 247)

This module uses carnival games to introduce geometric models for determining probability.

## Discussion

(page 247)

- a. Sample response: The spinner has a better chance because it appears to win half the time. The dart looks like it has less than a 50% chance of winning.
- b. Sample response: It is necessary to use the tip so that a single point determines the part of the board in which the spinner lands. If the entire head of the arrow is considered, the spinner could land in more than one part of the board.
- c.
  1. Students may mention tossing rings onto the necks of pop bottles, throwing a ball through an opening, shooting at a target with an air rifle, using a sledgehammer to ring a bell, shooting a basketball through a hoop, throwing darts at balloons, knocking over bottles with a ball, playing a raffle, or catching metal disks with a magnet attached to a fishing pole.
  2. Some games, such as shooting baskets or throwing darts, involve at least some degree of skill. Others, such as raffles or bingo, are based on chance. Still others involve a mix of chance and skill.
- d. Answers will vary. Students may prefer the dart game because at least some skill is involved. Others may choose the spinner game, arguing that the chances of winning seem better. **Note:** Students are not expected to calculate the probabilities of winning at this time. This situation is revisited in Problem 1.5.
- e. Since most carnival games are designed for both entertainment and profit-making, they are usually not fair games in the mathematical sense. Many carnival games allow few winners, or offer inexpensive prizes, to keep the expected value low. Some students may argue that foul play also makes some games unfair (for example, bottles that can't be knocked over or bent rifle barrels).

## *Activity 1*

In this activity, students use geometric models to determine probability.

### Materials List

- dimes (one per group)
- quarters (one per group)
- centimeter graph paper (one sheet per group)
- rulers (one per group)
- cardboard (four sheets per group)
- tape

### Teacher Note

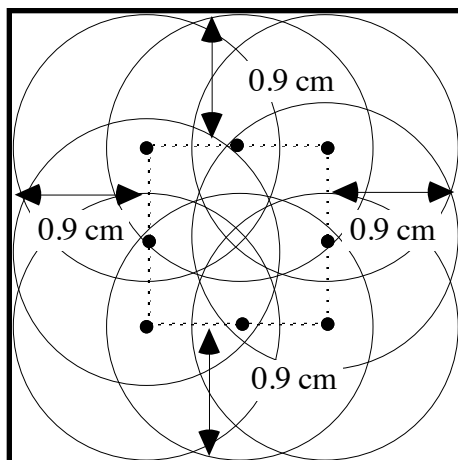
This exploration works well for groups of two students. After one student drops the coin onto the grid, the other can judge and record wins or losses.

### Exploration

(page 248)

Students play the coin-drop game and record the experimental outcomes.

- a. Students create a game board with four walls around it.
- b–c. **Note:** If dimes are dropped from too great a height, they may bounce over the walls. Placing a notebook under the grid can reduce bouncing.
- d. Experimental probabilities will vary. The theoretical probability of winning is approximately 0.16.
- e.
  1. Answers may vary. Given a dime with a radius of 0.9 cm, the center of the dime must be at least 0.9 cm from each line on the grid.
  2. Using the center of the dime as a reference point, the region that models all winning outcomes is a square within the  $3\text{ cm} \times 3\text{ cm}$  square. The following diagram shows eight possible positions for winning dimes, the centers of these dimes, and the square formed by the centers of all winning coins.



The area of the region that models winning outcomes is approximately  $(3 - (2 \cdot 0.9))^2 = 1.44 \text{ cm}^2$ .

3. The theoretical probability can be determined by dividing the area of the inner square (representing all winning outcomes) by the area of the outer square (representing all possible outcomes):

$$\frac{\text{area of inner square}}{\text{area of outer square}} = \frac{1.44 \text{ cm}^2}{9 \text{ cm}^2} = \frac{4}{25} = 0.16$$

- f. Answers will vary. For a large number of trials, the experimental probability is likely to approximate the theoretical probability.

## Discussion

(page 250)

- a. Sample response: If any other point is used, the orientation of the coin would have to be considered when determining the area that represents a win. This would make the problem very hard to solve.
- b. Assuming that the walls are at least 0.9 cm from the grid, the probability of winning is the same for any number of squares ( $n$ ):
- $$\frac{n \cdot \text{area of inner square}}{n \cdot \text{area of outer square}} = \frac{1.44n}{9n} = \frac{1.44}{9} = \frac{4}{25} = 0.16$$
- c. Answers will vary. These probabilities may or may not be approximately equal, depending on the class results. For a large number of trials, the experimental probability is likely to be close to the theoretical probability.
- d. Sample response: The game could be made impossible to play by using a coin with a diameter that is greater than the dimensions of the squares, or by using squares with dimensions that are less than the diameter of the dime.

## Assignment

(page 251)

- 1.1 a.** The region that models all winning outcomes is the square formed by the centers of all winning quarters. Since the radius of a quarter is approximately 1.2 cm, the area of this region can be found as follows:  $(3 - (2 \cdot 1.2))^2 = 0.36 \text{ cm}^2$ .

- b.** The theoretical probability of winning the coin-drop game using a quarter is:

$$\frac{\text{area of inner square}}{\text{area of outer square}} = \frac{0.36 \text{ cm}^2}{9 \text{ cm}^2} = \frac{1}{25} = 0.04$$

- \*1.2 a.** The region that models all winning outcomes is the square formed by the centers of all winning dimes. Since the radius of a dime is approximately 0.9 cm, the area of this region can be found as follows:  $(4 - (2 \cdot 0.9))^2 = 4.84 \text{ cm}^2$ .

- b.** The theoretical probability of winning this version of the game is:

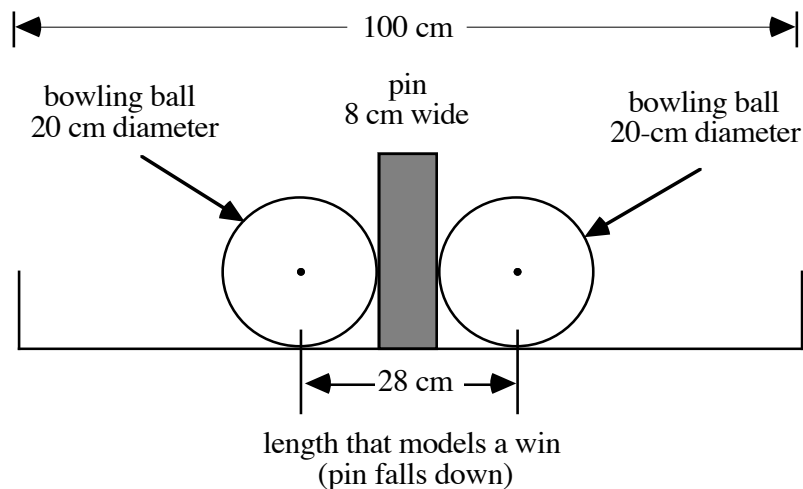
$$\frac{\text{area of inner square}}{\text{area of outer square}} = \frac{4.84 \text{ cm}^2}{16 \text{ cm}^2} = \frac{121}{400} = 0.3025$$

- 1.3 a.** Using a geometric model, the probability of landing in the yellow ring is the ratio of the area of the yellow ring to the area of the entire target:

$$\frac{\pi(10)^2 - \pi(7)^2}{\pi(13)^2} = \frac{\pi(51)}{\pi(169)} = \frac{51}{169} \approx 0.30$$

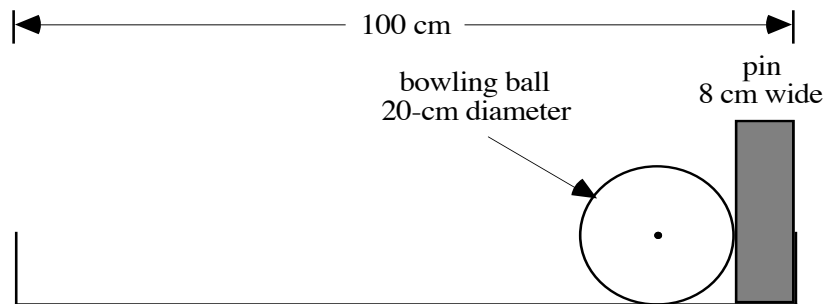
- b.** Sample response: No. It is not really a random event because players can aim the dart at a particular point.

- \*1.4 a.** As shown in the diagram below, students can consider all the positions of the ball's center that will knock down the pin:



Using this geometric model, the width of all possible paths of the ball's center down the lane is  $(100 - 2 \cdot 10)$ , or 80 cm. The width of the paths that result in knocking down the pin is 28 cm. The probability of winning is  $28/80 = 7/20 = 0.35$ .

- b. As shown in the diagram below, if the pin is placed against one rail, the ball can pass on only one side of the pin. Since the center of the ball can pass no closer than 10 cm from the rail, the width of the paths that result in knocking down the pin is 8 cm. This reduces the probability of winning to  $8/80 = 1/10 = 0.1$ .



- 1.5 Sample response: I would rather play the spinner game because the probability of winning is greater. Since each of the four central angles where the spinner is located measure  $90^\circ$ , the probability that the spinner lands in the unshaded portion of the board is:

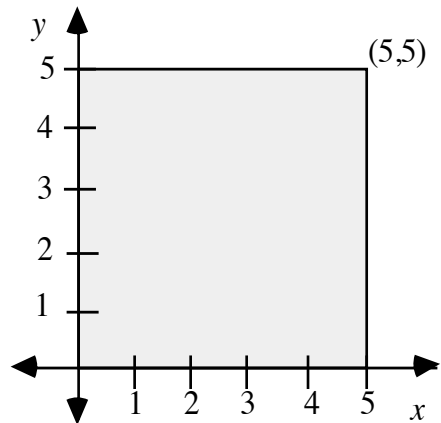
$$\frac{90^\circ + 90^\circ}{360^\circ} = 0.5$$

The probability of a dart landing in the unshaded portion of the board is the sum of areas of the two unshaded rectangles, divided by the total area of the board:

$$\frac{40 \cdot 70 + 40 \cdot 90}{130 \cdot 110} \approx 0.448$$

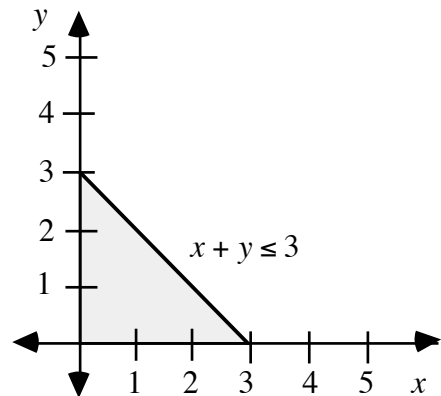
\* \* \* \* \*

**\*1.6** a. Sample graph:



b. The area of all the possible outcomes is  $25 \text{ units}^2$ .

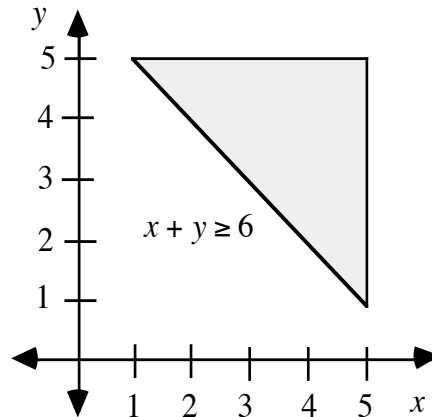
c. Sample graph:



The area of the shaded region is  $\frac{1}{2}(3)(3) = 4.5 \text{ units}^2$ .

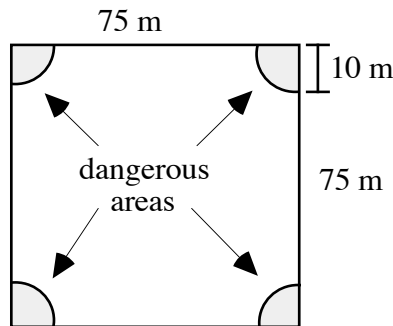
d. Using a geometric model, the probability that  $x + y \leq 3$  is  $\frac{4.5}{25} = 0.18$ .

- e. The following sample graph shows the region bounded by  $x + y \geq 6$ ,  $0 \leq x \leq 5$ , and  $0 \leq y \leq 5$ .



The area of the shaded region is  $\frac{1}{2}(4)(4) = 8 \text{ units}^2$ . Using a geometric model, the probability that  $x + y \geq 6$  is  $\frac{8}{25} = 0.32$ .

- \*1.7 a. Sample sketch:



- b. Sample response: The dangerous area is a circle with a radius of 10 m. Therefore, the area of the parking lot in which it is safe to land is:  $75^2 - \pi 10^2 \approx 5300 \text{ m}^2$ .
- c. 1. Using a geometric model, the probability that the parachutist lands safely can be found as follows:

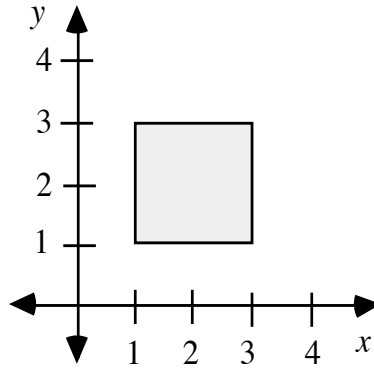
$$\frac{\text{safe area}}{\text{parking lot area}} \approx \frac{5300}{5625} \approx 0.94$$

2. The probability that the parachute's lines get tangled in a security light is:

$$\frac{\text{unsafe area}}{\text{parking lot area}} \approx \frac{\pi 10^2}{75^2} \approx \frac{314}{5625} \approx 0.06$$

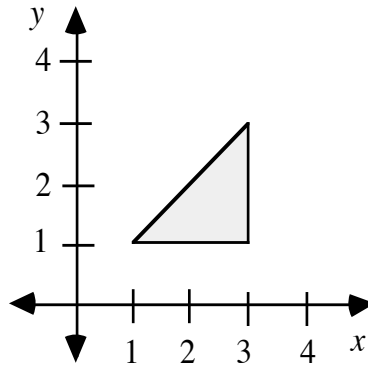
- d. Sample response: The probabilities in Part c add to 1 because they are complementary events.

1.8 a. Sample graph:



b. The area of the shaded region in the graph above is 4 units<sup>2</sup>.

c. Sample graph:



The area of the shaded region in the graph above is 2 units<sup>2</sup>.

d. Using a geometric model, the probability is  $2/4 = 0.5$ .

e. Using a geometric model, the probability is  $2/4 = 0.5$ . **Note:** The events described in Parts **d** and **e** are not complementary, since the points on the line  $y = x$  are included in both.

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## Activity 2

In this activity, students calculate expected value and determine whether or not a game is mathematically fair.

### Materials List

- dice (one per group)



## Exploration

(page 254)

- a–b. Students should recognize that the chances of winning each prize are not equally likely.
- c. The theoretical probability of winning each prize is shown below.

Value of Prize	Roll of Die	Probability
\$1.50	1	1/6
\$1.00	2	1/6
\$0.50	3	1/6
\$0.00	4	1/6
\$0.00	5	1/6
\$0.00	6	1/6

Since three of the outcomes result in a prize of \$0.00, its theoretical probability is  $1/6 + 1/6 + 1/6 = 3/6 = 1/2$ .

- d. The experimental probabilities may or may not approximate the theoretical probabilities.
- e. The expected value can be found as follows:

$$\frac{1}{6} \cdot \$1.50 + \frac{1}{6} \cdot \$1.00 + \frac{1}{6} \cdot \$0.50 + \frac{1}{2} \cdot 0 = \$0.50$$

## Discussion

(page 255)

- a. Sample response: No. You are not equally likely to win each prize. A non-winning roll of 4, 5, or 6 is three times more likely to occur than any single winning roll.
- b. Since the cost to play (\$1.00) and the expected value (\$0.50) are not equal, the game is not mathematically fair. To make the game fair, students may suggest reducing the cost to play, increasing the value of the prizes, or changing the probabilities of winning the prizes.
- c. Answers will vary. To pay wages, travel costs, and other expenses, carnival operators must make a profit from their games. It would be unreasonable to expect a mathematically fair game under these circumstances.

## Assignment

(page 256)

- \*2.1** To make this a fair game, the cost to play must equal the expected value:

$$\frac{1}{2} \cdot \$0.00 + \frac{1}{3} \cdot \$0.50 + \frac{1}{6} \cdot \$2.00 = \$0.50$$

- 2.2** a. The expected value can be calculated as follows:

$$\frac{21}{25} \cdot \$0.00 + \frac{4}{25} \cdot \$5.00 = \$0.80$$

- b. To make the game mathematically fair, the cost to play must be decreased to \$0.80.
- c. The value of the prize that makes the game mathematically fair can be calculated as follows:

$$\frac{21}{25} \cdot \$0.00 + \frac{4}{25} \cdot x = \$1.00$$
$$x = \$6.25$$

- \*2.3** a. The expected value can be calculated as follows:

$$\frac{24}{25} \cdot \$0.00 + \frac{1}{25} \cdot \$15.00 = \$0.60$$

- b. To make the game mathematically fair, the cost to play must be decreased to \$0.60.
- c. The value of the prize that makes the game mathematically fair can be calculated as follows:

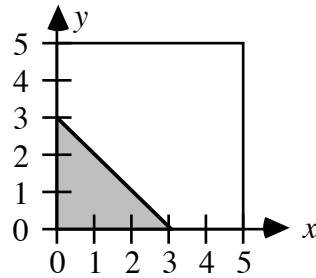
$$\frac{24}{25} \cdot \$0.00 + \frac{1}{25} \cdot x = \$1.00$$
$$x = \$25.00$$

- 2.4** To make this a fair game, the cost to play must equal the expected value:

$$\frac{1}{13} \cdot \$10 + \frac{3}{13} \cdot \$1 + \frac{9}{13} \cdot \$0 = \$1.00$$

\* \* \* \* \*

- 2.5 a. In the following sample graph, the shaded region represents all the possible pairs of numbers in the interval  $[0, 5]$  for which the sum is less than or equal to 3.



The probability of Juan winning is the area of the shaded region divided by the area of the square:

$$\frac{4.5}{25} = 0.18 = 18\%$$

- b. Since Juan theoretically wins \$0.50 in 18% of the games and loses \$0.20 in 82% of the games, the expected value can be calculated as follows:  $0.18 \cdot 0.50 + 0.82 \cdot -0.20 \approx -0.07$ .
- c. Sample response: Juan can expect to lose an average of approximately \$0.07 each time he plays. Therefore, he should not play the game.
- d. Answers will vary. Sample response: If Joan continues to pay Juan \$0.50 each time he wins, then solving the following equation for  $x$  gives the amount Juan should pay Joan to make the expected value the same for both players.

$$0.18 \cdot \$0.50 = 0.82x$$

$$\$0.11 \approx x$$

If Juan continues to pay Joan \$0.20 each time she wins, then solving the following equation for  $x$  gives the amount Joan should pay Juan to make the expected value the same for both players.

$$0.18x = 0.82 \cdot \$0.20$$

$$x \approx \$0.91$$

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## Research Project

(page 257)

After students have presented their games to the class, you may wish to ask them to discuss modifications that might make the games profitable at a carnival.

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### Activity 3

In this activity, students examine multistage probabilities.

#### Materials List

- ping-pong balls or other small objects (10 per group)
- red and blue markers (one each per group)
- paper bags or other opaque containers (one per group)

#### Technology

- random number generator

#### Exploration 1

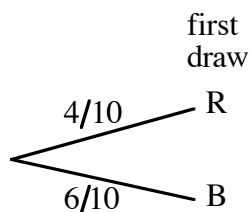
(page 258)

Students simulate the fishing derby by drawing objects from a bag. **Note:** To ensure random draws, the objects should be indistinguishable by touch.

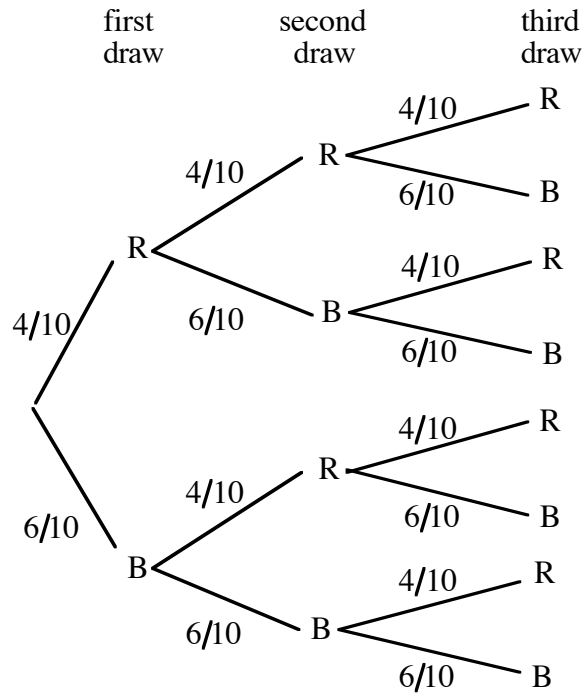
- a–b. Students simulate the fishing derby 10 times, then combine their results with the class. Sample data for 100 trials:

Event	Points	Number of Occurrences	Experimental Probability
3 red	6	8	0.08
2 red, 1 blue	5	25	0.25
1 red, 2 blue	4	46	0.46
3 blue	3	21	0.21

- c. Sample tree diagram:



- d. The eight possible outcomes are RRR, RRB, RBR, RBB, BRR, BRB, BBR, and BBB (where R represents red and B represents blue).  
Sample tree diagram:



- e. Students should calculate the following probabilities:  
 $P(\text{RRR}) = (4/10)^3 = 0.064$ ,  $P(\text{RRB}) = (4/10)^2 \cdot 6/10 = 0.096$ ,  
 $P(\text{RBR}) = (4/10)^2 \cdot 6/10 = 0.096$ ,  $P(\text{RBB}) = 4/10 \cdot (6/10)^2 = 0.144$ ,  
 $P(\text{BRR}) = 6/10 \cdot (4/10)^2 = 0.096$ ,  $P(\text{BRB}) = (6/10)^2 \cdot 4/10 = 0.144$ ,  
 $P(\text{BBR}) = (6/10)^2 \cdot 4/10 = 0.144$ , and  $P(\text{BBB}) = (6/10)^3 = 0.216$ .
- f. 1. The probabilities of winning each prize are as follows:  
 $P(6) = 0.064$ ,  $P(5) = 3 \cdot 0.096 = 0.288$ ,  $P(4) = 3 \cdot 0.144 = 0.432$ ,  
and  $P(3) = 0.216$ .
2. For a large number of trials, the experimental probabilities are likely to be close to the theoretical probabilities.

## Discussion 1

(page 259)

- a. There are eight possible outcomes.
- b.
  1. There is one way to get 3 points: BBB.
  2. Any combination of 2 blues and 1 red earns 4 points. There are three possibilities: BBR, BRB, and RBB.
  3. Any combination of 2 reds and 1 blue earns 5 points. There are three possibilities: RRB, RBR, and BRR.
  4. There is one way to earn 6 points: RRR.
  5. There is no way to earn 7 points.
- c. Sample response: Because there are fewer rings with red dots than rings with blue dots, the probability of drawing a red ring is less than that of drawing a blue one.
- d.
  1. Sample response: No. Since the object is returned to the container and mixed in with the others, the second draw is from the same set of objects as the first.
  2. Sample response: No, for the same reasons mentioned above.
- e. Some students may predict that the rabbit's foot is the prize most often won because it requires the fewest points. Considering theoretical probabilities, however, the celebrity poster should be the most frequently awarded prize.
- f.
  1. Answers may vary. Sample response: Generate random numbers from 1 to 10, inclusive. The numbers from 1 to 4 represent a red ring and earn 2 points. The numbers from 5 to 10 represent a blue ring and earn 1 point.
  2. Students may expect the simulations to produce the same results. Since each is repeated a relatively few number of times, however, the results are likely to vary. Over the long run, the simulations should produce similar results because they involve the same theoretical probabilities.
- g. Sample response: Because it contains more trials, the class data should better represent the long-term results of the game. The experimental probability should be closer to the theoretical probability as the number of trials increases.

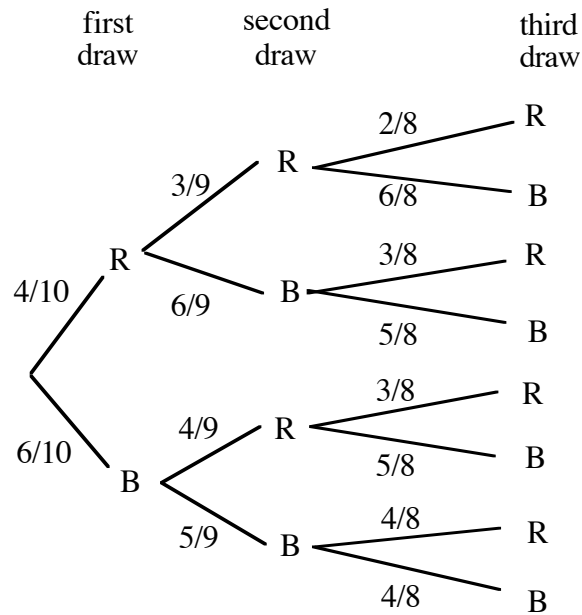
## Exploration 2

(page 260)

In this exploration, students examine conditional probabilities by simulating a version of the fishing derby without replacement.

- Responses will vary. Some students may observe that the probabilities of scoring 6 points (RRR) or 3 points (BBB) are decreased.
- The possible outcomes are the same as those in the original version of the game: RRR, RRB, RBR, RBB, BRR, BRB, BBR, and BBB.

Sample tree diagram:



- Students should determine the probabilities of each outcome as follows:  $P(\text{RRR}) = 4/10 \cdot 3/9 \cdot 2/8 = 1/30 \approx 0.03$ ,  
 $P(\text{RRB}) = 4/10 \cdot 3/9 \cdot 6/8 = 1/10 = 0.1$ ,  
 $P(\text{RBR}) = 4/10 \cdot 6/9 \cdot 3/8 = 1/10 = 0.1$ ,  
 $P(\text{RBB}) = 4/10 \cdot 6/9 \cdot 5/8 = 1/6 \approx 0.17$ ,  
 $P(\text{BRR}) = 6/10 \cdot 4/9 \cdot 3/8 = 1/10 = 0.1$ ,  
 $P(\text{BRB}) = 6/10 \cdot 4/9 \cdot 5/8 = 1/6 \approx 0.17$ ,  
 $P(\text{BBR}) = 6/10 \cdot 5/9 \cdot 4/8 = 1/6 \approx 0.17$ , and  
 $P(\text{BBB}) = 6/10 \cdot 5/9 \cdot 4/8 = 1/6 \approx 0.17$ .

Using these probabilities, the theoretical probabilities of obtaining each score are  $P(6) = 1/30 \approx 0.03$ ,  $P(5) = 3/10 = 0.3$ ,  
 $P(4) = 3/6 = 0.5$ , and  $P(3) = 1/6 \approx 0.17$ .

- d. Students may design their own simulations using either a container of red and blue objects or a random number generator. Sample response: Generate random numbers from 1 to 10, inclusive. The numbers 1 to 4 represent the red rings and are worth 2 points. The numbers 5 to 10 represent the blue rings and are worth 1 point. During a single round of the simulation, any number that comes up more than once is ignored and another number generated.
- e. The following table shows some sample data for 100 trials. In this case, the experimental probabilities closely approximate the theoretical probabilities.

Event	Points	Number of Occurrences	Experimental Probability
3 red	6	3	0.03
2 red, 1 blue	5	31	0.31
1 red, 2 blue	4	45	0.45
3 blue	3	21	0.21

## Discussion 2

(page 261)

- a. Sample response: When using a random number generator, ignoring any number that comes up more than once in a single round models its removal from the tank. When using red and blue objects, removing them from the container models rings removed from the tank.
- b. In the new version of the game, the theoretical probability of getting 6 points decreases from 0.064 to approximately 0.03.
- c. Sample response: Although the chances of winning a hat have improved slightly, the chances of winning a bear have decreased. Overall, the chances of winning one or the other has also decreased. In the original game, the chances of winning a hat or a bear are  $0.288 + 0.064 = 0.352$ . In the new version, the chances of winning a hat or a bear are approximately  $0.3 + 0.033 = 0.333$ .
- d. 1. Sample response: The probabilities are different because the first chip drawn is not replaced in the box. Since the numbers and kinds of chips in the box have changed, the probabilities must be adjusted.
2. Since there is only one blue chip in the game, the probability of drawing two blue chips is 0.



## Assignment

(page 261)

- \*3.1**
- Because all the outcomes result in a prize, and because the sum of their probabilities is 1, the probability of winning some prize in the fishing derby is 1.
  - Because no combination of three rings adds up to 7 points, the probability of earning 7 points is 0.
  - The expected value, in points, can be determined as follows:

$$\frac{8}{125} \cdot 6 + \frac{36}{125} \cdot 5 + \frac{54}{125} \cdot 4 + \frac{27}{125} \cdot 3 = 4.2$$

- \*3.2**
- The expected value, in points, is:

$$\frac{1}{30} \cdot 6 + \frac{3}{10} \cdot 5 + \frac{1}{2} \cdot 4 + \frac{1}{6} \cdot 3 = 4.2$$

- Answers will vary. The probabilities of winning a poster or hat increased slightly in the new version, while the probabilities of winning a key chain or teddy bear decreased.

- 3.3**
- Sample response: For the game to be fair, the expected value must equal the cost to play. The expected value of the original fishing derby is:

$$\frac{8}{125} \cdot \$2.00 + \frac{36}{125} \cdot \$0.80 + \frac{54}{125} \cdot \$0.60 + \frac{27}{125} \cdot \$0.25 \approx \$0.67$$

One way to make this a fair game would be to reduce the cost to play to \$0.67.

- Sample response: For the game to be fair, the expected value must equal the cost to play. The expected value of the new version is:

$$\frac{1}{30} \cdot \$2.00 + \frac{3}{10} \cdot \$0.80 + \frac{1}{2} \cdot \$0.60 + \frac{1}{6} \cdot \$0.25 \approx \$0.65$$

One way to make this a fair game would be to reduce the cost to play to \$0.65.

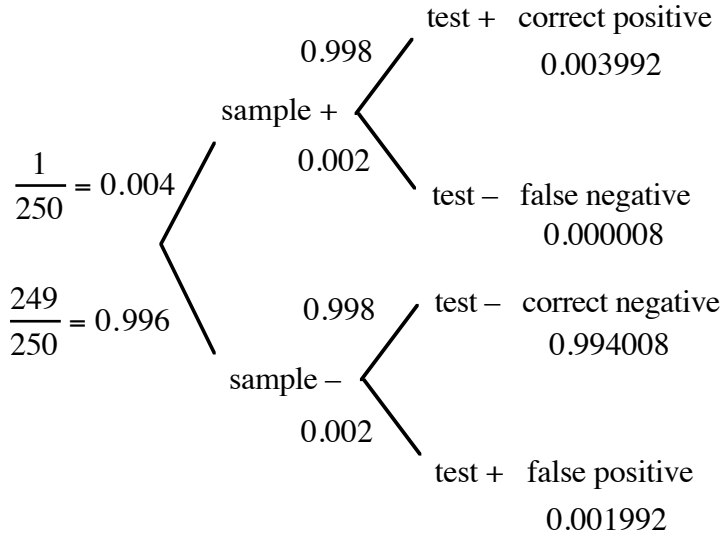
- \*3.4**
- There are four equally likely outcomes when tossing two coins: HH, HT, TH, and TT. Therefore, the probabilities of each result can be determined as follows:  $P(1 \text{ step}) = P(HT) + P(TH) = 1/2 = 0.5$ ,  $P(0 \text{ steps}) = P(TT) = 1/4 = 0.25$ , and  $P(-1 \text{ step}) = P(HH) = 1/4 = 0.25$ .
  - The expected value, in number of stairs, is:

$$\frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot -1 = \frac{1}{4} = 0.25$$

- c. Sample response: Since  $10 \cdot 0.25 = 2.5$ , Guinn is likely to be 2 or 3 steps up from the starting point.
- d. If the game starts on the middle of the 25 stairs (or the 13th stair), a player must move up 12 stairs to win. On average, Guinn would need 48 turns ( $12/0.25 = 48$ ) to get to the top step.

\* \* \* \* \*

- \*3.5** a. The following tree diagram shows the probability of each outcome:



- b. 1. 3992
- 2. 8
- 3. 994,008
- 4. 1992
- c. Sample response: Of the 1,000,000 people tested, I expect  $3992 + 1992 = 5984$  to test positive. Of these, only 3992 would actually have HIV, so  $P(\text{HIV}|\text{Test } +) = 3992/5984 \approx 0.667$ .

\* \* \* \* \*

## Answers to Summary Assessment

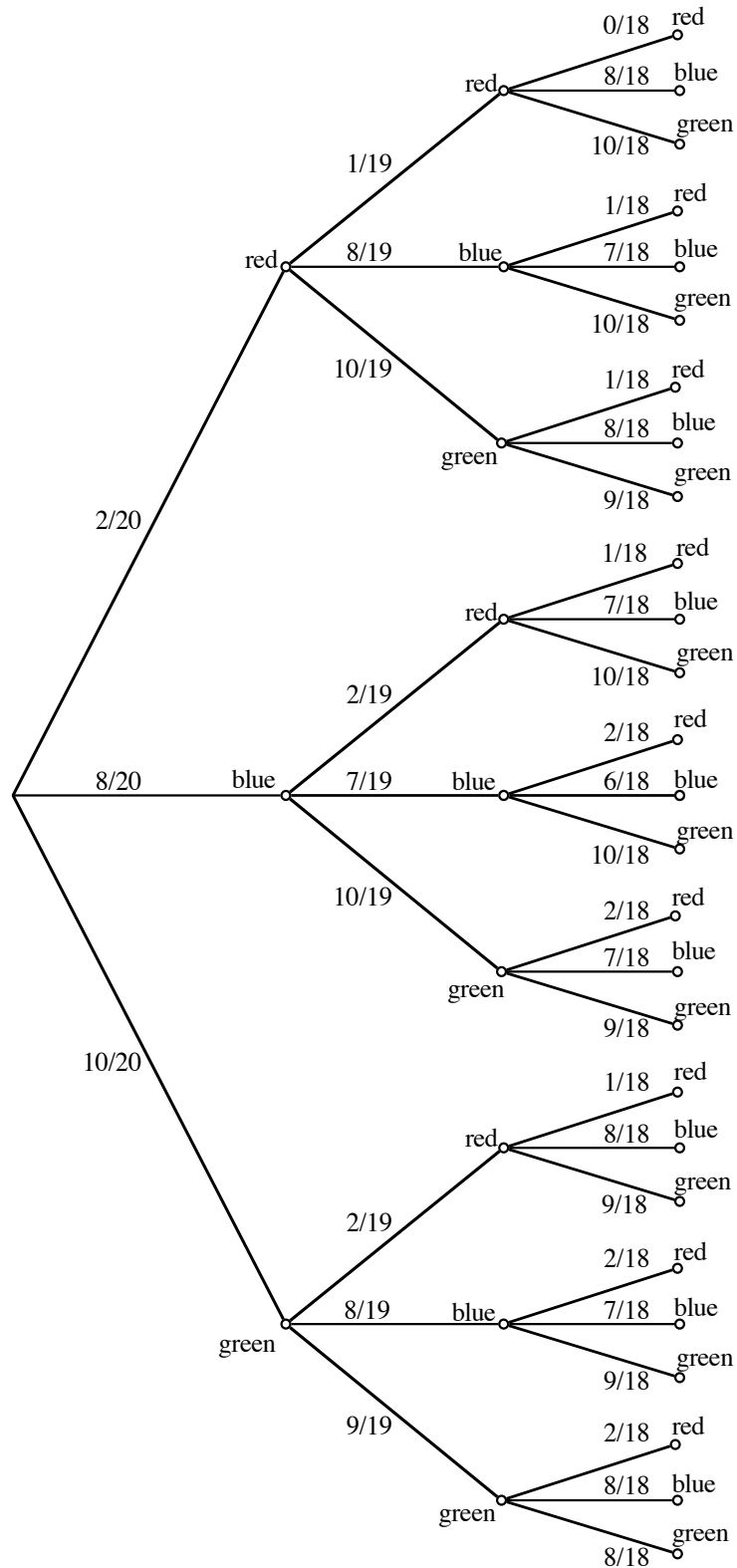
(page 264)

1. Student responses should take the form of a letter and include the following information.
  - a. The total area of the board is  $30,000 \text{ cm}^2$ . The area of the red zones is  $3(30 \cdot 35) = 3150 \text{ cm}^2$ . Assuming that the dart is equally likely to land anywhere on the board, the probability of landing in a red zone is  $3150/30,000 = 21/200 = 0.105$ .

The area of the blue zone is  $\pi(4^2) \approx 50 \text{ cm}^2$ . The probability of landing in the blue zone is approximately  $50/30,000 \approx 0.002$ . The probability of not landing in either winning zone is  $(30,000 - 3150 - 50.27)/30,000 \approx 0.893$ . **Note:** This may also be calculated by  $1 - P(\text{red zone}) - P(\text{blue zone})$ .
  - b. The expected value can be calculated as follows:
$$(0.893 \cdot \$0) + (0.105 \cdot \$25) + (0.002 \cdot \$10,000) \approx \$22.63$$
  - c. Assuming no other costs, the lowest price the charity could charge to play, and still hope to earn some profit, is \$23.00.
  - d. For the first 1000 players, the charity should plan to award about  $1000 \cdot 0.105 = 105$  miniature replicas, and  $1000 \cdot 0.002 = 2$  full-sized cars.
  - e. Since each game is an independent event, the probability that two players in a row win full-sized cars is approximately  $0.002^2 \approx 0.000004$ , or about 4 in 1,000,000.

Student recommendations will vary. At a cost to play of \$23.00, the charity could expect to earn an average of approximately \$0.37 per player. Based on this small per-player profit, some students may recommend against using this game as fund-raiser. They may also mention the possibility of having to award a full-sized car before earning enough money to pay for it, or suggest that the high cost of playing may discourage potential players.

2. Student responses should again take the form of a letter. To determine probabilities, some may construct a tree diagram like the one shown below:



The probabilities and point values for each possible draw of three balls are shown in the following table.

<b>Balls Drawn</b>	<b>Point Value</b>	<b>Probability</b>
3 red	15	$0/6840 = 0.000$
2 red, 1 blue	12	$48/6840 \approx 0.007$
2 red, 1 green	11	$60/6840 \approx 0.009$
1 red, 2 blue	9	$336/6840 \approx 0.049$
1 red, 1 blue, 1 green	7	$960/6840 \approx 0.140$
1 red, 2 green	8	$540/6840 \approx 0.079$
3 blue	6	$336/6840 \approx 0.049$
2 blue, 1 green	5	$1680/6840 \approx 0.246$
1 blue, 2 green	4	$2160/6840 \approx 0.316$
3 green	3	$720/6840 \approx 0.105$

The probabilities for winning each prize can be calculated by adding the probabilities for the appropriate point values. The expected value for the game is shown in the table below.

<b>Points</b>	<b>Prize Value</b>	<b>Probability</b>	<b>Product</b>
[12, 15]	\$3.00	$48/6840 \approx 0.007$	\$0.02
[9, 11]	\$1.75	$396/6840 \approx 0.058$	\$0.10
[6, 8]	\$0.40	$1836/6840 \approx 0.268$	\$0.11
[3, 5]	\$0.00	$4560/6840 \approx 0.667$	\$0.00
<b>Expected Value</b>			\$0.23

Over the long term, the carnival can expect to earn an average profit of \$0.77 per player. Based on this relatively large profit, students may recommend that the carnival add the game.

## *Module Assessment*

1. In one lottery game, each player purchases a ticket for \$5.00, then selects a three-digit number. Each digit must be between 0 and 9, inclusive, and can appear only once in the number.

Lottery officials then randomly select three digits, one at a time and without replacement. These digits are recorded in the order selected to form a three-digit number.

If the player's three-digit number matches the number selected by the lottery, the player wins.

- a. How many three-digit numbers are possible in this lottery game?
  - b. What is the probability of winning with one ticket?
  - c. If this were a fair game, how much should the lottery pay for matching the winning number? Justify your response.
2. A local convenience store is having a grand opening celebration. As part of its promotional activities, the store plans to offer prize tickets to everyone who enters the store. No purchase is necessary to win. The value of each prize and the probability of winning it are shown in the table below. What is the expected value for a prize ticket?

<b>Prize</b>	<b>Value</b>	<b>Probability</b>
soft drink	\$1	0.05
two lottery tickets	\$2	0.01
40 L of gasoline	\$15	0.001
weekend vacation	\$1000	0.000005
European vacation	\$10,000	0.00000005

3. Consider a game in which three balls—numbered 1, 2, and 3—are placed in a paper bag. Player 1 draws a ball, records its number, then replaces it in the bag. Player 2 does the same. For each version of this game described below, determine whether you would want to be player 1 or player 2. Justify your responses.
- a. If the sum  $x$  of the two numbers is odd, player 1 loses  $x$  points while player 2 wins  $x$  points. If the sum of the two numbers is even, player 2 loses 4 points, while player 1 wins 4 points.
  - b. If both players draw the same number, no points are won or lost. If the sum of the two numbers is even, the player with the higher number wins 5 points, while the other player loses 5 points. If the sum of the two numbers is odd, the player with the lower number wins 3 points, while the other player loses 3 points.

4. As described in Problem 3.4, Guinn and Ebdul are playing the stair-step game. To reduce the time required to declare a winner, they decide to change the rules of the game.

Ebdul suggests tossing four coins on each turn. If one head or three heads appear on a player's four coins, the player moves up one stair. If two or four heads appear, the player moves down one stair. If no heads appear, the player remains on the same step.

As before, the game starts on the middle step of a staircase 25 steps long. To win the game, a player must reach the top step.

- a. If movement is described in terms of the number of stairs moved upwards, then each player's turn in the game has three possible results: 1 step, 0 steps, and  $-1$  step.

What is the probability of each of these results?

- b. What is the expected value, in number of stairs moved upwards, for each turn in this version of the game?
- c. Will Ebdul's suggestion shorten the stair-step game? Justify your response.
5. In the cube game, players toss a cube with three green faces, one red face, one white face, and one blue face into a bucket. After you toss the cube, the game operator tells you that the green face is not showing. Given this information, what is the probability that the red face is showing? Justify your response.

## Answers to Module Assessment

1.
  - a. There are  $10 \cdot 9 \cdot 8 = 720$  possible numbers for this game.
  - b. The probability that any one ticket wins is  $1/720 \approx 0.001$ .
  - c. If this were a fair game, the expected value should equal the cost to play, or \$5.00. Therefore, the prize should be worth \$3600.
2. The expected value can be determined as follows:  
 $(0.05 \cdot \$1) + (0.01 \cdot \$2) + (0.001 \cdot \$15) + (0.000005 \cdot \$1000) + (0.00000005 \cdot \$10,000) + (0.93899495 \cdot \$0) \approx \$0.09$
3.
  - a. There are five possible sums: 2, 3, 4, 5, and 6. The probability of a 2 is  $1/9$ , of a 3 is  $2/9$ , of a 4 is  $3/9 = 1/3$ , of a 5 is  $2/9$ , and of a 6 is  $1/9$ . The expected value for player 1, in points, can be found as follows:

$$\frac{1}{9} \cdot 4 + \frac{2}{9} \cdot -3 + \frac{3}{9} \cdot 4 + \frac{2}{9} \cdot -5 + \frac{1}{9} \cdot 4 = \frac{4}{9}$$

The expected value for player 2 is:

$$\frac{1}{9} \cdot -4 + \frac{2}{9} \cdot 3 + \frac{3}{9} \cdot -4 + \frac{2}{9} \cdot 5 + \frac{1}{9} \cdot -4 = -\frac{4}{9}$$

In this case, player 1 has the advantage.

- b. The following table shows the number of points associated with each draw.

Number Drawn		Points Received	
Player 1	Player 2	Player 1	Player 2
1	1	0	0
1	2	3	-3
1	3	-5	5
2	1	-3	3
2	2	0	0
2	3	3	-3
3	1	5	-5
3	2	-3	3
3	3	0	0

Both players have three chances of winning 0 points; two chances of winning 3 points; two chances of winning -3 points; one chance of winning 5 points; and one chance of winning -5 points. The expected value for both players is:

$$\frac{3}{9} \cdot 0 + \frac{2}{9} \cdot 3 + \frac{2}{9} \cdot -3 + \frac{1}{9} \cdot 5 + \frac{1}{9} \cdot -5 = 0$$

In this case, neither player has an advantage.



4. a. There are 16 equally likely outcomes when tossing four coins: HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, and TTTT. Therefore, the probabilities of each result can be determined as follows:  $P(1 \text{ step}) = 1/2 = 0.5$ ,  $P(0 \text{ steps}) = 1/16 = 0.0625$ , and  $P(-1 \text{ step}) = 7/16 = 0.4375$ .
- b. The expected value, in number of stairs, is:
- $$\frac{1}{16} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{7}{16} \cdot -1 = \frac{1}{16} = 0.0625$$
- c. Sample response: The rule change would actually slow the game down. The expected movement per turn decreases from 0.25 steps to 0.0625 steps, a factor of  $1/4$ . Therefore, one would expect the new version to take four times as long as the original game.
5. Sample response: If the result of rolling the cube is not green, the roll must be either red, white, or blue. Each of these three colors appears once, so the probability of a red face is  $1/3 \approx 0.33$ .

## **Selected References**

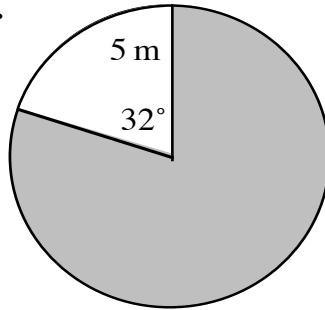
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# Flashbacks

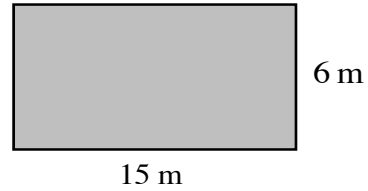
## Activity 1

1.1 Find the area of the shaded region in Parts **a–d** below.

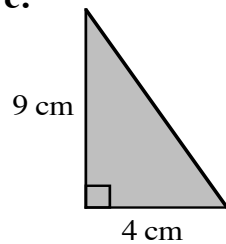
a.



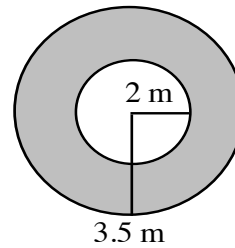
b.



c.



d.



1.2 Consider an experiment which involves tossing a coin and counting the number of times that it lands heads up. After 65 tosses, you record 31 heads.

a. What is the experimental probability of the coin landing heads up?

b. What is the theoretical probability of the coin landing heads up?

1.3 Graph the following inequality on a two-dimensional coordinate system:  $x + y \geq 1$ .

## Activity 2

- 2.1** Consider an experiment which involves drawing a card out of a standard deck of 52 playing cards. Determine the theoretical probability of drawing each of the following:
- a six
  - a heart
  - a face card
  - a club or a diamond.
- 2.2** Consider an experiment which involves rolling a six-sided die and recording the number that appears.
- List all the possible outcomes of this experiment.
  - What is the theoretical probability of each outcome?
  - What is the expected value of this experiment?
- 2.3** Solve each of the following equations for  $x$ .
- $\frac{2}{3}x = 5$
  - $\frac{1}{13}x = 2$

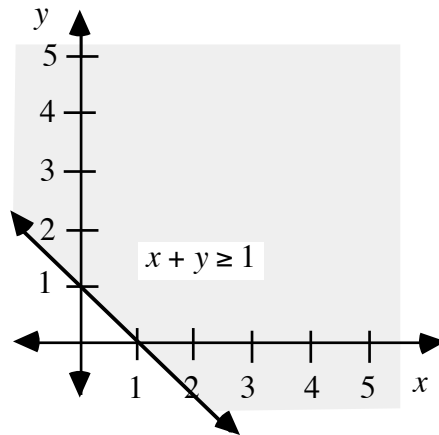
## Activity 3

- 3.1** Consider an experiment in which three cards—an ace, a king and a queen—are placed face down on a table. The cards are then turned over, one at a time.
- Draw a tree diagram that shows all the possible sequences in which the cards may be turned over.
  - How many outcomes are there in the sample space for this experiment?
  - What is the probability that the ace is the first card turned over?
  - What is the probability that the queen is the first card and the ace is the second card?
- 3.2** A bag contains 20 red marbles, 16 green marbles, and 12 blue marbles. Consider an experiment in which two marbles are drawn from the bag, one at a time and without replacement.
- What is the probability that the first marble is blue?
  - What is the probability that the first marble is red?
  - If the first marble is green, what is the probability that the second marble is red?

## Answers to Flashbacks

### Activity 1

- 1.1 a.  $328/360 \cdot \pi \cdot 5^2 = 205\pi/9 \approx 71.56 \text{ m}^2$   
b.  $15 \cdot 6 = 90 \text{ m}^2$   
c.  $1/2 \cdot 4 \cdot 9 = 18 \text{ cm}^2$   
d.  $(\pi \cdot 3.5^2) - (\pi \cdot 2^2) \approx 25.92 \text{ m}^2$
- 1.2 a.  $31/65 \approx 0.48$   
b.  $1/2 \approx 0.5$
- 1.3 Sample graph:



### Activity 2

- 2.1 a.  $4/52 = 1/13 \approx 0.08$   
b.  $13/52 = 1/4 = 0.25$   
c.  $12/52 = 3/13 \approx 0.23$   
d.  $26/52 = 2/4 = 1/2 = 0.5$

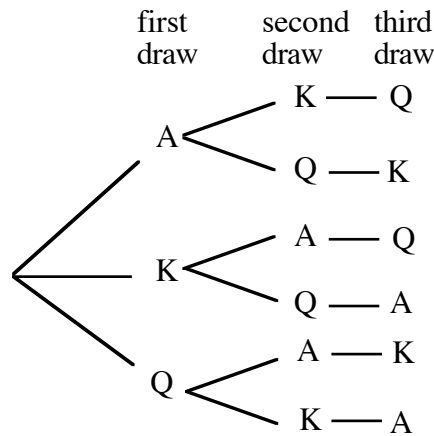
- 2.2 a. There are six possible outcomes: 1, 2, 3, 4, 5, and 6.  
 b. The theoretical probability of each outcome is  $1/6 \approx 0.17$ .  
 c. The expected value is:

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{7}{2}$$

- 2.3 a.  $x = 15/2 = 7.5$   
 b.  $x = 26$

### Activity 3

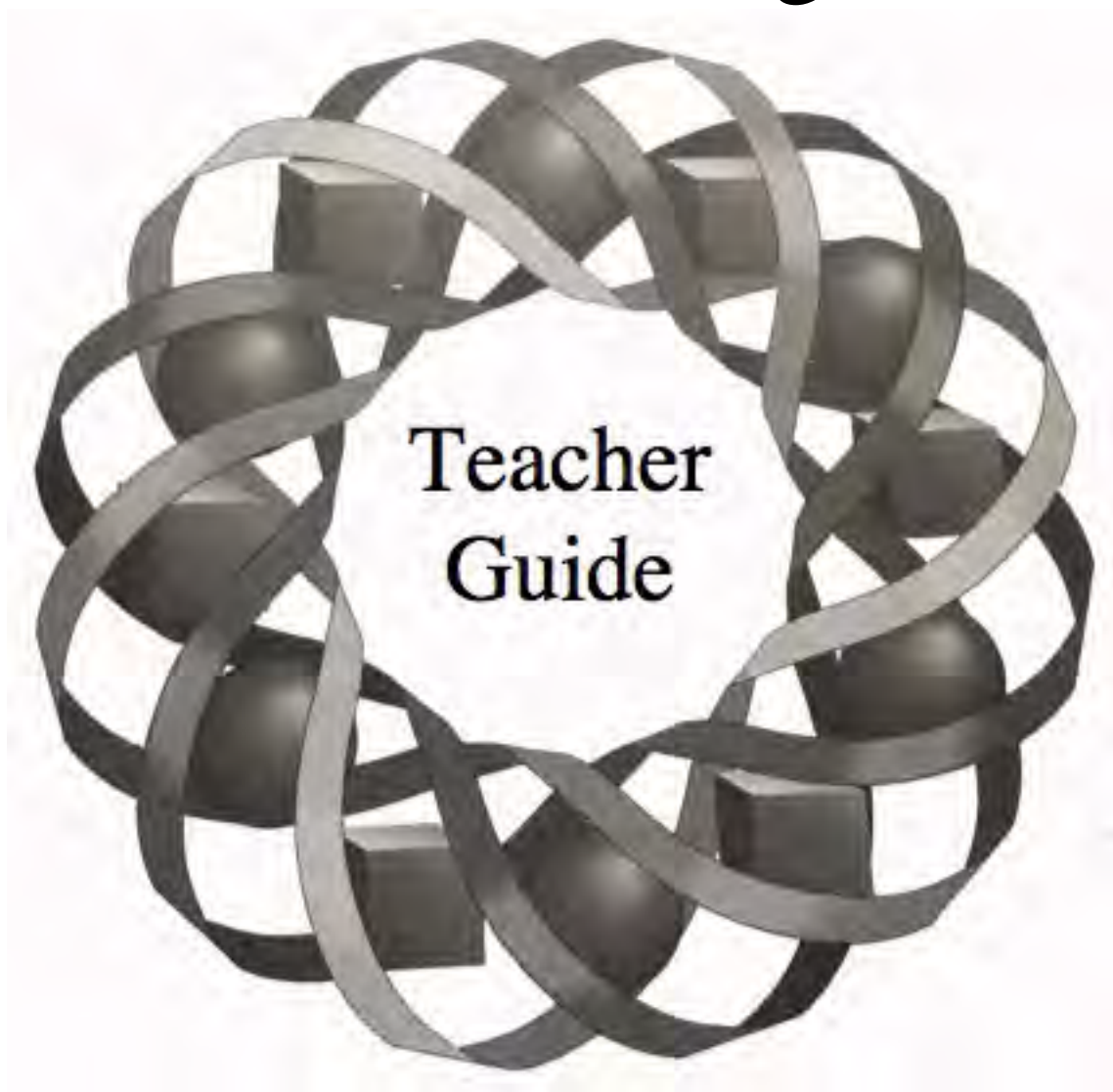
- 3.1 a. Sample tree diagram:



- b. The sample space contains 6 equally likely possibilities: AKQ, AQK, KAQ, KQA, QAK, and QKA.  
 c. Since the ace appears first in 2 of the 6 outcomes, the probability is  $2/6 \approx 0.33$ .  
 d. Since the queen appears first and the ace second in 1 of the 6 outcomes, the probability is  $1/6 \approx 0.17$ .
- 3.2 a.  $12/48 = 1/4 = 0.25$   
 b.  $20/48 = 5/12 \approx 0.42$   
 c.  $20/47 \approx 0.43$



# Atomic Clocks Are Ticking



You've found the remains of an old cooking fire. How do you tell if that lump of charcoal is 10 years old, or 10,000? In this module, you explore one method that archaeologists use to tell time.

*John Freal • Sandy Johnson • Anne Merrifield*



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## Teacher Edition

# Atomic Clocks Are Ticking

### Overview

This module uses radioactive decay and carbon dating to develop models of exponential decay and examine negative and fractional exponents. Exponential equations are solved graphically and by guess-and-check using a calculator.

### Objectives

In this module, students will:

- develop models of exponential decay
- examine the relationship between negative and positive exponents
- use equations containing negative exponents as mathematical models
- examine the relationship between rational exponents and roots
- develop properties of exponents.

### Prerequisites

For this module, students should know:

- how to interpret an exponential expression
- how to model exponential growth using equations of the form  $y = a \cdot b^x$
- scientific notation
- how to use a spreadsheet.

### Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	3	3	2	10

## Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
chips	X			
graph paper	X		X	
box with lid	X			
dice (optional)			X	

## Technology

Software	Activity			
	1	2	3	Summary Assessment
graphing utility	X	X	X	X
spreadsheet	X	X	X	

# Atomic Clocks Are Ticking

## Introduction

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The opening narrative introduces students to carbon dating. As the module progresses, students eventually gain a more accurate understanding of radioactive decay. In the summary assessment, students will use Professor Cordova's data to determine the age of charcoal found in the cave.

**Note:** Students should not perceive radioactive decay as the “death” of an atom. A radioactive atom typically emits a small particle (or particles) from the nucleus—resulting in a new nucleus with a different number of protons. The decay of an atom of carbon-14, for example, produces an atom of nitrogen.

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## *Activity 1*

In this activity, students use equations of the form  $y = a \cdot b^x$  to model data collected during a simulation of radioactive decay. In this simulation, each shake of the container represents a half-life. **Note:** In Activity 3, students use a simulation which produces half-life values that are not whole units.

## Materials List

- chips (32 per group)
- flat container with lid (one per group)

## Teacher Note

To conduct the exploration, each group will require 32 chips (such as poker chips) that can be marked on one side. As an alternative, you may wish to use pennies as chips. Pizza boxes or shoe boxes provide suitable containers.

## Technology

- graphing utility
- spreadsheet (optional)

## Exploration

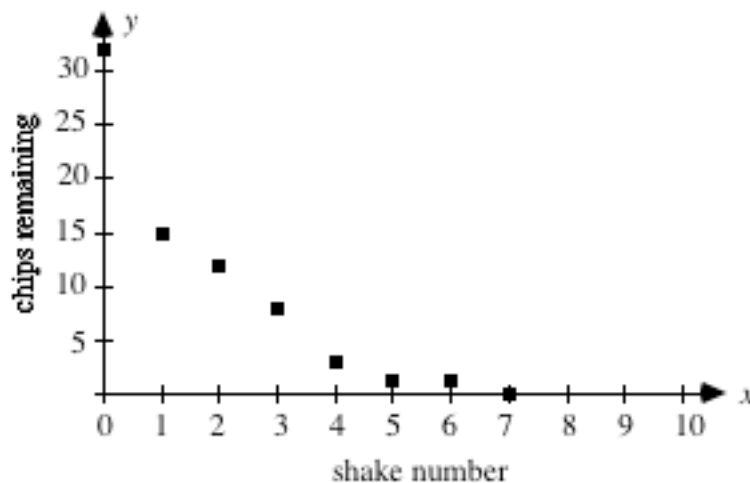
(page 271)

**Note:** Students should save their data for use in the assignment.

- a–f.** Students simulate the process of radioactive decay. It should take from 4 to 7 shakes to remove all the chips. Sample data:

Shake	No. of Chips Remaining
0	32
1	15
2	12
3	8
4	3
5	1
6	1
7	0

- g.** A graph of the sample data appears below.



## Discussion

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- a.** Sample response: The number of remaining chips appears to decrease more slowly as the shake number increases.
- b.** Answers will vary. Sample response: The general shapes of the graphs are the same, with the number of chips decreasing as the shake number increases. However, different numbers of shakes were required to remove all the chips, and the graphs vary in steepness.
- c.** Since the probability of a chip landing with the mark up is  $1/2$ , students should expect half of the chips to be removed and half to remain.
- d.** When a positive number is multiplied by a factor between 0 and 1, the number decreases.

- e.
  1. Sample response: The value of  $y$  represents the worth of the machine at the end of half a year, or 6 months.
  2. Sample response: The value of  $y$  represents the worth of the machine at the end of one-third of a year, or 4 months.
- f. Answers will vary, depending on the calculator used.
  1. Sample response: Type “ $0.81^{(1/2)}$ ” then press the enter button.
  2. Sample response: Type “ $0.81^{(1/3)}$ ” then press the enter button.
- g. Students should realize that the value of  $(0.81)^{1/2}$  is the square root of 0.81, or 0.9.
- h. Students should conjecture that since an exponent of  $1/2$  in the expression  $(0.81)^{1/2}$  represents the square root, an exponent of  $1/3$  in the expression  $(0.81)^{1/3}$  represents the cube root of 0.81. This can be verified by cubing the value reported by the calculator for  $(0.81)^{1/3}$ .
- i.
  1. If the equation models exponential growth,  $b > 1$ .
  2. If the equation models exponential decay,  $0 < b < 1$ .

### Assignment

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- 1.1
  - a. 5
  - b. 3
  - c. 7
  - d. 2
- 1.2
  - a. The table below shows the percent decrease after each shake for the sample data given in the exploration:

Shake	No. of Chips Remaining	Percent Decrease
0	32	
1	15	47%
2	12	20%
3	8	33%
4	3	63%
5	1	67%
6	1	0%
7	0	100%

- b. 1. Answers will vary. The mean of the percents of decrease for the sample data shown above is 47%.
2. Sample response: I would expect the mean of the percents of decrease to be close to 50%, because each chip has a 50% chance of having its mark showing.
- c. Sample response: The mean of the percents of decrease can be used as the rate of decay, which is also a percentage, because they measure the same thing.
- d. Answers will vary. For the sample data, the rate of decay is  $-0.47$ . In this case,  $b = 1 - 0.47 = 0.53$ .
- e. Since the initial population is 32, the equation that models the data is  $y = 32 \cdot (0.53)^x$ .

1.3 A completed table is shown below.

	Initial Population	Rate of Growth or Decay	Model Equation of the Form $y = a \cdot b^x$
a.	500	8%	$y = 500 \cdot (1.08)^x$
b.	20,000	-11%	$y = 20,000 \cdot (0.89)^x$
c.	100	-32%	$y = 100 \cdot (0.68)^x$

1.4  $5.02 \cdot 10^{22} / 1 \cdot 10^{12} \approx 5 \cdot 10^{10}$  atoms

1.5 Using an equation of the form  $y = a \cdot b^x$ , the initial population  $a$  is  $4 \cdot 10^{14}$  chips,  $b$  is 0.5, and the number of shakes  $x$  is 3. Solving for  $y$  gives  $5 \cdot 10^{13}$  or 50 trillion chips.

Students also may solve this problem as follows: After the first shake, the number of chips is  $(4 \cdot 10^{14}) / 2 = 2 \cdot 10^{14}$ ; after the second shake, the number of chips is  $(2 \cdot 10^{14}) / 2 = 1 \cdot 10^{14}$ ; after the third shake, the number of chips is  $(1 \cdot 10^{14}) / 2 = 5 \cdot 10^{13}$ .

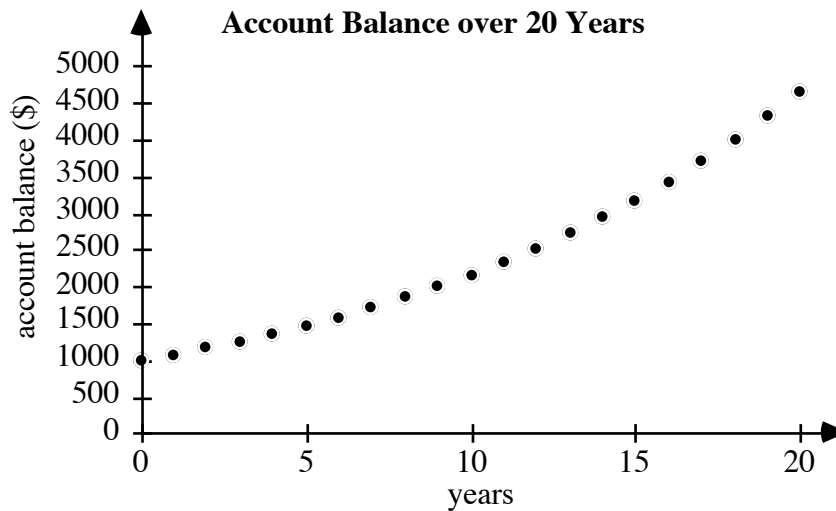
1.6 Sample response: To find the number of atoms in the previous year, multiply by 2:  $3.8 \cdot 10^{13}(2) = 7.6 \cdot 10^{13}$  atoms. **Note:** Some students may substitute  $-1$  for  $x$  in the equation  $y = (3.8 \cdot 10^{13})(0.5^x)$  to calculate an answer. Negative exponents are introduced in Activity 2.

1.7 Sample response: Starting with  $1 \cdot 10^{12}$  and repeatedly multiplying by 0.5 for each decade, it would take 10 decades before the number of atoms was less than  $1 \cdot 10^9$  (one billion).

- \*1.8**
- Since  $b > 1$ , this equation models exponential growth. In this case,  $1.055 = 1 + r$  or  $r = 0.055$ . As  $x$  increases, the graph increases gradually at first, then dramatically.
  - Since  $0 < b < 1$ , this equation models exponential decay. In this case,  $0.50 = 1 + r$  or  $r = -0.50$ . As  $x$  increases, the graph decreases dramatically at first, then gradually.
  - Since  $0 < b < 1$ , this equation models exponential decay. In this case,  $0.75 = 1 + r$  or  $r = -0.25$ . As  $x$  increases, the graph decreases dramatically at first, then gradually.

\* \* \* \* \*

- 1.9**
- The total amount of money in the account after  $x$  years may be modeled by the equation  $y = 1000 \cdot (1 + 0.08)^x = 1000 \cdot (1.08)^x$ .
  - The account balance after 5 years is  $y = 1000 \cdot (1.08)^5 = \$1469.33$ .
  1. Sample graph:



- The account balance will reach \$2000 in about 9 years.
  - Answers may vary. For a 16-year-old student, the account balance may be calculated as follows:  $y = 1000 \cdot (1.08)^{49} \approx \$43,427.42$ .
- 1.10**
- The town's initial population was 50,000.
  - The population decreased to 25,000 between year 6 and year 7.
  - The population at year 15 is about 11,000.
  - Answers may vary slightly. The rate of decrease is 10% per year.
  1. Sample response:  $y = 50,000 \cdot (0.90)^x$ .
  - The estimated value from the graph and the approximate value from the equation should be reasonably close. Using the sample equation given above,  $y = 50,000 \cdot (0.90)^{15} \approx 10,295$ .

\* \* \* \* \*

## Activity 2

In this activity, students use negative values for  $x$  in equations of the form  $y = a \cdot b^x$  to represent periods of time before the present. This allows them to explore negative exponents and continue their investigation of carbon dating.

### Materials List

- none

### Technology

- spreadsheet
- graphing utility

### Exploration 1

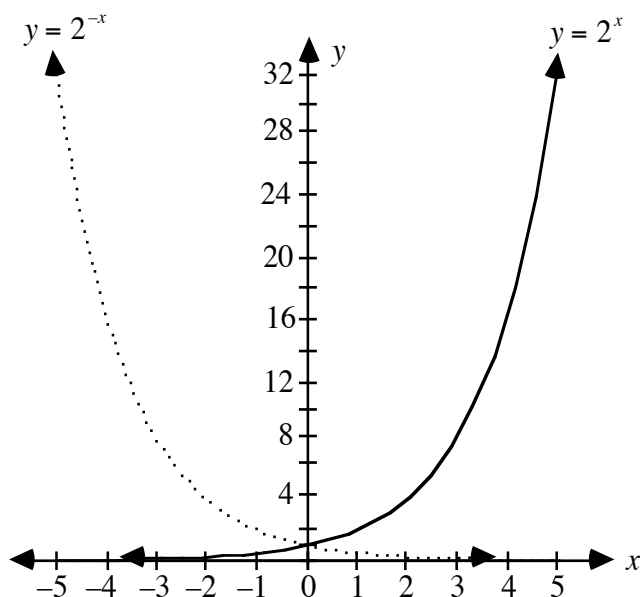
- a. Sample spreadsheet:

$x$	$2^x$	$(1/2)^x$	$1/2^x$	$2^{-x}$
4	16	0.0625	0.0625	0.0625
3	8	0.125	0.125	0.125
2	4	0.25	0.25	0.25
1	2	0.5	0.5	0.5
0	1	1	1	1
-1	0.5	2	2	2
-2	0.25	4	4	4
-3	0.125	8	8	8
-4	0.0625	16	16	16

- b. Sample response: The entries in the  $(1/2)^x$ ,  $1/2^x$ , and  $2^{-x}$  columns are the same. Since  $1/2^x$  is the reciprocal of  $2^x$ , then  $(1/2)^x$  and  $2^{-x}$  are also reciprocals of  $2^x$ .



c. Sample graph:



### Discussion 1

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- a.  $2^0 = 1$
- b. The expressions  $(1/2)^x$ ,  $1/2^x$ , and  $2^{-x}$  are equal.
- c. Sample response: The value of  $2^x$  is the reciprocal of the value of each of the other expressions.
- d. Sample response: There is no conflict. Because  $1^x = 1$  for any value of  $x$ ,  $(1/2)^x = 1^x/2^x = 1/2^x$ .

e. Sample response:

$$2^{-3} = \frac{1}{2^3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125$$

- f. Sample response: The graphs are reflections of each other in the y-axis.
- g. 1. Using negative exponents,  $2^3$  can be expressed as  $(1/2)^{-3}$  or  $1/2^{-3}$ .
2. Using negative exponents,  $a \cdot b^2$  can be expressed as  $a \cdot (1/b)^{-2}$ ,  $a \cdot 1/b^{-2}$ , or  $a/b^{-2}$ .
- h. 1. Without using negative exponents,  $2^{-3}$  can be expressed as  $(1/2)^3$  or  $1/2^3$ .
2. Without using negative exponents,  $a \cdot b^{-2}$  can be expressed as  $a \cdot (1/b)^2$ ,  $a \cdot 1/b^2$ , or  $a/b^2$ .

## Exploration 2

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- a–c. Students use a spreadsheet to experiment with various values for  $b$ ,  $x$ , and  $y$  in the expressions  $b^{x+y}$ ,  $b^{x-y}$ ,  $b^{x \cdot y}$ ,  $b^x \cdot b^y$ ,  $b^x/b^y$ , and  $(b^x)^y$ .

They should observe the following relationships:  $b^x \cdot b^y = b^{x+y}$ ,  $(b^x)^y = b^{x \cdot y}$ , and  $b^x/b^y = b^{x-y}$ .

## Discussion 2

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- a. Sample response: Yes, the relationships  $b^x \cdot b^y = b^{x+y}$ ,  $(b^x)^y = b^{x \cdot y}$ , and  $b^x/b^y = b^{x-y}$  appear to be true for all values tested.
- b. 1. Sample response: This equation indicates that  $4^{-5} \cdot 4^x = 4^{-5+x}$ . This means that  $-5 + x = 10$ , so  $x = 15$ .
2. Sample response: This equation indicates that  $(4^{-5})^x = 4^{-5x}$ . This means that  $-5x = 10$  so  $x = -2$ .
- c. Sample response: This equation indicates that  $(5^3)^x = 5^{3 \cdot x}$ . This means that  $3x = 1$  so  $x = 1/3$ .
- d. 1.  $16^{0.25} = 16^{25/100} = 16^{1/4} = \sqrt[4]{16} = 2$
2.  $16^{3/4} = (\sqrt[4]{16})^3 = \sqrt[4]{4096} = 8$
- e. Sample response: In this case, 4.3 represents the number of units of time that have passed since the initial level of radioactivity was measured.
- f. Sample responses:

$$b^{4.3} = b^4 \cdot b^{0.3} = b^4 \cdot b^{3/10} = b^4 \cdot (\sqrt[10]{b})^3 \text{ or } b^{4.3} = b^{43/10} = (\sqrt[10]{b})^{43}$$

- g. 1. Sample response:

$$\begin{array}{ll} (-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2 & (-8)^{1/3} = \sqrt[3]{(-8)^1} = -2 \\ (-8)^{2/6} = (\sqrt[6]{-8})^2 = \text{undefined} & (-8)^{1/3} = (\sqrt[3]{-8})^1 = -2 \\ (-8)^{2/6} = ((-8)^2)^{1/6} = 2 & (-8)^{1/3} = ((-8)^1)^{1/3} = -2 \\ (-8)^{2/6} = ((-8)^{1/6})^2 = \text{undefined} & (-8)^{1/3} = ((-8)^{1/3})^1 = -2 \end{array}$$

**Note:** Some forms of technology may give results that are inconsistent with this sample response. For example, some calculators may report that  $(-8)^{2/6} = -2$ .

2. Sample response: When the rational exponent is not in lowest terms, raising a base to that power may be undefined or may give conflicting values. **Note:** Another way of approaching this topic is to consider the rational exponent as a decimal. This works well when the decimal is terminating. If the decimal is non-terminating, however, one must consider the corresponding limit.

### Assignment

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- 2.1 a. 1. In 2 years,  $1/4$  of the radioactive atoms will remain.  
 2. Since  $x = 0$  indicates that no time has passed, the number of atoms would remain the same.  
 3. Two years earlier, the number of radioactive atoms was four times the present number.  
 b. Sample response: The  $-2$  indicates that you are calculating the number of atoms that were present two years earlier.
- 2.2 a. Sample response: This represents the number of bacteria present after 4 min.  
 b. Sample response: This represents the number of bacteria present 4 min before the initial population was counted.  
 c. The expression can be rewritten as  $32 \cdot (1/2^4)$  or  $32 \cdot (1/2)^4$ .
- 2.3 a. Sample response:  $y = 32 \cdot 2^{-x}$ .  
 b. Some students may solve this problem by graphing  $y = 32 \cdot (1/2)^x$  and using the trace feature on their graphing utility. Some may write 32 as  $2^5$  and use the properties of exponents; others may choose to use a spreadsheet.  
 1. 2 shakes  
 2. 4 shakes  
 3.  $-1$  shakes  
 4.  $-2$  shakes  
 5. 0 shakes
- 2.4 Sample response:

$$\left(\frac{a}{b}\right)^{-x} = \frac{a^{-x}}{b^{-x}} = \frac{1/a^x}{1/b^x} = \frac{b^x}{a^x} = \left(\frac{b}{a}\right)^x$$

- 2.5
- $3^{-3}$
  - $a^{-b}$
  - $\frac{3^2}{5^2} = \frac{9}{25}$
  - $(5/3)^{-2}$
  - $(3/4)^2$
- \*2.6
- Using  $5.12 \cdot 10^{14}$  for  $a$  and  $(1 - 0.5)$  for  $b$ , the equation is:  
 $3.2 \cdot 10^{13} = 5.12 \cdot 10^{14} \cdot (0.5)^x$ .
  - Sample response: The equation  $3.2 \cdot 10^{13} = 5.12 \cdot 10^{14} \cdot (0.5)^x$  can be simplified to  $0.0625 = (0.5)^x$ , or  $1/16 = (1/2)^x$ . In this case,  $x = 4$ . The radioactive atoms in the artifact began decaying 4 intervals of time ago.
- 2.7
- Sample response: Changing the initial population changes the value of  $a$ , which in turn changes the  $y$ -intercept. The graph appears less steep as the value of  $a$  approaches 0. **Note:** Although the graph no longer models decay, it curves downward when the value of  $a$  becomes negative and appears steeper as  $a$  continues to decrease. When  $a = 0$ , the graph of  $y = a \cdot b^x$  is the horizontal line  $y = 0$ .
  - Sample response: Since  $b = 1 + r$ , changing the rate of decay changes  $b$ . For values of  $b$  between 0 and 1, the graph becomes less steep for values closer to 0. When  $b$  is 0, the graph coincides with the  $x$ -axis. When  $b$  is 1, the curve coincides with the line  $y = a$ . **Note:** Although the graph no longer models decay, it curves upward and to the right instead of upward and to the left for values of  $b$  greater than 1. When  $b$  is less than 0, the graph alternates between positive and negative values.
- \*2.8
- Sample response: The equation indicates that the initial number of radioactive atoms in the object was 4000. The rate of decay is  $-3/20$ . This means that 15% of the radioactive material decays in each time interval.
- 2.9
- $(x^3)^{1/3} = x^{3 \cdot 1/3} = x^1 = x$
  - $x = 4$

- 2.10** a. Sample response: In this case,  $x = 3$ . This can be found by raising both sides of the equation to a power of  $1/6$ .

$$\begin{aligned}(x^6)^{1/6} &= 729^{1/6} \\ x^{6 \cdot 1/6} &= 729^{1/6} \\ x &= 729^{1/6} = 3\end{aligned}$$

- b. Sample response: In this case,  $x = 256$ . This can be found by raising both sides of the equation to the 8th power.

$$\begin{aligned}x^{0.125} &= 2 \\ x^{1/8} &= 2 \\ (x^{1/8})^8 &= 2^8 \\ x &= 256\end{aligned}$$

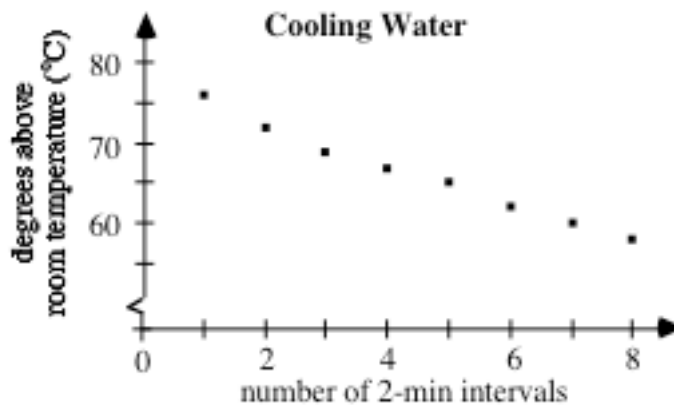
- c. Sample response:

$$\begin{aligned}10^5 \cdot x &= 10^{16} \\ x &= 10^{16}/10^5 \\ &= 10^{16-5} = 10^{11}\end{aligned}$$

- 2.11** a. Since  $b = 2$ ,  $r = 1$ . Because  $r$  is positive, this equation models a pattern of exponential growth.
- b. Since  $b \approx 1.1$ ,  $r \approx 0.1$ . Because  $r$  is positive, this equation models a pattern of exponential growth.
- c. Since  $b \approx 0.5$ ,  $r \approx -0.5$ . Because  $r$  is negative, this equation models a pattern of exponential decay.
- d. Since  $b \approx 0.8$ ,  $r \approx -0.2$ . Because  $r$  is negative, this equation models a pattern of exponential decay.

\* \* \* \* \*

- 2.12** a. Sample scatterplot:



- b. The mean of the percents of decrease in temperature is approximately 4.1% per 2-min interval.

2-min Interval	Degrees above Room Temperature (°C)	Percent Decrease
1	74	
2	70	5.1%
3	67	4.4%
4	65	3.1%
5	63	3.2%
6	60	5.0%
7	58	3.4%
8	56	3.6%

- c. Using the rate of decay from Part b, one equation that could be used to model the data is  $y = 77(0.959)^x$ , where  $x$  represents the number of 2-min intervals. Students may find the value of  $a$  in their models by trial and error or by examining residuals for various values. **Note:** The regression equation for the data is  $y \approx 76(0.962)^x$ .
- d. Using their model from Part c, students can estimate the room temperature by subtracting the value of the function at  $x = 0$  from  $100^\circ\text{C}$ . Using the equation  $y = 77(0.959)^x$ , the approximate room temperature is  $23^\circ\text{C}$ .
- e. Sample response: The exponential equation would not be a good model to predict temperatures before the water is poured because the function increases without bound as  $x$  decreases. This would indicate that the water continually increases in temperature as the time prior to pouring the liquid increases.

**2.13** The equation  $y = 5 \cdot (1 - 0.5)^3$  is equivalent to  $y = 5 \cdot (2)^{-3}$ . The equation  $y = 5 \cdot (2)^3$  is equivalent to  $y = 5 \cdot (1/2)^{-3}$ . The equation  $y = 5 \cdot (2)^{1/3}$  is equivalent to  $y = 5 \cdot (0.5)^{-1/3}$ .

\*\*\*\*\*

### Activity 3

This activity focuses on solving equations of the form  $y = a \cdot b^x$  for  $x$ .

#### Materials List

- graph paper
- numbered dice (or other random number generator; one per group)

#### Technology

- graphing utility
- spreadsheet (optional)

#### Exploration

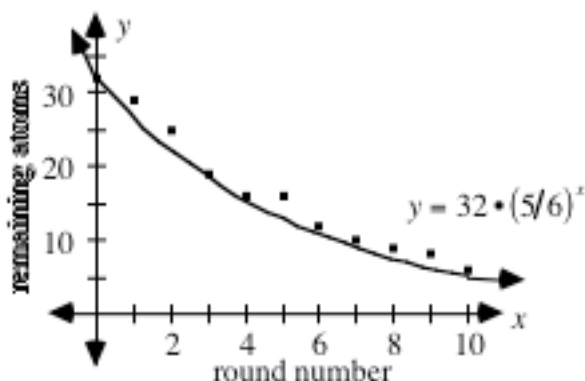
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- a–c.** The simulation requires students to use an appropriate random number generator. It should take about four rounds to reach the first half-life. Although students may need more than 10 rounds to remove all the atoms, 10 is sufficient to show the existing patterns. Sample data:

Round	Number of Radioactive Atoms
0	32
1	29
2	25
3	19
4	16
5	15
6	12
7	10
8	9
9	8
10	6

- d.** A graph of the sample data appears in Part e below.

- e.
1.  $y = 32 \cdot (5/6)^x$
  2. Sample graph:



- f.
1. Sample response: The half-life is about 4 rounds.
  2. Students may use the graph to find the intersection of  $y = 16$  and  $y = 32 \cdot (5/6)^x$ . They may also use the trace feature to estimate the value of  $x$  when  $y = 16$ . Students could also use a guess and check method in  $16 = 32 \cdot (5/6)^x$

## Discussion

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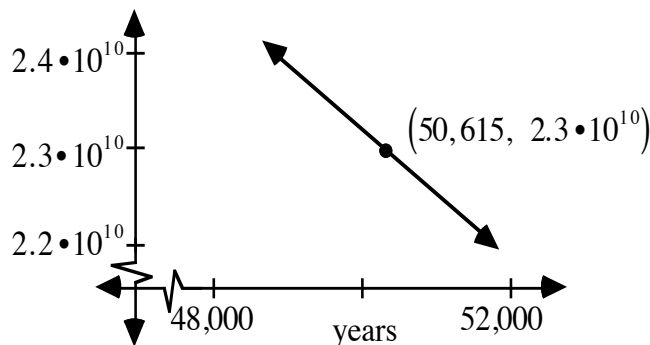
- a. Sample response: Since the probability of decay in the simulation is  $1/6$ , about  $5/6$  of the atoms should remain after each round.
- b. The half-life in this exploration is about 4 rounds while the half-life for the simulation in Activity 1 was 1 round (or shake).
- c.
  1. Since  $a = 32$  and  $b = 1 - (1/6) = 5/6$ , an appropriate model is  $y = 32(5/6)^x$ .
  2. Sample response: The value of  $a$  in both simulations is 32. The value of  $b$  in Activity 3 is  $5/6$ ; it was  $1/2$  in Activity 1. The value of  $r$  in Activity 3 is  $-1/6$ ; it was  $-1/2$  in Activity 1.



## Assignment

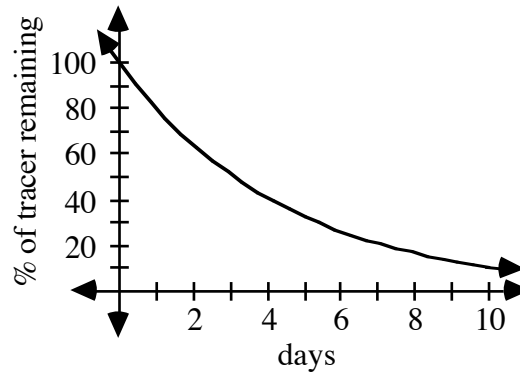
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- 3.1 a.  $y = 1.05 \cdot 10^{11} \cdot (0.99997)^x$
- b. The sample is about 51,000 years old. To find a solution, students may graph the equation or use a guess-and-check method. Sample graph:



- 3.2 a. 1.  $x = 2$   
2.  $x = 3$
- b. Sample response: The value of  $x$  must be between 2 and 3.
- c.  $x \approx 2.52$
- d.  $x \approx 3.27$
- 3.3 The half-life is approximately 3.80 rounds. See Exploration Part **f2** for solution methods.
- 3.4 a. In all three cases, the problem can be solved by substituting  $a/2$  for  $y$ . The value of  $x$  can then be found by guess-and-check using a calculator. Students also may graph each equation, trace until the  $y$ -value equals  $a/2$ , then record the corresponding  $x$ -value. (Students are not expected to use logarithms.)
- In each case, the half-life is about 1.4 time intervals.
- b. Sample response: Since all three equations have the same value for  $b$ , they must also have the same rate of decay. Since the half-life is the same in all three cases, the initial number of radioactive atoms does not appear to affect the half-life.
- \*3.5 **Note:** Radioactive tracers are useful in a number of applications, including engineering, biochemistry, and medicine. Iodine-131 and technetium-99m are two radioactive isotopes used in nuclear medicine. You may wish to consult a local hospital for more details about the use of radioactive tracers.
- a.  $y = 100 \cdot 0.8^x$

b. Sample graph:



c. The half-life occurs when  $y = 50\%$ . From the graph, this takes place after about 3 days.

d. Sample response: Using the equation  $y = 100 \cdot (0.8)^x$  and an exponent of  $-2$ , the newly manufactured sample's radioactivity was about 156% of the sample that Kim received.

**3.6** a.  $b = 0.5^{1/2.5} \approx 0.946$ ;  $r \approx -5.39\%/hour$

b.  $b = 0.5^{1/57.3} \approx 0.988$ ;  $r \approx -1.20\%/century$

**\*3.7** a.  $b^5 = 0.50$ ;  $b \approx 0.87$

b.  $r \approx -0.13 \approx -13\%/year$

c. Substituting 0.87 for  $b$  and 93 for  $y$  in the comparative equation  $y = 100b^x$  gives  $93 = 100(0.87)^x$ . Solving this equation for  $x$ , the sample is about 0.5 years old.

**\*3.8** a. The percentage of radioactivity remaining after each hour is equal to the value of  $b$  in the equation  $50 = 100 \cdot (b)^6$ . Solving this equation for  $b$  gives  $b \approx 0.89$ .

b.  $y \approx 100(0.89)^x$

c. Since 45 min equals 0.75 hours, students should write the equation  $y = 100(0.89)^{0.75}$ . This yields a value of approximately 92% for  $y$ .

d. After about 20 hours, 10% of the tracer will remain. This can be calculated by solving the equation  $10 = 100(0.89)^x$  for  $x$ .

e. The time 3 hours prior to the injection may be represented by  $x = -3$ . Solving the equation  $y = 100(0.89)^{-3}$  for  $y$  yields approximately 142. This means that the tracer was 42% more radioactive at the time of manufacture.

- 3.9** Sample response: Since 5730 years is 57.3 centuries, the comparative equation  $50 = 100(b)^{57.3}$  can be used to find the rate of decay in centuries. Solving for  $b$  yields  $b = 0.5^{1/57.3} \approx 0.988$ . Since  $b = 1 + r$ , the rate of decay is about  $-0.012$  or  $-1.2\%$  per century.

\* \* \* \* \*

**3.10**  $x = 0.25$  or  $1/4$

- 3.11 a.** Sample response: Since there are four multiples of 5 in the numbers from 1 to 20, the probability of generating a multiple of 5 is  $4/20$  or  $1/5$ . The number of atoms removed should be approximately  $1/5$  of the atoms remaining in the box from the previous round.

**b.**  $4/5$

**c.** The half-life is about 3 rounds.

**d.**  $y = 32 \cdot (0.8)^x$

- 3.12 a.** Approximately 97.6% of the initial radioactivity remains after 1 year. This can be calculated by finding the value of  $b$  using the known half-life of strontium-90:

$$50 = 100 \cdot (b)^{28}$$

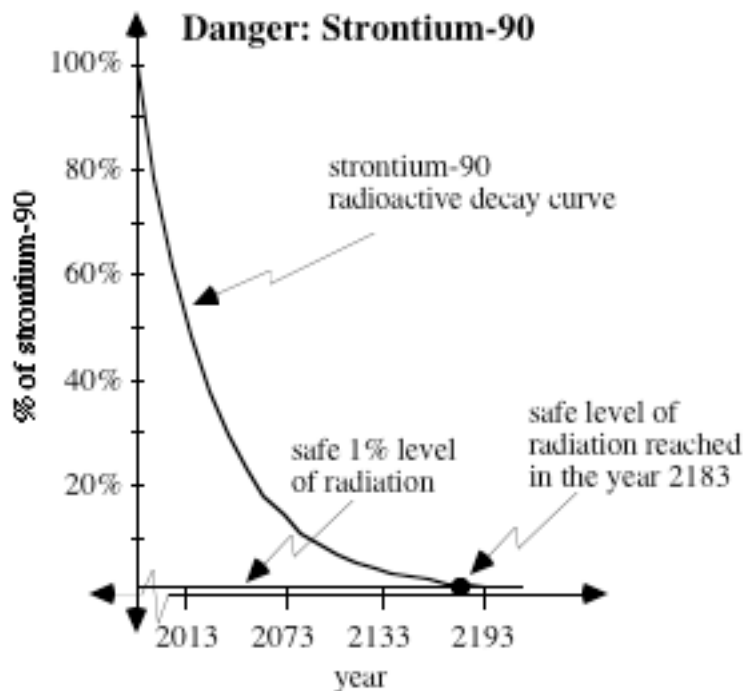
$$0.5^{1/28} = b$$

$$0.976 \approx b$$

- b.** Solving the equation  $1 = 100(0.976)^x$  for  $x$ , it will take about 190 years for the level of radioactivity to drop below 1%.

- c.** Solving the equation  $y = 100(0.976)^{-5}$  for  $y$  yields about 113. This means that the material was approximately 13% more radioactive 5 years ago.

- d. **Note:** The labels on the  $x$ -axis in the following sample response were generated by adding the year 2002 to the value of  $x$ . Students should add the present year to  $x$ .



- 3.13 a. To find the growth rate, students should solve the following equation for  $b$ :

$$0.65 = 0.25 \cdot b^{24}$$

$$b = \sqrt[24]{2.6}$$

$$b \approx 1.04$$

$$r \approx 4\%$$

- b. The following sample equation models the price of the candy bar where  $x$  represents the number of years after 1972:

$$y = 0.25 \cdot (1.04)^x$$

- c. Solving the equation  $0.10 = 0.25 \cdot (1.04)^x$  for  $x$  yields approximately  $-23.4$ . This corresponds with the year 1948.
- d. Solving the equation  $1.00 = 0.25 \cdot (1.04)^x$  for  $x$  yields approximately  $35.3$ . This corresponds with the year 2008.

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## Answers to Summary Assessment

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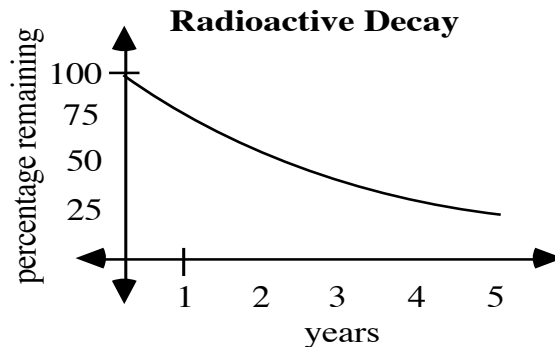
1. Students can solve this problem efficiently by using the trace function on a graphing calculator. Professor Cordova's maximum estimate of 12,000 years (120 centuries) should give students an idea of the appropriate domain. Using the value of  $b$  found in Problem 3.6, students may use the equation  $y = 100(0.988)^x$ .

Layer	% Radioactive C-14	Age (in centuries)
1	91.6	7.3
2	83.3	15.1
3	81.6	16.8
4	78.5	20.0
5	75.0	23.8
6	71.2	28.1
7	65.4	35.2
8	47.4	61.8
9	42.9	70.1
10	39.9	76.1
11	32.1	94.1

2. Because 11,500 years is about two half-lives ago, about 25% of the original carbon-14 should remain. This solution can also be found by substituting 115 for  $x$  in the equation  $y = 100(0.988)^x$ .
3. a. Using the comparative equation  $y = 100b^x$  and the value of  $b$  for carbon-14 found in Problem 3.6,  $112 = 100(0.988)^x$ . Solving for  $x$ ,  $x \approx -9.4$  centuries. The negative sign indicates that the carbon-14 in the innermost ring has been decaying 940 years longer than the carbon-14 in the outermost ring.
- b. The age of the tree using carbon dating is the same as the age determined by counting the rings.

## Module Assessment

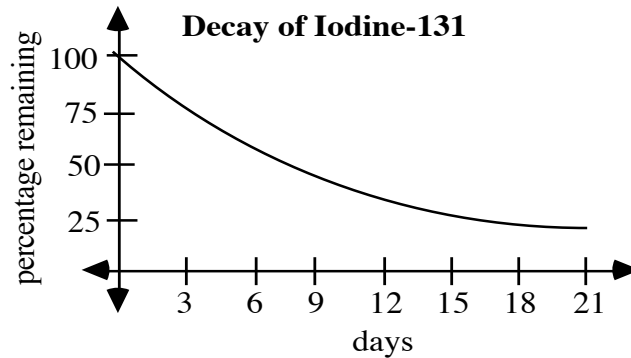
- The following graph shows the change in radioactivity for a substance over time. Use the graph to estimate the half-life of the substance.



- Give an example of a function of the form  $y = a \cdot b^x$  that models each of the following situations and explain why your example is appropriate:
  - exponential growth
  - exponential decay.
- Find the value of  $x$ , to the nearest integer, in each of the following equations:
  - $48 = 24(0.5)^x$
  - $1/2 = (4/5)^x$
- Former U.S. President George Bush was successfully treated for thyroid disease with a radioactive form of iodine. When the thyroid gland absorbs a dose of iodine-131, the process of radioactive decay kills the diseased cells.
  - Iodine-131 has a half-life of 8 days. What percentage of radioactivity in a dose of iodine-131 remains after 1 day? Explain your response.
  - Write an equation of the form  $y = 100b^x$  that models the decay of iodine-131.
  - Graph the decay of a dose of iodine-131 over 3 weeks.
  - What percentage of radioactivity in a dose of iodine-131 remains after 1 week?
  - Iodine therapy can lose its effectiveness when the level of radioactivity in a dose falls below 20%. After how many days does this occur? Explain your response.

## Answers to Module Assessment

1. The half-life is about 2.5 years.
2.
  - a. Sample response: In exponential growth, the rate is positive. Since  $b = 1 + r$ , the value for  $b$  must be greater than 1. One possible equation is  $y = 100 \cdot 2^x$ .
  - b. Sample response: In exponential decay, the rate is negative, so the value of  $b$  must be less than 1 (but greater than 0). One possible equation is  $y = 100(0.5)^x$ .
3.
  - a.  $x = -1$
  - b.  $x \approx 3$
4.
  - a. Solving the equation  $50 = 100b^8$  for  $b$  gives  $b \approx 0.917$ . The percentage of radioactivity remaining after 1 day is about 91.7%.
  - b.  $y = 100(0.917)^x$
  - c. Sample response:



- d.  $100(0.917)^7 \approx 54.5\%$
- e. Using the trace function on a graphing calculator, the graph of  $y = 100(0.917)^x$  drops below the line  $y = 20$  between the 18th and 19th day.

### **Selected References**

Libby, W. F. *Radiocarbon Dating*. Chicago, IL: University of Chicago Press, 1952.

Taylor, R. E. *Radiocarbon Dating: An Archaeological Perspective*. Orlando, FL: Academic Press, 1987



# Flashbacks

## Activity 1

- 1.1 When flipping a fair coin, what is the probability of getting tails?
- 1.2 Simplify each of the following expressions.
  - a.  $(6.33 \cdot 10^4) / (5 \cdot 10^2)$
  - b.  $(4.18 \cdot 10^{13}) \cdot 2$
- 1.3 Express each of the following numbers as a power of 10.
  - a. one hundred
  - b. one million
  - c. one

## Activity 2

- 2.1 Solve for  $x$  in each equation below.
  - a.  $3 \cdot x = 1$
  - b.  $2 \cdot x = 1$
- 2.2 Simplify each of the following expressions.
  - a.  $2^3$
  - b.  $(1/2)^3$
  - c.  $1/2^3$
- 2.3 Find the value of  $x$  in each of the following equations.
  - a.  $2^x = 4$
  - b.  $3^x = 27$
  - c.  $0.5^x = 0.0625$
- 2.4 The equation  $y = 5 \cdot 2^x$  models the annual growth in a population. In this model, describe what is represented by each of the following:
  - a.  $y$
  - b. 5
  - c. 2
  - d.  $x$

### Activity 3

- 3.1** A jar contains 200 jelly beans. Write an equation that describes the number of jelly beans remaining in the jar for each of the following situations:
- a.  $\frac{1}{3}$  of the jelly beans are removed each minute
  - b. 20% of the jelly beans are removed each minute
  - c. 60% of the jelly beans are removed each minute.
- 3.2** Write each of the following in the form  $a/b$  where  $a$  and  $b$  are integers:
- a.  $2^{-2}$
  - b.  $3^{-3}$
  - c.  $0.6^{-2}$
- 3.3**
- a. Graph the equation  $y = 2000(0.35)^x$ .
  - b. Find the value of  $x$  when  $y$  is 100.

## Answers to Flashbacks

### Activity 1

- 1.1** 50% or 0.5 or  $\frac{1}{2}$
- 1.2** a.  $1.266 \cdot 10^2$   
b.  $8.36 \cdot 10^{13}$
- 1.3** a.  $10^2$   
b.  $10^6$   
c.  $10^0$

### Activity 2

- 2.1** a.  $x = 1/3$   
b.  $x = 1/2$
- 2.2** a. 8  
b.  $1/8$   
c.  $1/8$
- 2.3** a.  $x = 2$   
b.  $x = 3$   
c.  $x = 4$
- 2.4** a. The variable  $y$  represents the population after  $x$  years.  
b. The quantity 5 represents the initial population.  
c. The quantity 2 represents  $1 + r$ , where  $r$  is the growth rate.  
d. The variable  $x$  represents the time in years.

### Activity 3

3.1 a.  $y = 200 \cdot (2/3)^x$

b.  $y = 200 \cdot (0.8)^x$

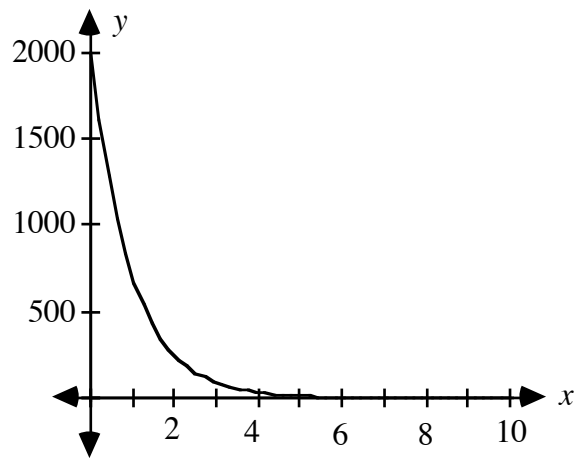
c.  $y = 200 \cdot (0.4)^x$

3.2 a.  $1/4$

b.  $1/27$

c.  $25/9$

3.3 a. Sample graph:



b. When  $y = 100$ ,  $x \approx 2.85$ .



# And the Survey Says . . .



How can television newscasters predict the next president of the United States long before the votes are counted? In this module, you explore some of the basics of sampling and surveys.

*Staci Auck • Kyle Boyce • Tom Teegarden*



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# Teacher Edition

## And the Survey Says . . .

### Overview

In this module, students explore methods of sampling. They examine potential sources of bias and experiment with different types of sampling. Histograms are used to represent data and to predict characteristics of a population. Confidence statements and margins of error are introduced as methods for interpreting the results of a survey.

### Objectives

In this module, students will:

- use a variety of sampling techniques
- predict the characteristics of a population based on samples
- explore the role that biases play in sampling
- use histograms to estimate probabilities and make predictions
- investigate how sample size affects a survey's reliability
- explore confidence statements and margins of error.

### Prerequisites

For this module, students should know:

- how to calculate mean and standard deviation
- how to create frequency tables
- how to construct histograms.

### Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	1	2	3	2	2	10

## Materials Required

Materials	Activity				Summary Assessment
	1	2	3	4	
graph paper		X	X		
paper bags	X		X		
dry beans	X		X		
template of town map		X			

## Teacher Note

A blackline master of the template appears at the end of the teacher edition FOR THIS MODULE.

## Technology

Software	Activity				Summary Assessment
	1	2	3	4	
statistics package			X	X	X
spreadsheet		X	X	X	X
graphing utility				X	
random number generator		X			
programmable calculator (optional)			X	X	X



## And the Survey Says . . .

### Introduction

(page 295)

You may wish to ask students to discuss some recent opinion polls.

(page 295)

### *Activity 1*

Using a population of two different kinds of beans, students explore the effects of biased sampling methods on attempts to characterize a population. **Note:** This activity introduces students to some terms used throughout the module: *population, parameter, sample, statistic, census, and bias.*

### Materials List

- two varieties of dry beans that are significantly different in size (about 150 beans per group)
- paper bags or other opaque containers (one per group)

### Teacher Note

Each population should consist of a total of about 150 beans of two different sizes—for example, kidney beans and lentils. (The actual proportion of the smaller beans to larger beans is unimportant.)

### Exploration

(page 296)

- When students take samples by the handful, the size difference in the two kinds of beans creates a bias in favor of the larger beans.
- The following sample data was collected using a population of 30% kidney beans and 70% lentils.

Sample	Kidney Beans	Lentils
1	77%	23%
2	60%	40%
3	57%	43%
4	38%	63%
5	64%	36%
<b>Mean</b>	59%	41%

- c. Answers will vary. Students may use the mean percentages from their five samples to predict the population percentages. (See sample data given in Part b.)
- d. This actual proportions of beans in the population may differ considerably from student predictions.

## Discussion

(page 296)

- a.
  - 1. The percentages of each type of bean determined in the census are parameters.
  - 2. The percentages of each type of bean in a sample (as well as the means of these percentages) are statistics.
- b. Sample response: The data collected by sampling predicted a higher percentage of the larger beans and a lower percentage of the smaller beans than are actually in the population.
- c. The sampling method—in combination with the difference in the size of the beans—creates bias. When sampling by the handful, larger beans have a better chance of being selected since smaller ones tend to slip through the fingers.
- d. Students should suggest sampling techniques that give each bean the same chance of being selected. For example, using a cup to select the sample may help to reduce the bias in favor of the larger beans.
- e. Real-life surveys often ignore segments of the population through biased sampling. For example, telephone surveys eliminate anyone who does not own a phone. If conducted during normal business hours, they may also eliminate people who do not work at home. Other polls solicit voluntary responses and therefore count only those people with strong opinions. Since many magazines target specific audiences, their polls are limited to a select readership. Polls taken on downtown streets or in shopping malls exclude large segments of the general population. The wording of questions, especially regarding sensitive topics, may also bias a survey.
- f. Because of the descriptive adjectives used to praise the eagle and belittle the bison, this survey question is biased in favor of the eagle. To reduce bias, the question may be rephrased as follows: “Would you prefer the eagle or the bison as the school mascot, or do you have no opinion?”

## Assignment

(page 297)

- 1.1** Answers will vary. Such surveys do not provide random samples for several reason. Many dial-in surveys charge a small fee for the call, which creates bias against those unwilling to pay. Often, only those people who have strong opinions about the topic will call. The most significant bias, of course, is against people who are not watching the program.
- 1.2** Responses may vary. This type of survey does not give everyone in the population an equal chance to be in the sample. The polling method excludes those who do not visit shopping malls. Of those who are questioned, several members of the same household might answer affirmatively, although each individual does not own a separate VCR.
- \*1.3**
- a.** The population is the 2380 Washington High School students.
  - b.** Students should describe methods that survey every student in the school population. Sample response: Send a person to poll every first-period class in the school. Check records for any students without a first-period class and find them in the next available class.
  - c.** Answers will vary. Three sample responses are listed below.
    - 1.** Choose the first person who walks into the cafeteria at lunch time and every 10th person after that.
    - 2.** Select all the sophomores in one homeroom class.
    - 3.** Write a computer program to generate random numbers that correspond with student ID numbers. Sample those students whose ID numbers match the computer-generated numbers.
  - d.** The following list describes some advantages and disadvantages to the sample responses in Part **c.**
    - 1.** This type of sample is easy to administer. However, not all students may eat lunch in the cafeteria. The students who don't will be excluded from the sample.
    - 2.** This type of sample is also easy to administer. But since all students other than sophomores are excluded, as well as all other students not in that homeroom, the sample is extremely biased.
    - 3.** Since every student has an equal chance of being selected, this method produces a random sample. It may take some time and effort, however, to write a computer program that will generate appropriate numbers.

- 1.4 a. In 1936, many people did not own a car or a telephone. A significant portion of the voting population was therefore excluded from the survey.
- b. The pollsters should have designed a survey that gave every registered voter an equal chance of being selected. (**Note:** George Gallup did an accurate poll for this election, and his career as a pollster blossomed.)

1.5 Sample response: In a survey on providing housing for the homeless, the following question would be biased in favor of a “yes” response: “If some friends of yours suddenly became homeless, would you want them to be provided with housing?” By including the clause about friends, the question creates bias. Most people would not want to have friends in such a desperate situation. If the question was rewritten as follows — “Would you vote for a bond issue to provide housing for the homeless?” — the number of affirmative answers might change dramatically.

\* \* \* \* \*

- 1.6 a. Sample response: No. Jordan’s survey does not necessarily provide a representative sample of the population, since it contains only his classmates and friends. **Note:** You may wish to point out that although a random sample also may not necessarily provide a representative sample of a population, the manner in which it was selected is unbiased.
- b. Jordan’s survey question appears to be relatively free of bias. Some students may argue, however, that the question is biased towards rock and roll, since this category is listed first.

Since music preferences tend to be similar among friends, his survey technique introduces a bias against the preferences of other students in the school.

- c. To minimize bias in the sampling method, students may suggest that Jordan randomly select 50 people from a list of the entire school population.
- 1.7 a. Sample response: Because of her use of the word *confined*, Susanne appears to favor the proposal.
- b. Sample response: “Do you want to have open campus for the lunch period even though it would lengthen the school day by 15 minutes?”
- c. Sample response: “Are you in favor of or against an open campus lunch, or do you have no opinion?”

\* \* \* \* \*

## **Research Project**

(page 298)

- a. Student summaries should include a description of the population, the sample size, and the sampling technique, as well as a discussion of potential bias in the survey.
- b. Students may need some guidance on possible topics and on the format of their presentations. You may wish to consult with a history teacher for suggestions.

(page 299)

### **Activity 2**

This activity focuses on different types of sampling techniques, including stratified sampling and systematic sampling.

#### **Materials List**

- template of town map (one per student; a blackline master appears at the end of this teacher edition)

#### **Technology**

- statistics package
- random number generator

#### **Exploration**

(page 299)

The exploration provides hands-on activities that allow students to examine different sampling techniques. **Note:** Students sample from a map of a town divided into districts. This is a much simplified version of an actual district map. Typical voting districts or precincts often have irregular boundaries (and populations that are not multiples of 100).

- a. Allowing students to select their own samples may illustrate the need for sampling techniques that eliminate unintentional biases. Students often choose one small, one large, and two medium-sized districts. This creates bias, since after one small district is chosen, for example, the other small districts do not have an equal chance of being selected in the sample. Students may also tend to choose only rectangular districts, which creates bias against those with other shapes.
  1. The populations of individual districts vary from 100 to 600 people.
  2. Answers will vary. Many students will calculate the mean population of the four districts in their sample. The actual mean is 207.5.

3. Estimates will vary. Students may multiply the mean number of people per district by the total number of districts (40), multiply the number of people in their four samples by 10, or devise other methods.

b. The actual population of the town is 8300.

c. Student results will vary. A random sample of districts 25, 40, 39, and 32 results in the following mean population per district:

$$\frac{200 + 300 + 100 + 200}{4} = 200$$

This corresponds with a population estimate of  $200 \cdot 40 = 8000$ .

d. 1. Answers will vary. Some students may group districts by size and select one at random from each subgroup. Others may group districts by location or district number. For example, the two highways divide the town into four regions. Each region could be considered a stratum.

2. Using the method in Part 1, the following four districts might be selected: 4, 15, 21, and 37. For this sample, the mean population per district is:

$$\frac{300 + 200 + 300 + 400}{4} = 300$$

This corresponds with a population estimate of  $300 \cdot 40 = 12,000$

e. 1. Sample response: Starting with region 5, select every 10th region after that. The sample contain regions 5, 15, 25, and 35.

2. The mean population per district for the sample described in Step 1 is:

$$\frac{200 + 200 + 200 + 200}{4} = 200$$

This corresponds with a population estimate of  $200 \cdot 40 = 8000$ .

## Discussion

(page 301)

a. Since different samples can be selected using the same method, identical sampling techniques do not necessarily produce identical results. The class data should illustrate this point.

b. An intuitive sampling method will not guarantee that every district has the same chance of being selected.

Since each district has the same chance of inclusion in a simple random sample, this method has no apparent bias.

Depending on how strata are determined or on how samples are selected from each stratum, bias may or may not appear in a stratified sampling method.

A systematic sampling method might introduce bias through the selection of the starting point and the value of  $n$ .

- c. The sampling method based on intuition allows the most bias. Simple random sampling has the least potential for bias because every district has an equal chance of being selected.

## Assignment

(page 301)

- 2.1**
- a. This method does not produce a simple random sample because everyone in the school may not eat in the cafeteria and because it selects only those who arrive early.
  - b. This method produces a simple random sample because every student has an equal chance of selection.
  - c. This method produces a simple random sample of boys and a simple random sample of girls, but not a simple random sample of students, since selections depend on gender. This is an example of stratified sampling by gender.
  - d. This method produces a simple random sample because every student has an equal chance of selection.
  - e. This method does not produce a simple random sample because it is biased against students who do not attend the pep assembly. This is an example of systematic sampling.
  - f. This method produces a simple random sample only if student ID numbers are randomly issued (and some ID numbers end with 0).
- 2.2** Since it creates subgroups by gender, the method described in Part **c** represents stratified sampling. Because it selects every  $n$ th unit of the population, starting with the first student, the method described in Part **e** represents systematic sampling
- \*2.3**
- a.
    1. Sample response: Write the names of all students on identical slips of paper, put the slips into a bin, mix them thoroughly, and draw out 20 slips.
    2. Sample response: Select five names at random from each of the four classes: seniors, juniors, sophomores, and freshmen.
    3. Sample response: Select the first person who walks into the next school-wide assembly and every 10th person after that, until you have selected 20 people.

- b. The following answers correspond with the sample responses given in Part a.
1. Since this is a simple random sample, where each name has the same chance of being selected in the sample, there is little potential for bias.
  2. Some bias may occur in the stratified approach since each student is not selected independently. The proportions of students from each class in the sample may not reflect their proportions in the population. **Note:** When the sample sizes are proportional to the sizes of the strata, this type of stratified sampling is known as *proportional sampling*.
  3. This method selects from only those students who attend the assembly. (Also, if fewer than 200 students attend, it will not provide the desired sample size.)

2.4 Answers will vary. Students should not feel compelled to use simple random samples. Depending on the purposes of the survey, a different method—such as stratified sampling—could eliminate other biases, even though it does not produce a simple random sample. In a survey of dating preferences, for example, it might be appropriate to guarantee equal participation by males and females and by each class.

2.5 Sample response: For a magazine article on public attitudes about hunting, a group that seeks to ban hunting might sample only an urban population. Since many urban residents may have only limited experience with hunting, the sample would not accurately represent the views of the population as a whole.

\* \* \* \* \*

- 2.6
- a. Sample response: Starting with the 2nd voter on list, select every 14th voter.
  - b. Sample response: Randomly generate 100 integers in the interval [1, 1400] and select voters corresponding to these numbers.
  - c. Sample response: Since 53% of the population is female, randomly select 53 female voters and 47 male voters.

- 2.7
- a. The proposed method will not provide a simple random sample, since only the potential buyers who request a test drive are sampled.
  - b. Sample response: One advantage of the proposed method is that the sample would be easy to collect. This may make it more cost-effective for the manufacturer. One disadvantage is that the sample is biased against potential buyers who do not ask for a test drive.

\* \* \* \* \*



## **Activity 3**

In this activity, students explore sampling as a tool for describing a population. Using frequency histograms, they investigate the distribution of sample proportions.

### **Materials List**

- two varieties of dry beans of different colors but similar sizes (approximately 150 beans per group)
- paper bags or other opaque containers (one per group)
- markers (optional; one per group)

### **Teacher Note**

The two types of beans should be distinguishable by color but not by size—for example, kidney beans and pinto beans. As an alternative, students may place a clearly discernible mark on 40% of the beans. Objects other than beans also may be used as long as they are of similar size and shape and can be divided into two distinct groups.

### **Technology**

- statistics package
- spreadsheet (optional)
- programmable calculator (optional)

### **Teacher Note**

In Part **f** of the exploration, students use technology to simulate sampling. If a statistics package that allows sampling using binomial trials is not available, students may use a spreadsheet or a programmable calculator.

For example, the following program was written for the TI-92 calculator. It stores the results of each sample as an element of a list in the variable *list1*. (Each student should input a different seed number before running the program.)

```

:Sampling( )
:Prgm
:ClrIO:DelVar list1
:Input "Known Percentage?", p
:Input "Number of samples?", n
:Input "Sample size?", s
:0→c
:For i,1,n
:Disp "Sample number", i
:For j,1,s
:rand(100)→r
:If r<p Then
:  c+1→c
:EndIf
:EndFor
:int(100*(c/s)+0.5)→list1[i]
:0→c
:Disp "Percent marked:", list1[i]
:EndFor
:EndPrgm

```

To use a spreadsheet to simulate sampling from a population in which 40% have a particular characteristic, complete the following steps.

- Enter "1" in the first 40 rows in column 1.
- Enter "0" in the next 60 rows in column 1.
- Enter random numbers in each of the first 100 rows of column 2.
- Sort the random numbers, allowing each value in column 1 to follow its corresponding random number.
- To select a sample of size  $n$ , find the sum of the values in the first  $n$  rows of column 1. This sum represents the number of individuals in the sample with the given characteristic.

The spreadsheet below shows how this method was used to simulate a sample of size 10 from such a population.

0	0.97677807	
0	0.94181197	
0	0.9394662	sum = 4
1	0.91934078	
1	0.91402806	
0	0.88979734	
0	0.87726297	
1	0.85729261	
1	0.82452538	
0	0.80195173	
⋮	⋮	

## Exploration

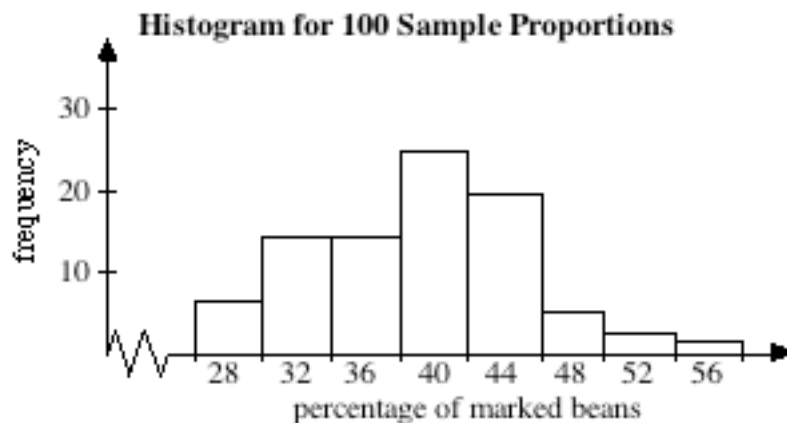
(page 303)

This exploration is intended for small groups. In Part e, the class will combine their data to obtain a larger pool of samples.

- a. Students do not need to create a population of 2380 beans. Approximately 150 beans is sufficient, as long as the population proportion is 40%.
- b. Since 40% of 25 is 10, it would be reasonable to expect about 10 beans to be marked.
- c–d. Sample data:

Sample	Percentage of Marked Beans
1	48
2	32
3	36
4	48
5	48
6	60
7	56
8	48
9	40
10	36

- e.
  1. Sorting the data should simplify the task of determining the number of sample proportions that fall within a given interval.
  2. The mean of the sample proportions should be close to the population proportion of 40%.
  3. Sample histogram for class data:



4. In the sample data shown above, 23% of the sample proportions are less than or equal to 32%.
- f. The results of a simulation using technology should be reasonably close to the results obtained using a population of beans.

## Discussion

(page 304)

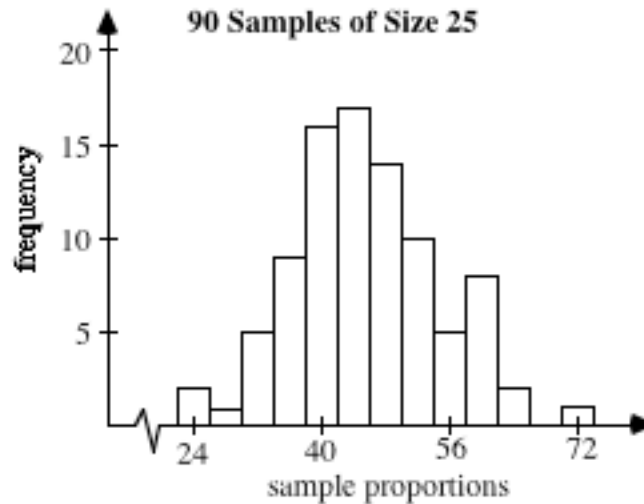
- a. Sample response: Both graphs have peaks near the mean of the sample proportions, then gradually taper off on each side.
- b. The mean of the sample proportions should be close to 40%, the population proportion.
- c.
  1. Sample response: The histograms would be similar in shape, but would be centered about the population proportion of 52%.
  2. If twice as many samples were taken, the histogram would still be centered about the mean and have the same basic shape.
- d. Students should observe that the percentage of sample proportions less than or equal to 32% provides an estimate of the probability of obtaining such a sample from a population in which the population proportion is 40%.
  1. Responses will vary. In one simulation of 90 samples of size 25, 23% were less than or equal to 32%. In another simulation, 26% were less than or equal to 32%.
  2. Sample response: The simulation models a situation in which Amelia is preferred by 40% of student voters. The selected sample showed only 32% support for her candidacy. Since the simulated samples indicate that obtaining a sample with 32% support from a population with 40% support is unlikely, Amelia should not be concerned.
- e. To obtain a more accurate characterization of the population, students may suggest increasing the sample size.

## Assignment

(page 304)

- \*3.1**
- a. Of those surveyed, 36% (9 of 25) indicated that they would vote for the pool.
  - b. There are 300 samples in the simulation.
  - c. The mean percentage of “yes” votes may be estimated by dividing the sum of the products of each frequency and its corresponding percentage by the total number of samples. The mean of the sample proportions in the simulation is about 52%.
  - d. Sample response: The architect should not be hired. The simulation models a population that would just barely pass the proposal. The town’s single sample of 25 voters revealed only 36% support, which is at the extreme low end of the simulated samples from a population with 51% support. The town’s sample is very unlikely to have come from a population with enough votes to pass the proposal.

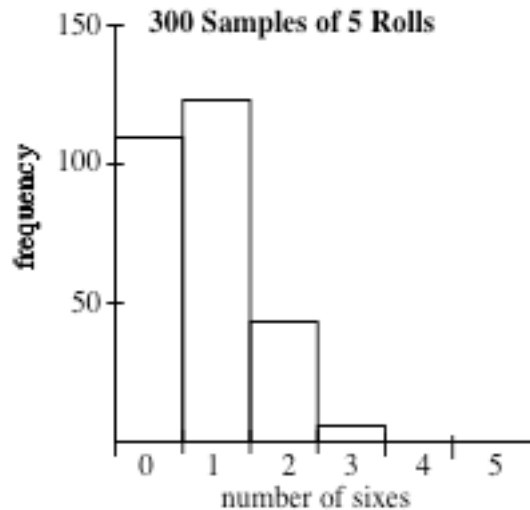
- 3.2**
- a.** Sample response: Since 10% of the population will not vote, Amelia must obtain a majority of the remaining 90%. Therefore, she would need approximately 46% to win.
  - b.** The following sample histogram shows the results of 90 samples of size 25 taken from a population with an assumed parameter of 46%.



- c.** In the histogram in Part **b**, 8 of the 90 samples, or approximately 9%, have a sample proportion of less than 32%.
- d.** Sample response: If Amelia actually has 46% support, the probability of selecting a sample of 25 students with 8 or fewer in favor of Amelia is fairly small. Amelia's chances of winning the election are not encouraging. Since there were a large number of undecided voters in the census, however, Amelia still has a chance of winning.

\* \* \* \* \*

- 3.3 a–b. The following histogram shows the results for 300 samples of size 5 for a simulation of a fair die.



- c. Answers will vary. In the sample data shown in Parts **a** and **b**, the estimated probability is 0 since in the 300 samples four sixes in five rolls never happened. (See graph.) **Note:** The theoretical probability of rolling a six 4 out of 5 times with a fair die is approximately 0.3%.
- d. Sample response: No. The die does not appear to be fair. Although rolling a six 4 out of 5 times is possible, it is very unlikely. It is not clear if the die is fair or not fair. Theoretically (see Part **c** answer), there would only be approximately one sample in 300 in which four of the 5 people rolled a six. The histogram showed zero.

\*\*\*\*\*

(page 306)

### Activity 4

In this activity, students determine the maximum standard deviation of all possible sample proportions for samples of any size. They then use this value to describe the interval for a confidence statement.

#### Materials List

- none

#### Technology

- graphing utility
- spreadsheet
- statistics package
- programmable calculator (optional)

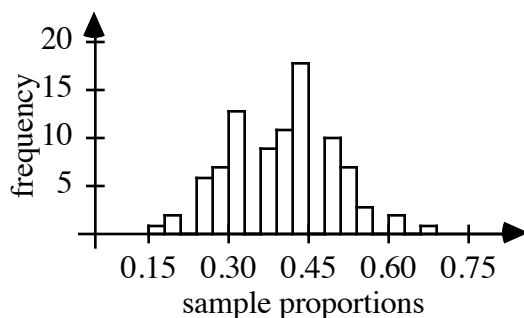
## Teacher Note

To save time in Exploration 1, you may wish to ask some students to complete Parts **a** and **b**, some to complete Part **c**, and some to complete Part **d**.

## Exploration 1

(page 307)

- a.**
1. Using a statistics package that allows binomial simulations, students should generate 90 samples of size 25, with a probability of success of 0.40. (See teacher note in Activity 3 if this technology is not available.)
  2. Sorting the data should simplify the task of determining the percentage of sample proportions that fall within 1 or 2 standard deviations of the population proportion.
  3. Sample histogram:



- b.**
1. Using the formula given in the mathematics note, the standard deviation of all possible sample proportions is:

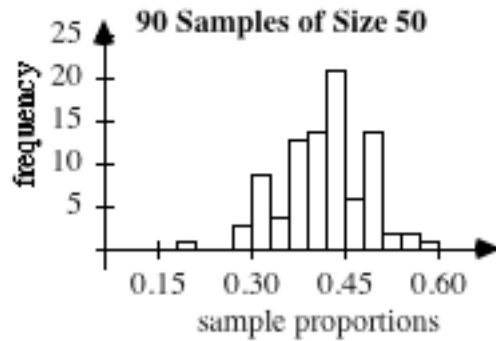
$$\sqrt{\frac{0.4(1-0.4)}{25}} \approx 0.098$$

or about 9.8%.

In the sample data shown in Part **a**, 69% of the sample proportions were contained in the interval [30.2, 49.8].

2. In the sample data shown in Part **a**, 98% of the sample proportions were contained in the interval [20.4, 59.6].

- c. The following histogram shows the results of 90 samples of size 50.

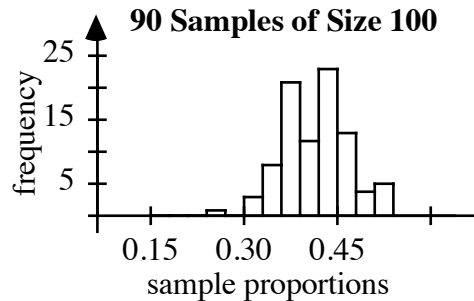


Using the formula given in the mathematics note, the standard deviation of all possible sample proportions is:

$$\sqrt{\frac{0.4(1-0.4)}{50}} \approx 0.069 = 6.9\%$$

For the sample data shown above, 69% of the sample proportions were contained in the interval [33.1,46.9], and 98% were contained in the interval [26.2,53.8].

- d. The following histogram shows the results of 90 samples of size 100.



Using the formula given in the mathematics note, the standard deviation of all possible sample proportions is:

$$\sqrt{\frac{0.4(1-0.4)}{100}} \approx 0.049 = 4.9\%$$

For the sample data shown above, 68% of the sample proportions were contained in the interval [35.1, 44.9], and 94% were contained in the interval [30.2,49.8].



## Discussion 1

(page 307)

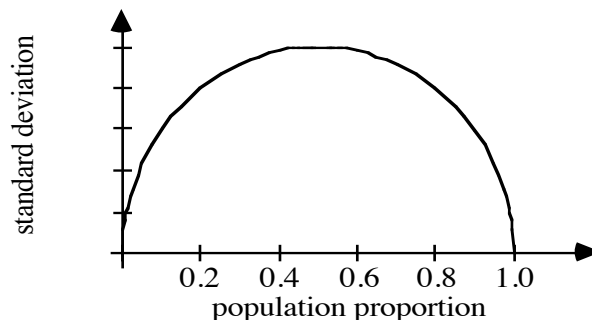
- a. The histograms should display approximately the same mean of the sample proportions (40%), although their shapes may vary.
- b.
  1. As the sample size increases, the sample proportions tend to be closer to the mean.
  2. As the sample size increases, the standard deviation of all possible sample proportions decreases.
  3. As the sample size increases, the interval that represents values within 1 standard deviation of the population proportion becomes narrower.
- c.
  1. For a sample size of 25, the percentages may vary. They should be relatively close—approximately 68% and 95%—for the larger sample sizes.
  2. Sample response: They indicate an estimated probability that the sample proportion will be within 1 or 2 standard deviations of the population proportion.
- d. Answers will vary. For a sample size of 100, a sample proportion of 32% is more than 1 standard deviation from the population proportion. In the sample data given in Exploration 1, only 5% of the samples yielded a proportion of 32% or less. This suggests that Amelia should be very concerned about her support, since the sample is not likely to have come from a population in which 40% favored her.

## Exploration 2

(page 308)

In this exploration, students use the formula for the standard deviation of all possible sample proportions to examine the maximum standard deviation for a given sample size. This value is a conservative estimate of the standard deviation of all possible sample proportions for a situation in which the actual population proportion is unknown.

- a.
  1. Sample graph:



2. Students may use the trace function on a graphing utility to find that the maximum standard deviation is 0.05.
  3. The population proportion that corresponds with the maximum standard deviation is 0.5.
- b. After repeating Part **a** for other sample sizes, students should conjecture that the maximum standard deviation always occurs when the population proportion is 0.5. As the sample size increases, the maximum standard deviation decreases.
  - c–d. Students add and subtract 5% from each sample proportion to obtain 90 intervals, then determine how many of these intervals contain the known population proportion of 40%.
    - e. The percentage of intervals that contain the population proportion should be close to 68%.
    - f. For a sample size of 100,  $2 \cdot \text{MSD} = 10\%$ . The percentage of intervals that contain the population proportion should be near 95%.

## Discussion 2

(page 309)

- a. The domain of the function is  $0 \leq p \leq 1$ , the range is  $0 \leq \sigma \leq 0.05$ .
- b. Sample response: I would be more confident predicting that the population proportion is within 2 MSDs of the sample proportion. According to the results of the simulation, about 95% of those intervals contained the known population proportion.
- c. Sample response: It is important to use the maximum standard deviation because this value describes the largest possible interval, thereby reducing the chance of error.
- d. Sample response: Since the width of the intervals depends on the MSD, and the MSD is calculated using the formula  $0.5/\sqrt{n}$ , increasing the sample size would make the intervals narrower, while decreasing the sample size would make the intervals wider.
- e. Sample response: I would be more confident in predicting the population proportion using the larger sample size. For a sample size of 100, the MSD is 5%. This means the sample proportion would be within 10% of the true proportion about 95% of the time.

For a sample size of 400, the MSD is  $0.5/\sqrt{400}$  or 2.5%. The value of 2 MSDs is  $2(2.5\%)$  or 5%. This means that the sample proportion would be within 5% of the true proportion about 95% of the time.

- f.
  1. When the margin of error is twice the maximum standard deviation, it decreases as the sample size increases.
  2. Since the maximum standard deviation of all possible sample proportions is not affected by the size of the population, neither is the margin of error.
- g. Using a value of twice the maximum standard deviation, the margin of error for a sample size of 1500 is approximately 2.6%. Therefore, the pollster can be confident that the statistic is within 2.6% of the parameter.

**Assignment**

(page 310)

- 4.1
  - a. For a sample size of 400, the maximum standard deviation of all possible sample proportions is 2.5%. Therefore, it is highly likely that the proportion of the population that will vote for Amelia is between 44% and 54%.
  - b. Sample response: Since 10% of the student body does not intend to vote, Amelia only needs the support of 46% of the population in order to win. Since it is very likely (about 95% probability) that the population proportion falls within the 44% to 54% interval, her prospects are very good but not guaranteed.
- 4.2 Sample response: A confidence statement should not be made about this survey. The ratio of 4 to 5 does not give any indication of sample size.
- 4.3 Sample response: Although Senator Rodriguez is not assured of reelection, a victory is likely since the poll predicts a majority of at least 51% (54 – 3) with a high level of confidence.
- \*4.4 The margin of error for a sample of size 1500 is  $1/\sqrt{1500} \approx 0.026$ , which is less than 3%.
- \*4.5 Sample response: The margin of error for a sample of 100 is about 10%. Since Amelia only needs votes from 46% of the population to win, her election is highly likely.

\* \* \* \* \*

- 4.6 The value of twice the maximum standard deviation of all possible sample proportions for  $n = 40$  is:

$$2 \cdot \sigma = 2 \cdot \sqrt{\frac{0.5 \cdot (1 - 0.5)}{40}} = \frac{1}{\sqrt{40}} \approx 0.16$$

Using this value to determine the margin of error, the population proportion is likely to fall in the interval [0.67, 0.99].

- 4.7** The value of twice the maximum standard deviation of all possible sample proportions for  $n = 25$  is:

$$1/\sqrt{25} = 0.2$$

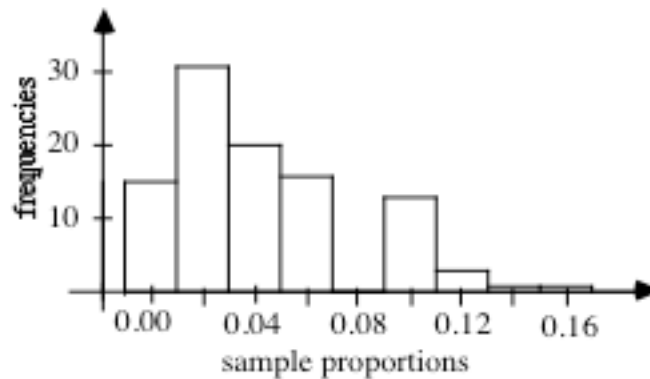
This value corresponds with a margin of error of 20%.

\* \* \* \* \*

## Answers to Summary Assessment

(page 312)

- Answers will vary. Sample response: To sample the bulbs, I would use a systematic sampling method. Assuming about 10,000 bulbs are produced per day, I would select every 250th bulb. This would create an approximate sample size of 40, which would be practical and efficient in terms of time and money.
- The following histogram shows the results of 100 samples of size 40 from a population in which  $p = 0.05$ :



- Using the value of twice the maximum standard deviation of all possible sample proportions, the margin of error for a sample size of 40 is:

$$\frac{1}{\sqrt{40}} \approx 0.158$$

- Judging from the results of the simulation, the experimental probability of obtaining a sample with at least 6% defective bulbs from a population in which  $p = 0.05$  is about 35%.

Given a sample proportion of 6.2%, the population proportion is highly likely to be contained in the interval  $[0, 22.0\%]$ . Since this interval is very wide, I would take another sample of the day's production using a larger sample size. If this sample also resulted in a proportion of defective bulbs greater than 5%, I would check to make sure that the equipment is running properly.

## ***Module Assessment***

1. The administration at Washington High is reviewing the student dress code. Before recommending any changes, the principal has agreed to allow the student council to conduct a survey of student opinion. Three members of the council have suggested sampling methods for the poll.

Proposal A involves selecting the first student that walks into the gym at the next school-wide assembly and every 15th student after that.

Proposal B uses a computer program to generate random numbers with the same number of digits as student identification numbers. Those students whose identification numbers match the computer-generated numbers will be selected for the poll.

Proposal C recommends selecting 10 freshmen, 10 sophomores, 10 juniors, and 10 seniors for the poll.

- a. Identify the strengths and weaknesses of each proposal.
  - b. Which proposal seems most fair? Justify your choice.
  - c. Describe any bias that might exist in the proposal you chose in Part **b**.
2.
    - a. Consider an experiment that involves flipping a fair coin 50 times. Use a simulation to estimate the probability that less than 40% of the flips will be heads.
    - b. Imagine that you flipped a dime 400 times and recorded 170 heads. Would you conclude that this dime is not a fair coin? Explain your response.
  3. Which statistic would you have more confidence in: a proportion from a simple random sample of 200 people or one from a sample of 500 people? Justify your response using the standard deviation of all possible sample proportions.
  4. An independent polling company asked a random sample of 588 teenagers the following question: “What brand of court shoes do you prefer?” Of those surveyed, 174 responded “Eagle Wing.” Write a confidence statement describing the results of the poll.

## Answers to Module Assessment

1. a. Proposal A is an example of systematic sampling. This method provides an easy way to gather data. However, since all students who do not attend the assembly are automatically excluded, it creates a biased sample. If all students do attend, then it gives a relatively large sample—158 students. Using a value of twice the maximum standard deviation of all possible sample proportions, the margin of error for that sample size is about 8%.

Proposal B is an example of random sampling. Although it may involve some time and effort to create the computer program, it should generate an unbiased sample. However, there is no way to know how many students will be selected for the sample, and no way to estimate the margin of error.

Proposal C is an example of stratified sampling. Although it gives each class equal representation in the sample, it may not generate a sample that is representative of the school population. For example, if the freshman class is substantially larger than the senior class, the sample is biased against freshmen. It also provides a relatively small sample—only 40 students. The margin of error for that sample size is about 16%.

- b–c. Answers will vary. Since it has the least potential for bias, some students may select proposal B and specify a sample size. For a large sample, the margin of error will be relatively small.
2. a. Answers will vary. In one simulation, the proportion of heads was less than 40% in 7 of 90 samples. This corresponds with an estimated probability of 8%.
- b. Sample response: The sample proportion is  $170/400$  or 42.5%. Using  $1/\sqrt{n}$  to determine a margin of error, it is highly likely that the true proportion is between 37.5% and 47.5%. Therefore, I would guess that either this dime is not a fair coin, or the flipping process is somehow biased.
3. Students should be more confident in the results obtained using a sample size of 500. The value of twice the maximum standard deviation of all possible sample proportions for this sample size is approximately 4.5%. This means that the population proportion is very likely to be within 4.5% of the sample proportion. The corresponding margin of error for a sample size of 200 is about 7.1%.
4. The sample proportion is  $174/588$  or about 30%. Using  $1/\sqrt{n}$  to determine a margin of error, it is highly likely that between 26% and 34% of all teenagers prefer Eagle Wing brand court shoes.

## **Selected References**

- Gallup, G. *The Sophisticated Poll Watcher's Guide*. Ephrata, PA: Princeton Opinion Press, 1972.
- Landwehr, J. M., J. Swift, and A. E. Watkins. *Exploring Surveys and Information from Samples*. Palo Alto, CA: Dale Seymour Publications, 1988.
- Mizrahi, A., and M. Sullivan. *Finite Mathematics with Applications for Business and Social Sciences*. New York: John Wiley & Sons, 1973.
- Moore, D. S. *Statistics: Concepts and Controversies*. New York: W. H. Freeman and Co., 1991.
- Sincich, T. *Statistics by Example*. San Francisco, CA: Dellen Publishing Company, 1982.



# Flashbacks

## Activity 1

**1.1** Find the mean and standard deviation of the following set of numbers:

$$\{345, 642, 521, 745, 548, 367\}$$

**1.2** A box contains 100 kidney beans, 64 black beans, and 96 lima beans.

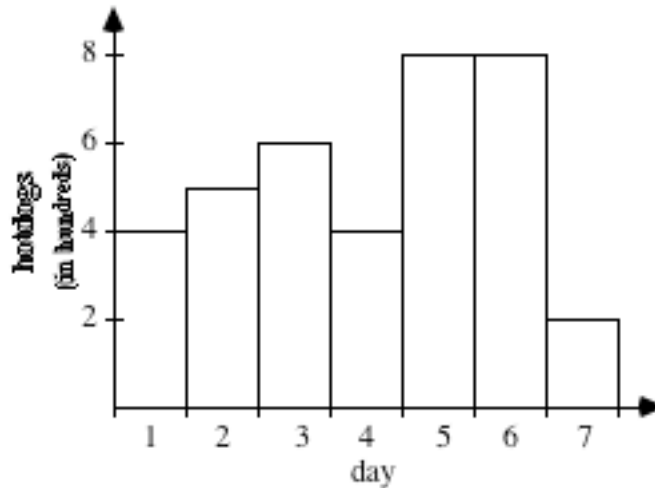
- a. What percentage of the beans in the box are black beans?
- b. What percentage are kidney beans?

## Activity 2

**2.1** Describe a method of randomly generating 20 integers from 1 to 10.

### Activity 3

- 3.1** What number is 40% of 360?
- 3.2** Write each of the following percentages as a decimal.
- a. 34%
  - b. 0.4%
- 3.3** Write each of the following decimals as a percentage.
- a. 0.52
  - b. 0.0334
- 3.4** The following histogram shows the number of hot dogs sold by a vendor each day for seven days.



- a. Find the mean number of hot dogs sold per day.
- b. Calculate the percentage of hot dogs sold on or before day 3.

### Activity 4

- 4.1** Terry caught 18 fish on her first day of fishing, 16 on her second day, 22 on her third day, 10 on her fourth day, and 14 on her last day. Create a bar graph that shows the number of fish she caught each day.
- 4.2** List the integers that are included in the interval  $[3.1, 7.2]$ .

# Answers to Flashbacks

## Activity 1

- 1.1 The mean of the numbers is 528; the standard deviation is approximately 141.
- 1.2 a. approximately 25%  
b. approximately 38%

## Activity 2

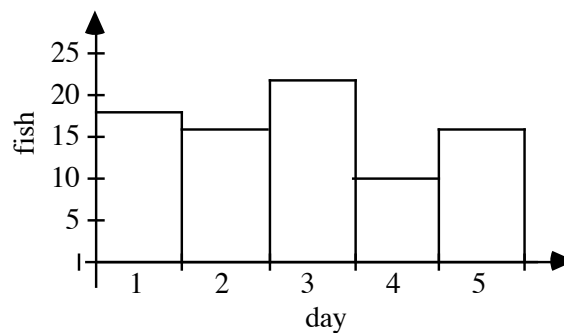
- 2.1 To randomly generate integers from 1 to 10, students may suggest drawing numbered slips of paper from a bowl, rolling a 10-sided die, or using technology. On a TI-92 calculator, for example, the command “rand(10)” generates a random integer between 1 and 10.

## Activity 3

- 3.1 144
- 3.2 a. 0.34  
b. 0.004
- 3.3 a. 52%  
b. 3.34%
- 3.4 a.  $\frac{3700}{7} \approx 529$  hot dogs  
b.  $\frac{15}{37} = \frac{1500}{3700} \approx 41\%$

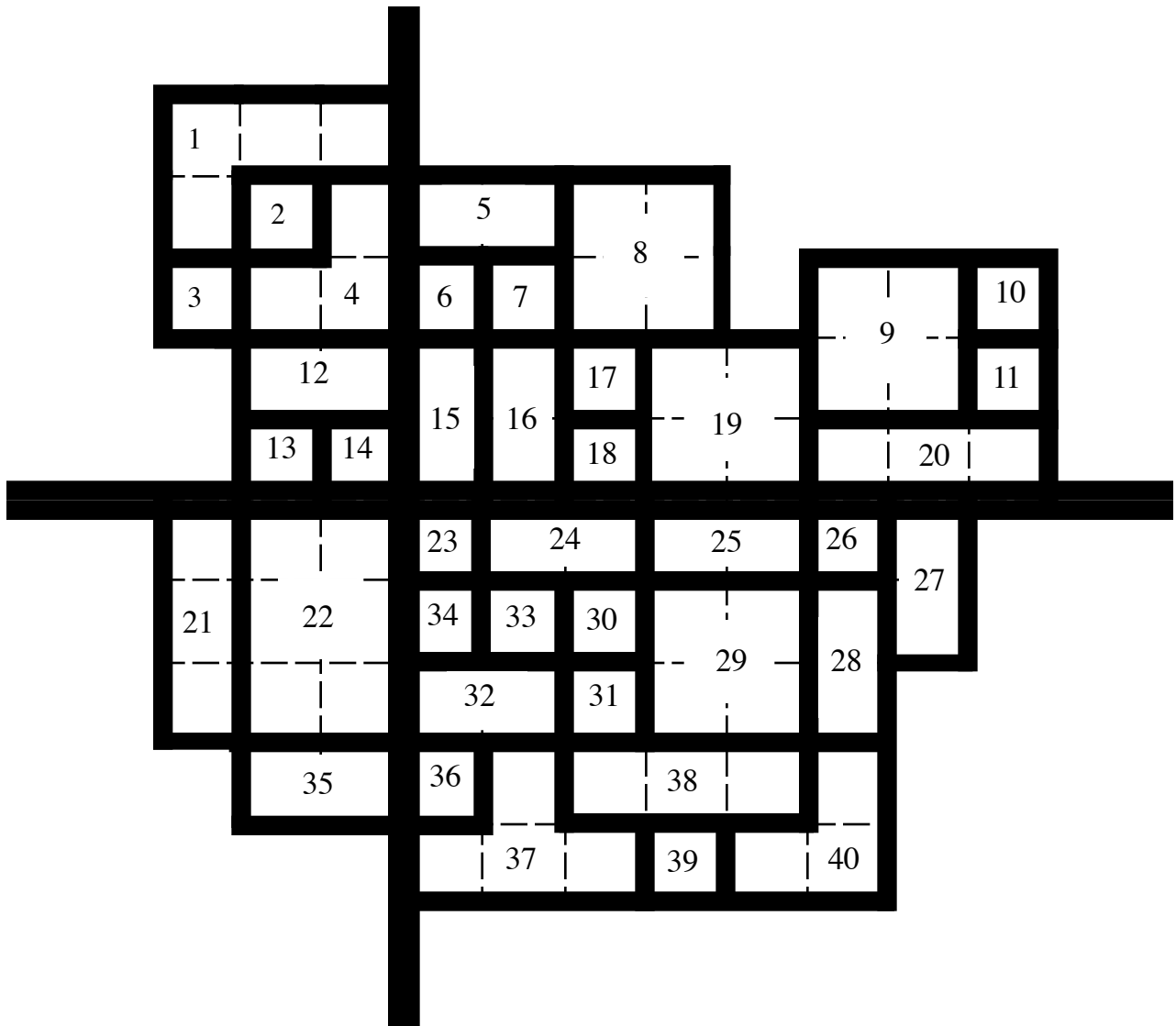
## Activity 4

- 4.1 Sample graph:



- 4.2 4, 5, 6, 7

# Town Map





# Traditional Design



In this module, you examine the geometric properties of some traditional American Indian art forms.

*Todd Fife • Anne Merrifield*



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## Teacher Edition

# Traditional Design

### Overview

This module uses traditional American Indian art to examine basic geometric principles. Students use the lone-star quilts of the Assiniboine and Sioux tribes to explore parallel and perpendicular lines. They then use the medicine wheels of the Northern Cheyenne to explore properties of tangents and secants in relation to a circle. Finally, they use Navajo sandpaintings to explore transformations.

### Objectives

In this module, students will:

- use paper-folding constructions to examine angle bisectors, perpendicular lines, parallel lines, and midpoints
- explore properties of angles formed by parallel lines and a transversal
- explore geometric rep tiles
- identify relationships between tangents and secants
- examine properties of parallelograms (specifically rhombi)
- examine similar triangles created by dilations.

### Prerequisites

For this module, students should know:

- the definition of supplementary angles
- line and rotational symmetry
- properties of a square
- properties of reflections, dilations, rotations, and translations.

### Time Line

Activity	Intro.	1	2	3	Summary Assessment	Total
Days	1	2	2	2	1	8

## Materials Required

Materials	Activity				
	Intro.	1	2	3	Summary Assessment
straightedge	X	X	X	X	X
scissors		X			
protractor	X			X	
template A	X				
template B		X			
template C		X			
template D			X		
template E			X		
template F			X		
template G				X	
template H				X	
template I					X

## Teacher Note

Blackline masters of the templates appear at the end of the teacher edition FOR THIS MODULE.

## Technology

Software	Activity				
	Intro.	1	2	3	Summary Assessment
geometry utility		X	X	X	



# Traditional Design

## Introduction

(page 319)

Students develop paper-folding methods to construct angle bisectors, perpendicular lines, parallel lines, and midpoints.

## Materials List

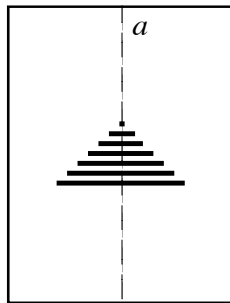
- template A (one copy per student)
- straightedge (one per student)
- protractor (one per student)

## Exploration

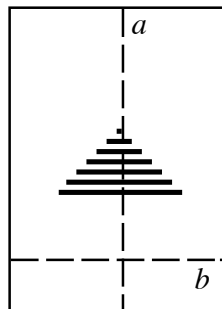
(page 319)

Each student will require a copy of template A to complete the exploration.

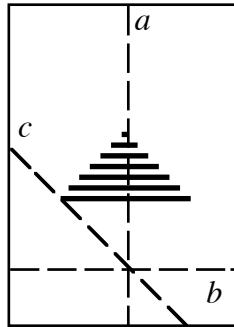
- a. 1. Student papers should resemble the following diagram:



2. Each bar is divided in half by the crease.  
3. The angles appear to be  $90^\circ$ .  
4. Sample response: The line of symmetry is the perpendicular bisector of each bar in the symbol.
- b. 1. Students should fold the paper so that the line of symmetry coincides with itself, creating a crease parallel to the bars. Student papers should resemble the following diagram:



2. Student papers should resemble the following diagram:

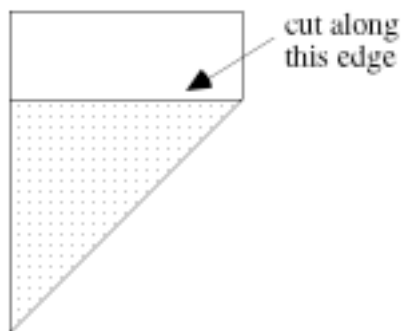


- c.
1. The two perpendicular distances should be equal.
  2. The measure of each angle is  $45^\circ$ .
  3. Sample response: Crease  $c$  is the bisector of an angle formed by creases  $a$  and  $b$ .

### Discussion

(page 321)

- a.
1. Sample response: An object can be reflected by folding along a line of reflection, then tracing to show the reflected image.
  2. The perpendicular bisector of a segment can be found by folding one of the segment's endpoints onto the other.
  3. The angle bisector can be found by folding one side of an angle so that it coincides with the other side of the angle. (See Figure 3 in the student edition.)
  4. The midpoint of a line segment can be found by creating a perpendicular bisector. The intersection of the perpendicular bisector and the segment is the midpoint.
- b. As shown in the following diagram, a square can be formed from a rectangular sheet of paper by creating the angle bisector of one of the  $90^\circ$  corners. The part of the paper that is not doubled can then be cut off, leaving a square.



## *Activity 1*

Students use the design of a star quilt to review the properties of angles formed from parallel lines cut by a transversal. They also examine the properties of a rhombus and explore how to create geometric rep tiles.

### Materials List

- straightedge (one per student)
- pair of scissors (one per student)
- copy of template B (one per student)
- copy of template C (one per student)

### Discussion 1

(page 322)

- a. Other than the eight-pointed star, students should recognize some parallelograms—including rhombi—in the design. They may also mention other shapes, such as the octagon that can be made by connecting the star’s eight outer vertices.
- b.
  1. A parallelogram—specifically, a rhombus—is the basic shape used to construct the star.
  2. A rhombus is a parallelogram with congruent sides. **Note:** You may want students to discuss the properties of parallelograms as well. In the following exploration, students should also discover these properties of a rhombus: opposite angles are congruent, the diagonals are perpendicular bisectors of each other, and the diagonals bisect the angles.

### Exploration

(page 322)

Students use paper folding to form a rhombus and explore some of its properties. Although students also may complete this construction using a geometry utility, this activity is designed to reflect the historical methods used by quiltmakers.

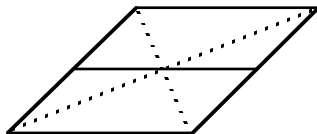
- a. A square can be formed from a rectangular sheet of paper by creating the angle bisector of one of the  $90^\circ$  corners. The part of the paper that is not doubled is then cut off. (See Part **b** of the discussion in the introduction.)
- b. Student papers should resemble the shape shown in Figure **6b** of the student edition.
- c. As shown in Figure **7**, the resulting shape is a rhombus.

- d. 1. Students should not have to use protractors to measure these angles. The measure of each of the two smaller angles of the rhombus is the sum of two angles whose measures are:

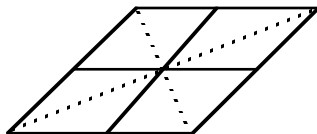
$$\frac{1}{2} \left( \frac{1}{2} \cdot 90^\circ \right) = 22.5^\circ$$

The sum is therefore  $45^\circ$ . The adjacent angles of the rhombus are supplementary to these  $45^\circ$  angles. Thus, each has a measure of  $135^\circ$ . **Note:** Students should recall supplementary angles from middle school. You may wish to remind them of the definition, or to draw and measure a few such angles on a geometry utility.

2. The opposite angles of a rhombus (and, in general, a parallelogram) are congruent.
- e. The diagonals of the rhombus are perpendicular bisectors of each other. As seen from the construction of this rhombus, the diagonals also bisect the interior angles.
- f. 1. The midpoint of a line segment can be found by folding one of the segment's endpoints onto the other to create the segment's perpendicular bisector. The intersection of the perpendicular bisector and the segment is the midpoint.
2. Two congruent parallelograms are formed. Student papers should resemble the diagram below.



3. Student papers should resemble the following diagram.



- g. Four smaller congruent rhombi are formed, each similar to the original rhombus.
- h. The measures of the interior angles of the smaller rhombi are the same as those of the original rhombus:  $45^\circ$  and  $135^\circ$ .

## Discussion 2

(page 324)

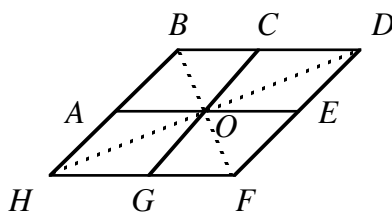
- a. Sample response: Yes, the shapes are similar to the original rhombus. The corresponding sides are proportional and the corresponding angles are congruent. **Note:** You may want to ask students what information is necessary to determine if two rhombi are similar. If one angle of a rhombus is congruent to one angle of another rhombus, then the two rhombi are similar.

- b. The two parallelograms are congruent. Since they are formed by joining the midpoints of opposite sides of a rhombus, the corresponding sides are congruent and the corresponding angles are congruent.
- c. The diagonals of the rhombus are perpendicular bisectors of each other. As seen from the construction of the rhombus, the diagonals also bisect the interior angles.
- d.
  1. The opposite angles of a rhombus (and, in general, a parallelogram) are congruent. Adjacent angles are supplementary.
  2. Sample response: One way to show congruence is to tear off the angles from the paper models and match them. Another way is to measure.

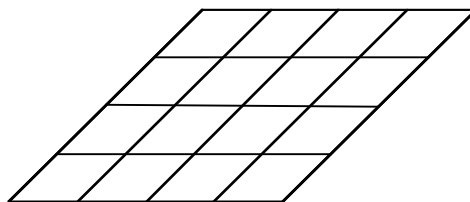
One way to show supplementary angles is to tear off the angles and place them together at the vertices. The non-adjacent sides will form a straight line.

**Note:** All of these methods are demonstrations. To prove that these relationships are true, students must work with congruent figures.

- e.
  1. The measures of angles 1 and 3 are equal.
  2. The measures of angles 2 and 4 are equal. Angles 2 and 4 also have the same positional relationship as angles 1 and 3 because they are two non-adjacent angles formed by two intersecting lines.
  3. Vertical angles are two non-straight, non-adjacent angles formed by two intersecting lines.
  4. Students should identify many pairs of vertical angles. In the diagram below, these pairs include  $\angle AOB$  and  $\angle EOF$ ,  $\angle BOD$  and  $\angle FOH$ ,  $\angle COD$  and  $\angle GOH$ ,  $\angle AOD$  and  $\angle EOH$ .



- f. The result will resemble the diagram below. Each smaller rhombus is divided into four congruent rhombi. Each of these rhombi is similar to the original rhombus as well as the four intermediate rhombi from which they were formed.



- g. Sample response: Yes, because the original rhombus is divided into congruent rhombi, each similar to the original. **Note:** The self-similar nature of rep tiles is a feature common to many fractals. You may want to use this opportunity to introduce fractals through Sierpinski's triangle. (Students will examine Sierpinski's triangle in a research project in the Level 2 module, "Take It to the Limit.")

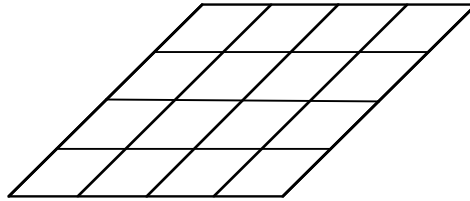
### Teacher Note

To complete Problem 1.3, each student will need a copy of template B. To complete Problem 1.4, students will need a copy of template C. Blackline masters appear at the end of the teacher edition for this module.

### Assignment

(page 326)

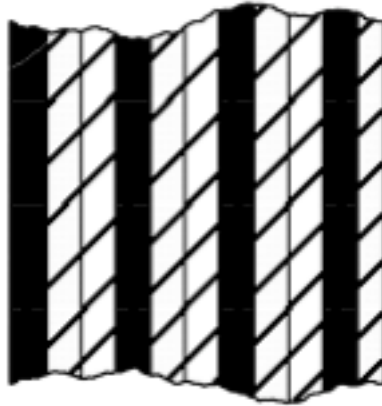
- 1.1 a. Some students may want to simulate the paper folding on a geometry utility. Sample response:



- b. 1. Pairs of corresponding angles formed along a transversal of parallel lines are congruent.
2. There are four pairs of corresponding angles:  $\angle 1$  and  $\angle 5$ ,  $\angle 2$  and  $\angle 6$ ,  $\angle 4$  and  $\angle 8$ , and  $\angle 3$  and  $\angle 7$ .
3. Sample response: Corresponding angles are in the same relative position relative to parallel lines cut by a transversal. **Note:** Students should observe that a pair of corresponding angles can be above the parallel and on the same side of a transversal, or below the parallel and on the same side of the transversal. You may want to discuss the possibility of calling angles "corresponding" even if the lines are not parallel. In this case, however, the angles are not congruent.
- c. There are four pairs of vertical angles:  $\angle 1$  and  $\angle 3$ ,  $\angle 4$  and  $\angle 2$ ,  $\angle 5$  and  $\angle 7$ , and  $\angle 6$  and  $\angle 8$ . The two angles in each pair are congruent.
- d. 1. The measures of the two angles are equal.
2. The measures of the two angles are equal.
3. There are two pairs of alternate interior angles:  $\angle 4$  and  $\angle 6$ , and  $\angle 3$  and  $\angle 5$ . There are two pairs of alternate exterior angles:  $\angle 1$  and  $\angle 7$ , and  $\angle 2$  and  $\angle 8$ .

4. Sample response: Alternate interior angles are between the parallel lines and on opposite sides of a transversal. Alternate exterior angles are outside the parallel lines and on opposite sides of a transversal. With parallel lines, the respective pairs of angles are congruent. **Note:** You may want to discuss the fact that alternate interior angles formed when two nonparallel lines are cut by a transversal are not congruent. The alternate exterior angles in this situation are also not congruent.
- 1.2
- a. Angle 1 and angle 3 are vertical angles. Angle 1 and angle 2 are supplementary angles. Angles 1 and 4 are also supplementary angles.
  - b. Sample response: Angles 1 and 5 are congruent because they are corresponding angles of two parallel lines cut by a transversal. Likewise, angles 1 and 13 are corresponding angles.  
 Angle 1 has the same measure as angle 9 because angle 9 and angle 5 are corresponding angles, or because angle 1 and angle 3 are vertical angles, while angle 3 and angle 9 are opposite angles of the rhombus.
  - c. Sample response: You only have to measure one angle to determine the measures of all of the angles at one vertex. Given the angles at one vertex, you can then determine the measures of all 16 angles in the diagram. Therefore, the measures of all the angles can be found if you measure just one angle.
  - d. Sample response: It would be necessary to measure four angles, one angle at each vertex.
- \*1.3
- a. Students should observe that the two pieces are congruent. Some may want to cut the star along the line to compare pieces. **Note:** Students are not limited to existing lines in the design. They may draw any line that passes through the center of the star.
  - b. The two pieces have  $180^\circ$  rotational symmetry about the center of the star. They may also have symmetry about the line.
  - c. The two pieces will have  $180^\circ$  rotational symmetry about the center of the star for any line that passes through the center. If the line does not pass through the center and one of the outer vertices of the star, the pieces will not have symmetry with respect to the line.
  - d. Sample response: A line that passes through the center of the lone star design divides the star into two pieces that are the same size and the same shape. The two pieces are identical in design and are related by  $180^\circ$  rotational symmetry about the center of the star. Only those lines that pass through the center and one of the outer vertices of the star are lines of symmetry.

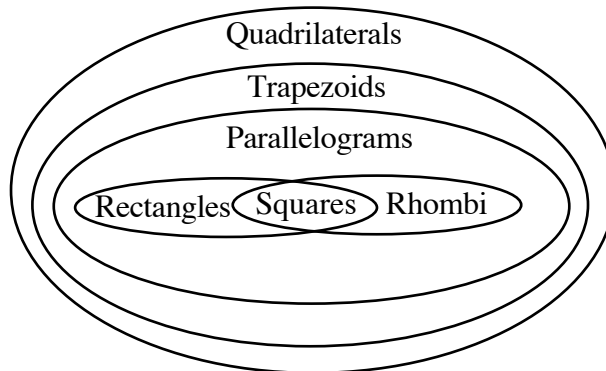
- \*1.4**
- Students may use lined notebook paper to help them draw sets of parallel lines. If the lines are perpendicular to the quilt strips, then the figures created are rectangles.
  - If the parallel lines are not perpendicular to the quilt strips, then the figures created are parallelograms.
  - $360^\circ/8 = 45^\circ$
  - Sample response: The quilter should draw a set of parallel lines that form  $45^\circ$  angles with the strips. To form rombi, these lines must be spaced the same distance apart as the vertical strips.



Cutting along these parallel lines would create strips of rhombi. These new strips of material could then be shifted up or down one color and sewn to create the desired pattern. The eight large rhombi could then be cut from this new pattern. **Note:** Students should simulate this process using the paper template, scissors, and tape.

- Sample response: The stitching may have been neither straight nor parallel. If the lines of stitches are not parallel, there is no guarantee that the corresponding angles will be congruent. As a result, the quilt may start to buckle or curl. To correct this, the seams would have to be taken out and re-sewn.

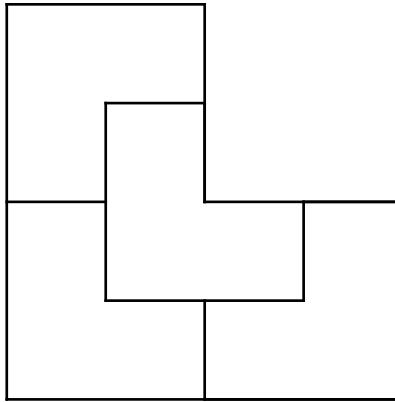
**1.5** Student responses should resemble the following diagram:



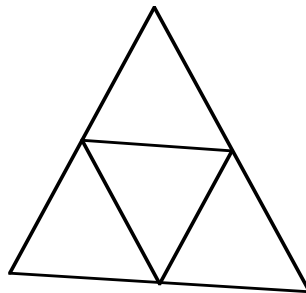


- 1.6**
- a. The diagonals do not bisect each other and are not perpendicular. They are not equal in length unless the trapezoid is isosceles.
  - b. The diagonals are not perpendicular but they do bisect each other. They are not equal in length.
  - c. The diagonals are equal in length. They are not perpendicular but they do bisect each other.
  - d. The diagonals are the perpendicular bisectors of each other. They also bisect the interior angles. The diagonals are not equal in length.
  - e. The diagonals are equal in length. They are the perpendicular bisectors of each other and also bisect the interior angles.

- 1.7**
- a. Sample drawing:



- b. Sample response: I started with a triangle, found the midpoints of the sides, and connected them. This formed segments parallel to the corresponding sides of the original triangle and, in turn, four smaller triangles similar to the original. They are similar using the measures of corresponding and alternate interior angles as well as the sum of the measures of the angles of a triangle.



I then rotated the original triangle  $180^\circ$  about the midpoint of a side so that the original and the image formed a parallelogram. To make the quilt pattern, I used groups of parallelograms in the same way that the rhombus is used in the lone star quilt in Figure 5.

\* \* \* \* \*

- 1.8** Student responses may include some of the following methods.
1. Show that both pairs of opposite sides are the same length.
  2. Show that both pairs of opposite sides are parallel.
  3. Show that the diagonals bisect each other.
  - 4 Show that both pairs of opposite angles are congruent.
  5. Show that one pair of opposite sides is parallel and congruent.
- 1.9** Sample response: Measure the sides to determine if opposite sides have the same length. If they do, the quadrilateral is a parallelogram. Measure the diagonals to see if they are the same length. If they do, the parallelogram is a rectangle. **Note:** Students may also use the Pythagorean theorem to verify that the floor is a rectangle.
- 1.10**
- a. parallelogram, rhombus, rectangle, square
  - b. rhombus, square
  - c. rhombus, square

\* \* \* \* \*

(page 330)

## ***Activity 2***

Students use medicine wheel designs to explore properties of tangents and secants. They also use the properties of a circle to create their own medicine wheels.

### **Materials List**

- copy of template D (one per student)
- copy of template E (one per student)
- copy of template F (one per student)
- straightedge

### **Technology**

- geometry utility

## Discussion 1

(page 331)

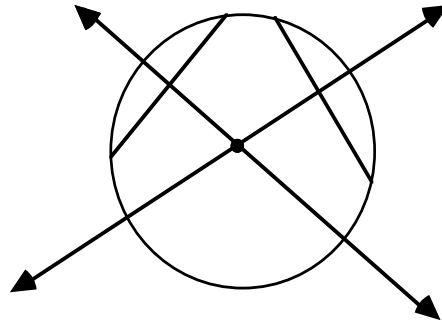
- a. The medicine wheel displays rotational and line symmetries.
- b. Sample response: Half of the symbol in the center could be constructed and reflected to create the other half. The larger circle is a dilation of the smaller one with center at the center of the circle. The quadrilaterals between the two concentric circles are all rotational images of another about the center. Each feather along the bottom of the wheel might be considered a translated image of the feather on the far right.

## Exploration

(page 332)

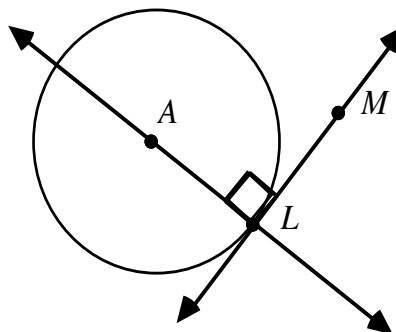
Students use a geometry utility to investigate properties of chords, secants, and tangents. (This exploration also can be completed with a compass and straightedge.)

- a. For best results, the diameter of the circle should be at least two-thirds the width of the screen.
- b.
  1. See sample drawing given in Step 3 below.
  2. Predictions may vary.
  3. Sample drawing:



4–5. Students should observe that the perpendicular bisectors of chords intersect at the center of the circle.

- c. Students should observe that when the angle has a measure of  $90^\circ$ , the line intersects the circle at a single point and is therefore a tangent line. The final sketch should resemble the following:



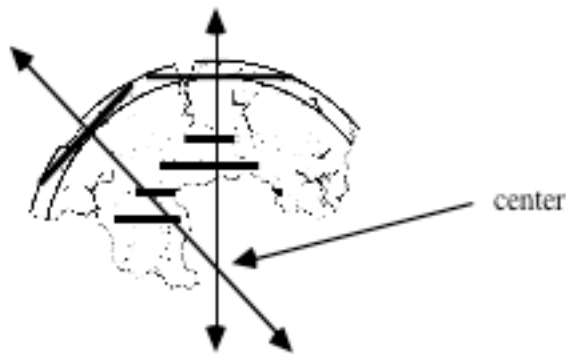
## Teacher Note

Before discussing Part **d** below, you may wish to distribute copies of template D (one per student).

## Discussion 2

(page 333)

- a. 1.  $90^\circ$   
2. The angle cannot measure  $90^\circ$ .
- b. Sample response: A line tangent to a circle can be created by first folding a circle onto itself through a line containing the center of the circle. This fold represents a secant containing a diameter. Next, without unfolding the first fold, fold the crease onto itself where the first fold intersects the circle. This second fold creates a tangent line perpendicular to a diameter, through the point of tangency.
- c. Sample response: The tangents are parallel. Two lines in the same plane perpendicular to the same line must be parallel to each other because alternate interior angles are congruent.
- d. Sample response: The center of the circular plate can be found by drawing two chords and their perpendicular bisectors as shown in the sketch below. The intersection of the perpendicular bisectors is the center of the circle.



- e. Sample response: Fold to create two chords on the circle. Fold to create the perpendicular bisector of each chord. The intersection of the perpendicular bisectors is the center of the circle.
- f. The point of intersection of the perpendicular bisectors is the point equidistant from the four endpoints of the two chords. The center of a circle is the point that is equidistant from all points on the circle.
- g. The altitude of an isosceles triangle is the perpendicular bisector of the base.

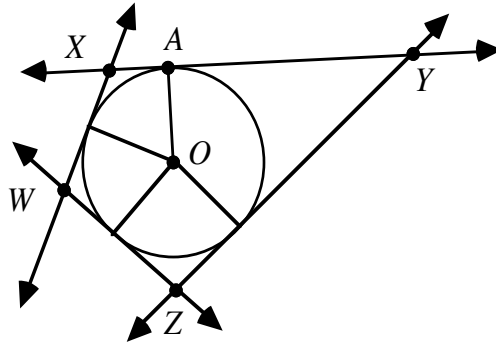
### Teacher Note:

Each student will need a copy of template E to complete Problem 2.3 and a copy of template F to complete Problem 2.4.

### Assignment

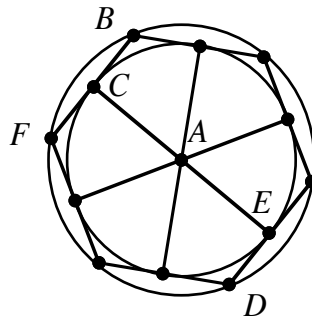
(page 334)

- 2.1 a–b. Student methods will vary, depending on the technology used. Sample response: To draw each tangent, the radius containing both the point of the tangency and the center of the circle is constructed first. The tangent is then constructed as the line perpendicular to the radius and containing the point of tangency.

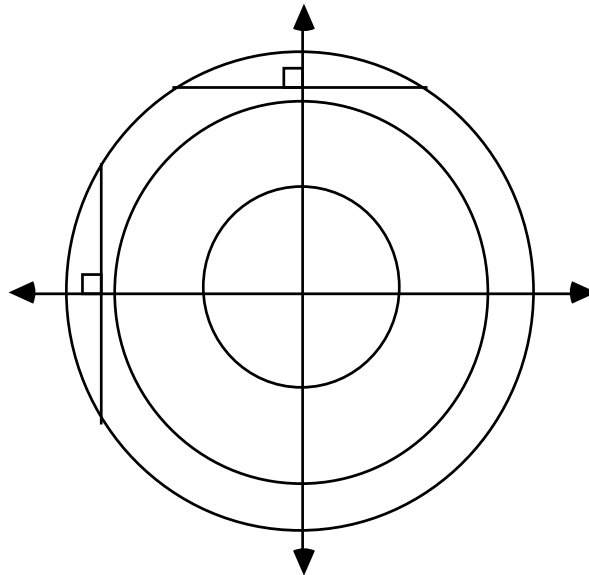


- \*2.2 a. The following sample response refers to the labeled points in the diagram below: First, I constructed the circle with the larger radius. Then I constructed a point  $B$  on the circle. Point  $F$  was constructed by transforming  $B$  using a rotation of  $60^\circ$  around center  $A$ . Four more points were constructed using the same type of transformation for a total of six points. These form the vertices of a hexagon.

Next, I constructed the six sides of the hexagon (such as  $\overline{BF}$ ) and the midpoints of each side (such as  $C$ ). Then I drew the three segments connecting opposite pairs of midpoints (such as  $\overline{CE}$ ). Finally, I constructed the circle with center  $A$  and radius  $\overline{CA}$ .

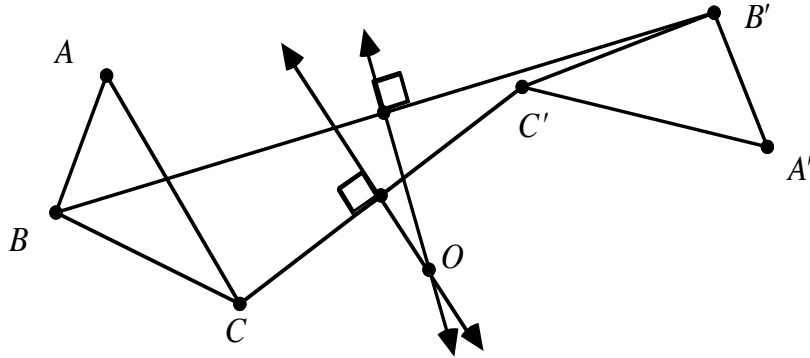


- b. 1.** The drawing has symmetry with respect to six lines: three that pass through the center of the circles and a pair of opposite vertices of the hexagon (such as *B* and *D* in the sample diagram above); and three that pass through the center of the circles and a pair of midpoints of the sides of the hexagon (such as *C* and *E* in the diagram above).
- 2.** The drawing has rotational symmetry around the center of the circles for any rotation of  $n \cdot 60^\circ$ , where  $n$  is an integer.
- \*2.3 a–b.** Students should find the center of the circle by constructing the perpendicular bisectors of at least two chords. Sample drawing:



- c.** Selections will vary. Your school may already have a mascot or symbol.
- d–e.** Encourage students to be creative. They may want to display their wheels as a class.

- 2.4 a. Sample response: The center of the circle must be on the perpendicular bisector of  $\overline{CC'}$  because all points contained on it are equidistant from  $C$  and  $C'$ .
- b. Sample response: Draw the perpendicular bisector of  $\overline{CC'}$  and the perpendicular bisector of  $\overline{BB'}$ . Their point of intersection is the center of rotation  $O$ .



\*\*\*\*\*

- 2.5 a. Since both chords include the center of the circle, they are diameters of the circle.
- b. 1. Sample response: Since  $\overleftrightarrow{FI}$  and  $\overleftrightarrow{GH}$  are tangent to the circle at  $B$  and  $E$ , respectively, the measures of  $\angle JBA$  and  $\angle LEA$  are both  $90^\circ$ . Since these two angles are congruent, they are alternate interior angles of the two parallel lines  $\overleftrightarrow{FI}$  and  $\overleftrightarrow{GH}$ .
2. Using a similar argument to the one given above, students can show that  $\overleftrightarrow{FG}$  and  $\overleftrightarrow{IH}$  are parallel.
- c. Although angle measures may vary among constructions, students should observe that  $\angle KGP \cong \angle CGE$  and  $\angle KGC \cong \angle CFB$ .  
Sample response:  $\angle KGP$  and  $\angle CGE$  are vertical angles;  $\angle KGC$  and  $\angle CFB$  are alternate interior angles; and  $\angle KGP$  and  $\angle KGC$  are supplements.
- d. Sample response:

Vertical Angles	Alternate Interior and Exterior Angles	Corresponding Angles
$\angle BID \cong \angle NIM$	$\angle BID \cong \angle IHL$	$\angle NIM \cong \angle DHL$
$\angle MID \cong \angle BIN$	$\angle MID \cong \angle DHE$	$\angle NIB \cong \angle DHE$
$\angle EHQ \cong \angle DHL$	$\angle NIB \cong \angle LHQ$	$\angle MID \cong \angle LHQ$
$\angle EHD \cong \angle QHL$	$\angle NIM \cong \angle EHQ$	$\angle BID \cong \angle EHQ$

- e. Sample response: All four segments are congruent. Allowing for rounding errors, measurement supports this conjecture.

- f. Sample response: Yes,  $GFIH$  is a rhombus because the lengths of  $\overline{FI}$ ,  $\overline{IH}$ ,  $\overline{HG}$ , and  $\overline{GF}$  are the same,  $\overline{FI}$  is parallel to  $\overline{HG}$ , and  $\overline{IH}$  is parallel to  $\overline{GF}$ .

2.6

- a. They are congruent isosceles triangles.
- b. 1. The two smaller angles are congruent. For example, consider the two angles formed by  $\overline{CB}$ . Since  $\overline{CB}$  is a radius, it is perpendicular to the tangent at  $B$ . The incoming angle and outgoing angle are congruent. The sum of the incoming angle and  $\angle ABC$  is  $90^\circ$ , as is the sum of the measures of the outgoing angle and  $\angle DBC$ . Therefore,  $\angle ABC \cong \angle DBC$ . A similar argument can be made at each of the other vertices.
2. angle bisector
- c. Sample response: The outgoing angle at point  $B$  and  $\angle BDA$  are alternate interior angles, where  $\overleftrightarrow{BD}$  is a transversal. Since both angles measure  $60^\circ$ , the incoming ray and the tangent at point  $D$  must be parallel.

\* \* \* \* \*

(page 339)

### Activity 3

Students use Navajo sandpaintings—along with similar triangles—to explore dilations.

#### Materials List

- template G (one copy per student)
- template H (one copy per student)

#### Technology

- geometry utility

#### Discussion 1

(page 339)

- a. Sample response: Ignoring the feathers on the head, the design appears to have a vertical axis of symmetry. Some individual parts of the sandpainting also display line symmetry.
- b. This design was constructed using two basic figures: rectangles and trapezoids.



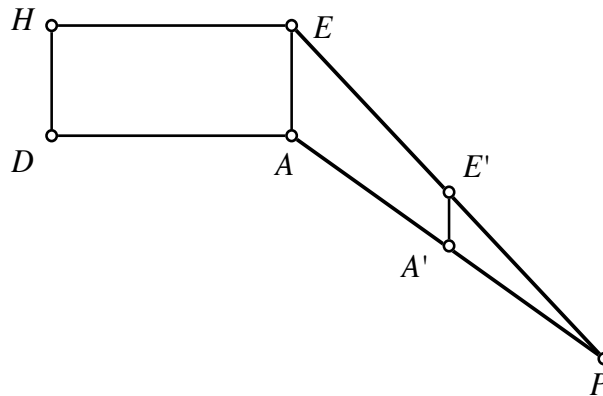
- c. Many of the geometric shapes which make up the sandpainting are similar to each other. For example, the trapezoids that make up the headdress are similar to each other and to the trapezoids that make up the hands, the arms, and the feet. These trapezoids are also similar to ones that form the garment worn around the waist of the figure, the design on the garment, and the tassels on the edge of the garment. The parallelograms that form the feathers in the headdress are similar to those that are found in the legs.

### Exploration

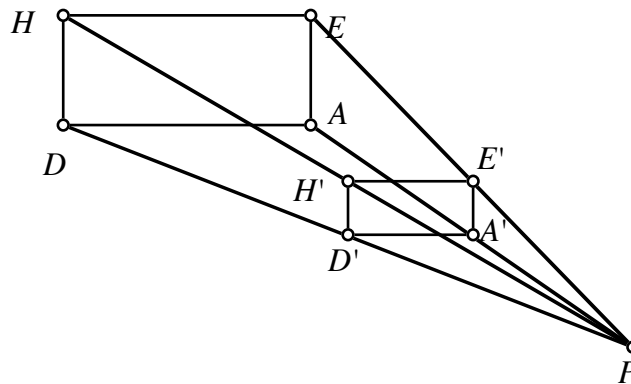
(page 340)

In this exploration, students use similar triangles to explore the properties of dilations.

- a. Sample drawing:



- b. Because corresponding sides are proportional and corresponding angles are congruent,  $\triangle PEA \sim \triangle PE'A'$ .
- c. Students complete a sketch of the image of  $HEAD$  under a dilation with center at  $P$ .



- d. Dilations of polygons yield similar polygons. Angle measures and parallelism are preserved.
- e. Sample response: Only the location of the image changes; its size stays the same.

- f. Sample response: The lengths of the sides of the preimage,  $HEAD$ , would be reduced by a scale factor of  $1/4$  in the image, instead of  $1/2$ .

## Discussion 2

(page 341)

- a. 1. The scale factor is 0.5.  
2. The scale factor would be 0.25.
- b. Changing the location of the center of dilation changes only the location of the image; it does not change the dimensions of the image.
- c. Sample response: The image is larger than the original if the scale factor is greater than 1. For instance, in Part **b** of the exploration,  $E'$  would be on  $\overrightarrow{PE}$  but  $E$  would be between  $P$  and  $E'$ .
- d. Reflections, rotations, and translations are isometries because congruence is preserved. Although the measures of angles and the ratios between adjacent sides are preserved under dilations, they are not necessarily isometries. The preimage and image are congruent under a dilation only if the scale factor is 1.

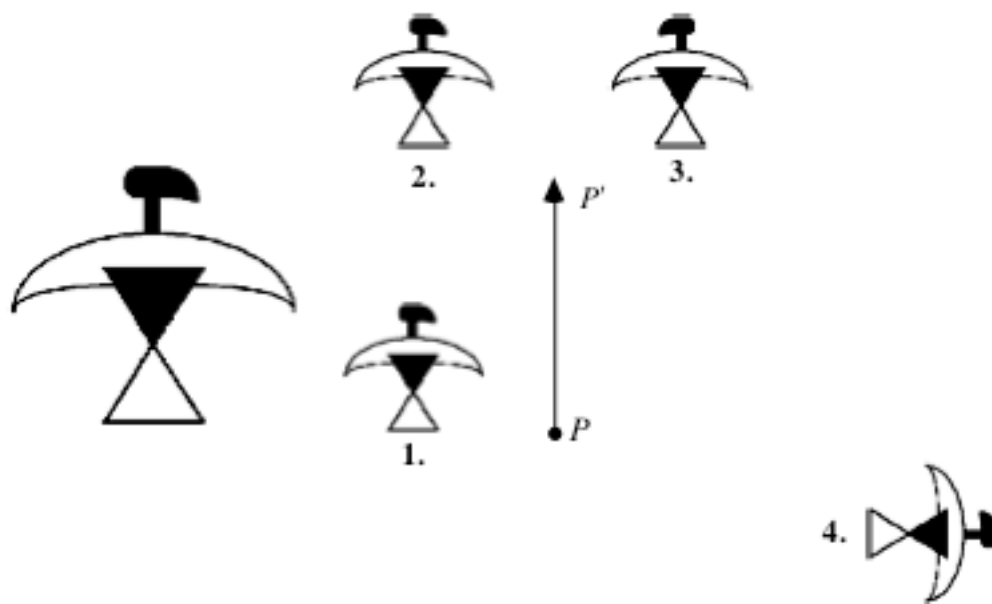
## Teacher Note

To complete Problems 3.1 and 3.2, students will need copies of templates G and H. Blackline masters appear at the end of the teacher edition for this module.

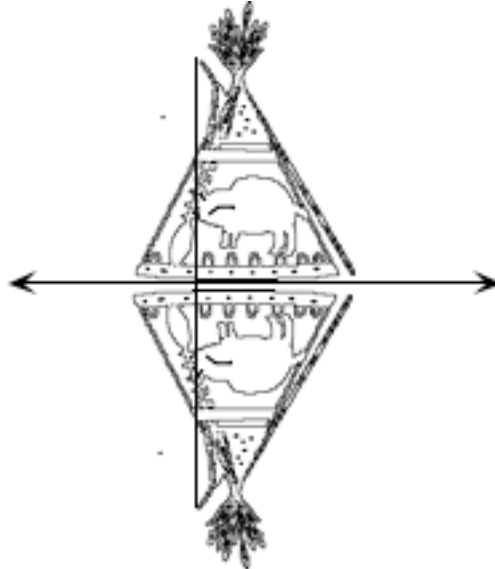
## Assignment

(page 341)

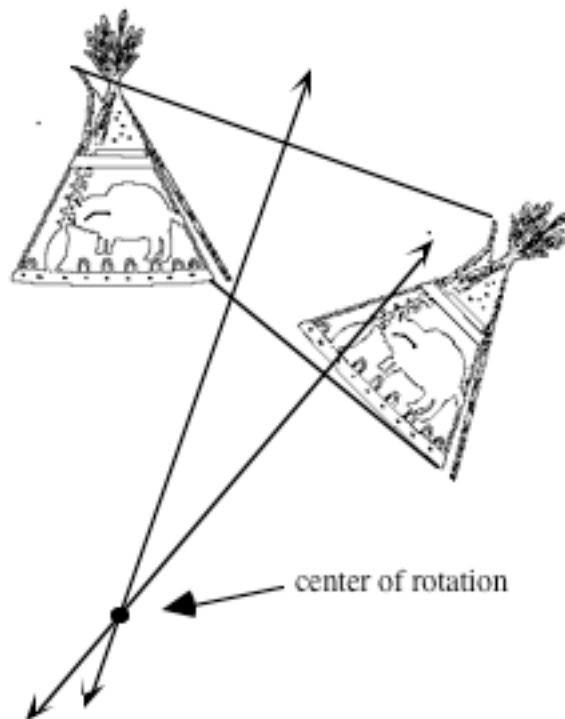
- \*3.1 Sample response:



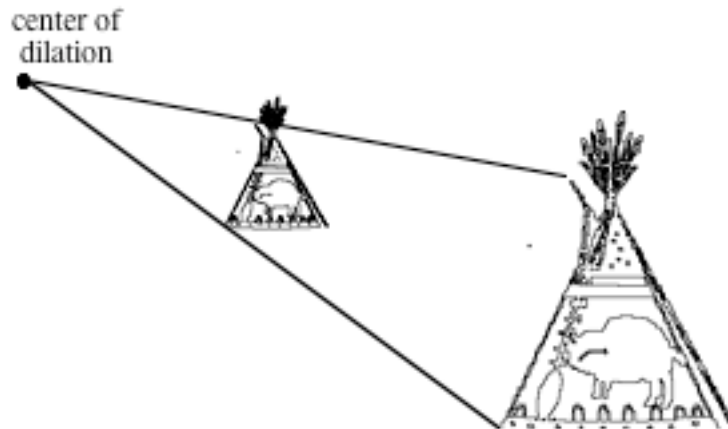
- \*3.2** a. The line of reflection can be found by constructing the perpendicular bisector of a segment whose endpoints are corresponding points on the image and preimage (or by paper folding).



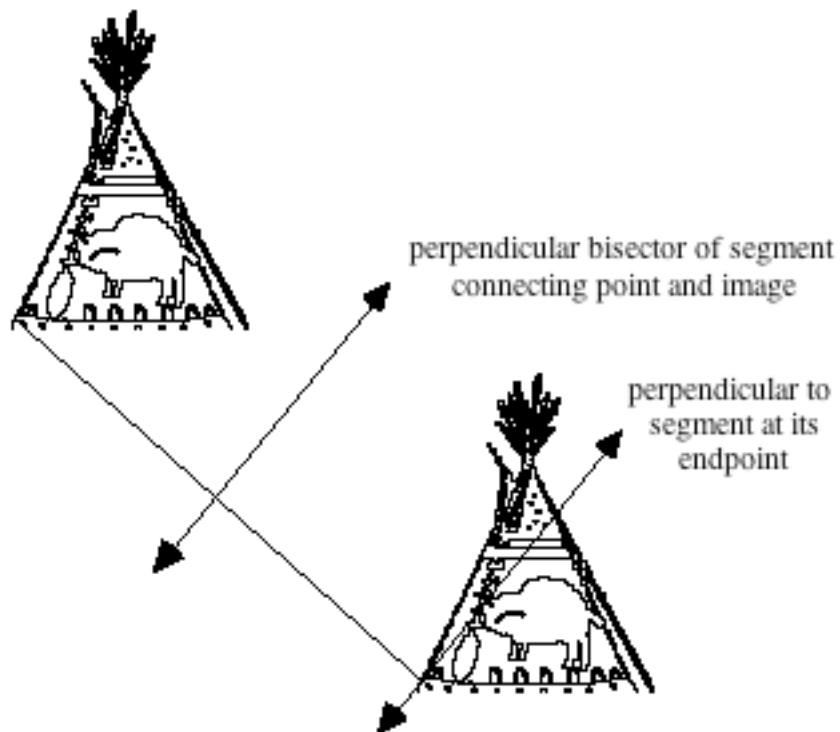
- b. The center of rotation can be found by determining the perpendicular bisectors of two segments whose endpoints are corresponding points on the image and preimage. The intersection of the perpendicular bisectors is the center of rotation.



- c. The vector may be indicated by connecting any point and its image, in the direction from the preimage to the image.
- d. The center of dilation can be found at the intersection of two lines connecting corresponding points on the image and preimage. The scale factor is approximately 2.



- e. Two possible reflection lines are perpendicular to a line connecting any point and its image. The distance between the reflection lines is half the distance between the point and its image.



- 3.3 a. To help students consider possible designs, you may wish to provide some reference materials.
- b. You may prefer to ask students to give oral presentations.

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## Research Project

(page 343)

Students have many choices for both culture and historical period. Besides the examples listed in the student edition, several movements in modern and contemporary art make dramatic use of geometric shapes and lines.

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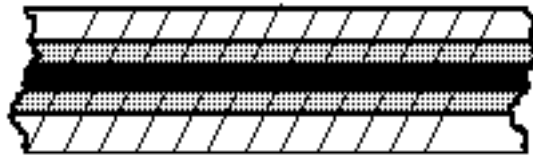
- 3.4 The following table provides sample descriptions for each figure.

Figure	Description of Transformation
2.	Not an isometry and not similar. Not an obvious type of transformation.
3.	Not an isometry, but similar. Dilation using $C$ as the center. Scale factor appears to be 0.5.
4.	Not an isometry, but similar. Dilation using $A$ as the center. Scale factor appears to be 3.
5.	An isometry and therefore also similar. A translation by a vector equivalent to the vector from $F$ to $G$ , with a length of approximately 4.3 cm, in a direction which can be described as about $300^\circ$ , or a bearing of about $150^\circ$ , or approximately southeast.
6.	Not an isometry, but similar. Two transformations: first a dilation identical to figure 3, then a reflection in the line containing $\overline{DE}$ .
7.	An isometry and therefore also similar. A rotation of $90^\circ$ about point $B$ .
8.	Not an isometry and not similar. Not an obvious type of transformation.

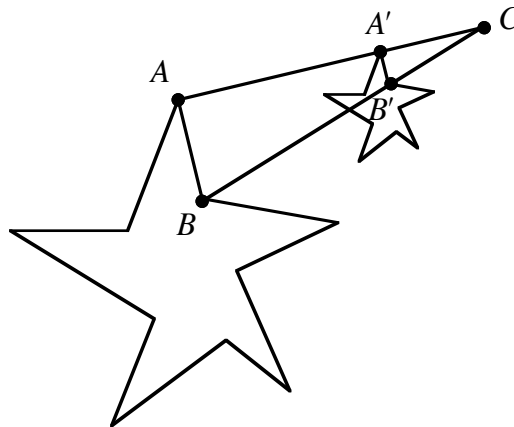
## Answers to Summary Assessment

(page 345)

1. a. Sample response: Parallel lines could be cut as shown in the sketch below. The resulting strips can be shifted one color and re-sewn.

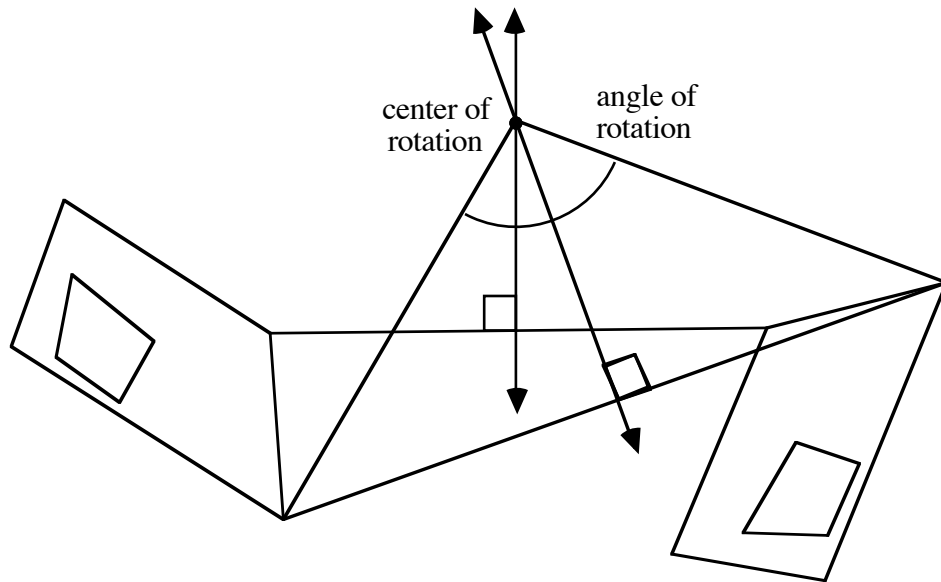


- b. Answers will vary. Cutting the fabric along parallel lines creates congruent corresponding angles. Re-aligning the new strips along the parallels creates the pattern of parallelograms.
2. a–b. Sample sketch:

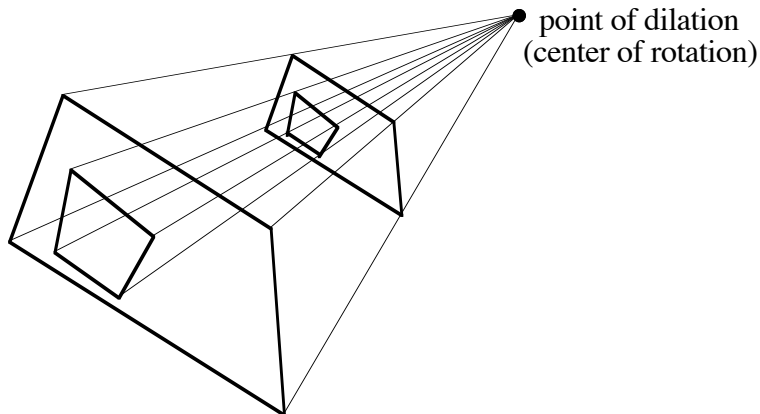


- c. The following sample response refers to the labeled points in the diagram above: Since the transformation is a dilation,  $\triangle ABC \sim \triangle A'B'C$ . Therefore,  $\angle BAC \cong \angle B'A'C$ . Since these are corresponding angles,  $\overline{AB}$  is parallel to  $\overline{A'B'}$ . A similar argument could be used for every pair of corresponding sides.

3. a. The center of rotation can be found by constructing the perpendicular bisectors of segments connecting corresponding points on the image and preimage, as shown below.



- b. Sample sketch:

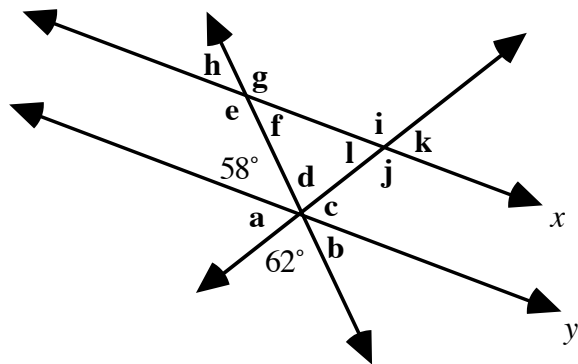


### Teacher Note

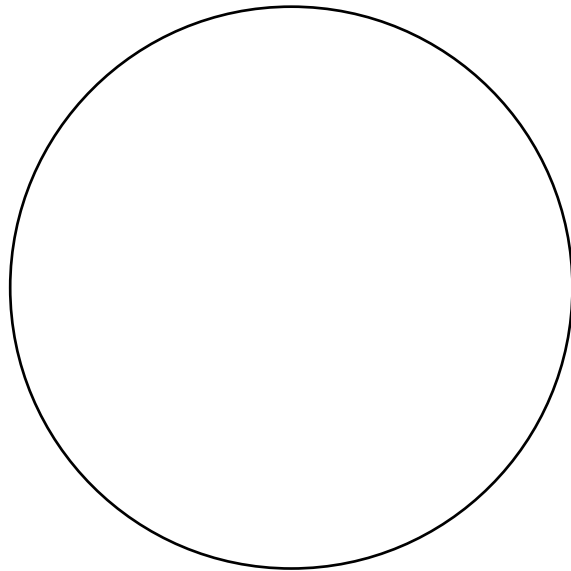
To complete Problem 3 in the module assessment, each student will need one copy of template I.

**Module  
Assessment**

1. In the diagram below, line  $x$  is parallel to line  $y$ . Find the measure of each labeled angle.

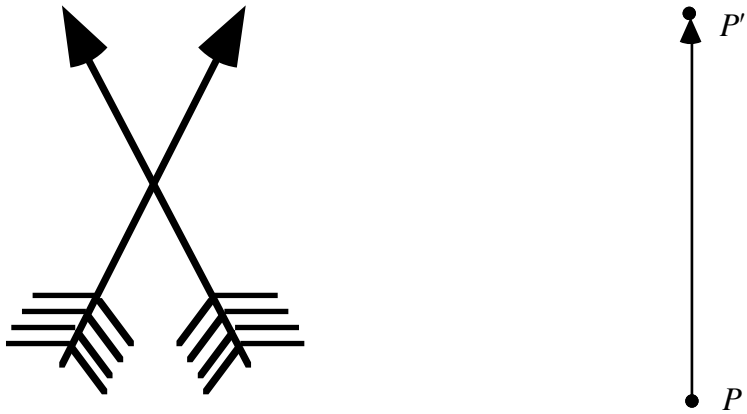


2. Find the center of the circle below. In a paragraph, describe the procedure that you used.





3. The crossed arrows in the diagram below symbolize friendship in some traditional American Indian artworks.

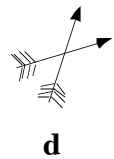
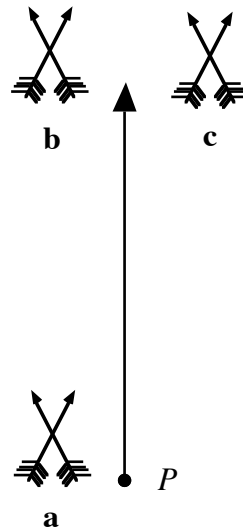
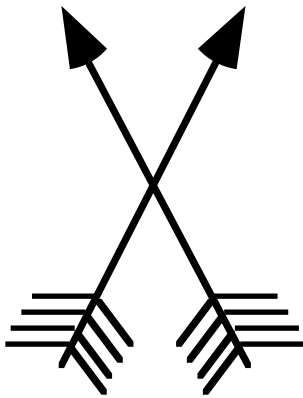


Use a copy of this diagram to complete the following transformations.

- a dilation with center at point  $P$  and a scale factor of  $1/4$
- a translation of the image found in Part **a** using the vector  $PP'$
- a reflection of the image found in Part **b** in a line containing  $P$  and  $P'$ .
- a  $45^\circ$  clockwise rotation of the image found in Part **c** about point  $P$

## Answers to Module Assessment

1.
  - a.  $60^\circ$
  - b.  $58^\circ$
  - c.  $60^\circ$
  - d.  $62^\circ$
  - e.  $122^\circ$
  - f.  $58^\circ$
  - g.  $122^\circ$
  - h.  $58^\circ$
  - i.  $120^\circ$
  - j.  $120^\circ$
  - k.  $60^\circ$
  - l.  $60^\circ$
2. The center of the circle can be found by constructing the perpendicular bisectors of any two chords. The intersection of these chords is the center of the circle.
3. Sample sketch:



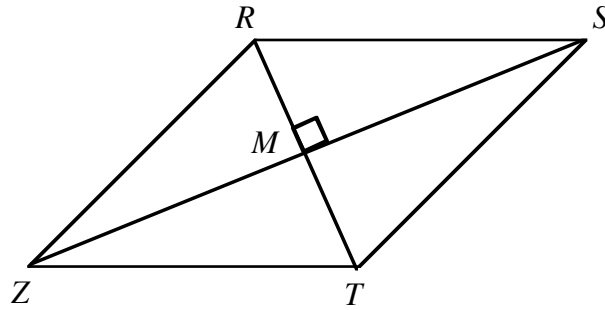
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# Flashbacks

## Activity 1

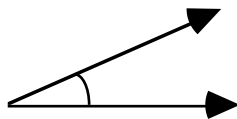
1.1 Use the parallelogram in the diagram below to complete Parts a–e.



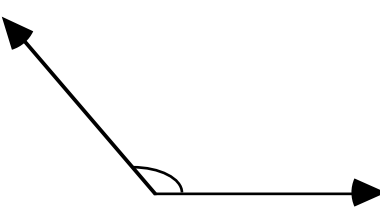
- Name the interior angles of the parallelogram that are adjacent to  $\angle SRZ$ .
- Name the interior angle of the parallelogram that is opposite  $\angle SRZ$ .
- If  $\angle RZM \cong \angle TZM$ , then  $\overline{ZS}$  is called an \_\_\_\_\_.
- If  $\overline{RM} \cong \overline{TM}$ , then  $\overline{ZS}$  is called a \_\_\_\_\_.
- Identify a line of symmetry for the parallelogram.

1.2 Use a protractor to measure the following angles.

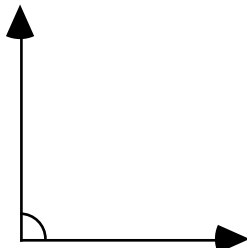
a.



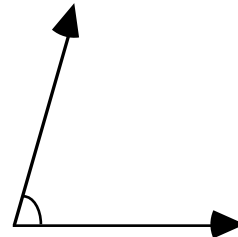
b.



c.

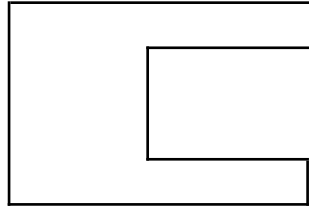


d.



## Activity 2

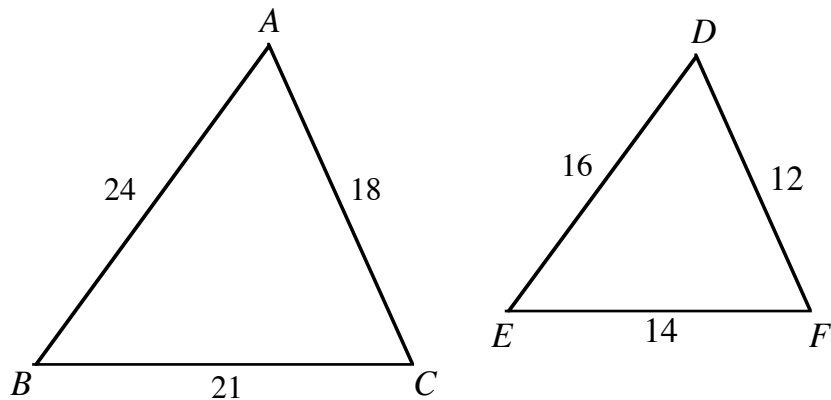
- 2.1 Using the figure below as the preimage, sketch its image for each transformation listed in Parts **a–d**.



- a reflection
  - a rotation
  - a dilation
  - a translation
- 2.2
- Draw a triangle  $ABC$  inscribed in a circle.
  - Draw a circle inscribed in a triangle  $ABC$ .

## Activity 3

- 3.1 When are two objects similar?
- 3.2 Find the value of  $x$  in each of the following proportions.
- $\frac{3}{x} = \frac{6}{10}$
  - $\frac{5}{6} = \frac{x}{36}$
- 3.3 Consider the two similar triangles below.



- What is the value of the scale factor that creates triangle  $DEF$  from triangle  $ABC$ ?
- What is the value of the scale factor that creates triangle  $ABC$  from triangle  $DEF$ ?

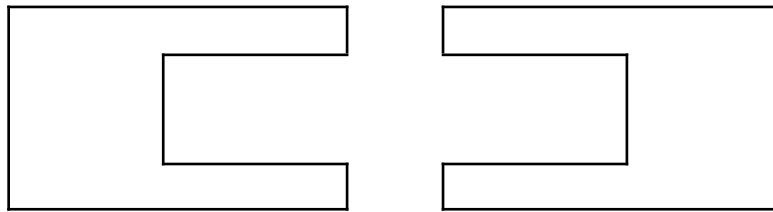
# Answers to Flashbacks

## Activity 1

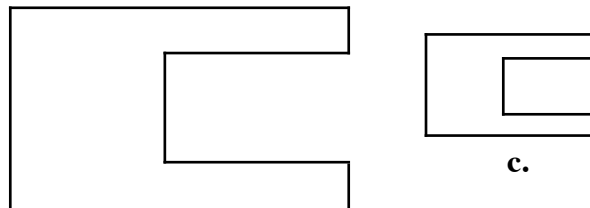
- 1.1
- a.  $\angle RST$  and  $\angle TZR$
  - b.  $\angle ZTS$
  - c. angle bisector
  - d. perpendicular bisector
  - e. line  $SZ$  or line  $RT$
- 1.2
- a.  $30^\circ$
  - b.  $120^\circ$
  - c.  $90^\circ$
  - d.  $75^\circ$

## Activity 2

- 2.1 The diagram below shows a sample response for each transformation.

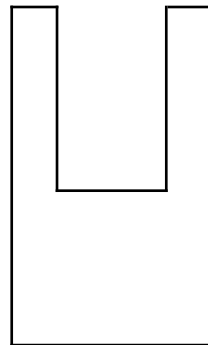


a.



c.

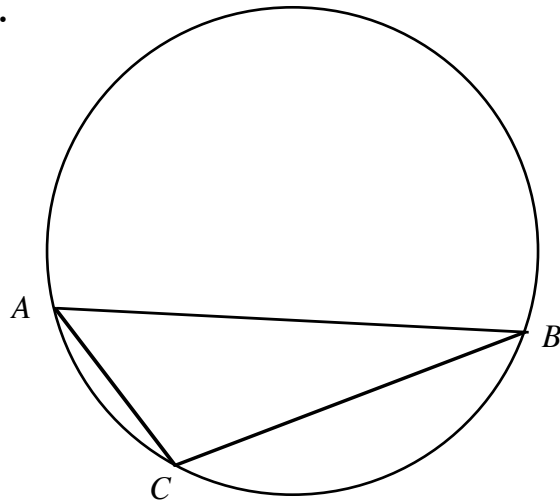
d.



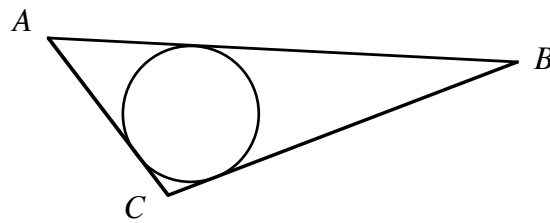
b.

2.2 Sample sketches:

a.



b.



**Activity 3**

3.1 Sample response: Two objects are similar if the measures of corresponding angles are equal and the lengths of corresponding sides are proportional.

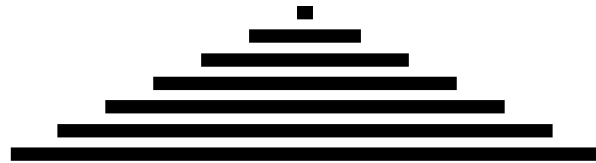
3.2 a.  $x = 5$

b.  $x = 30$

3.3 a.  $2/3$

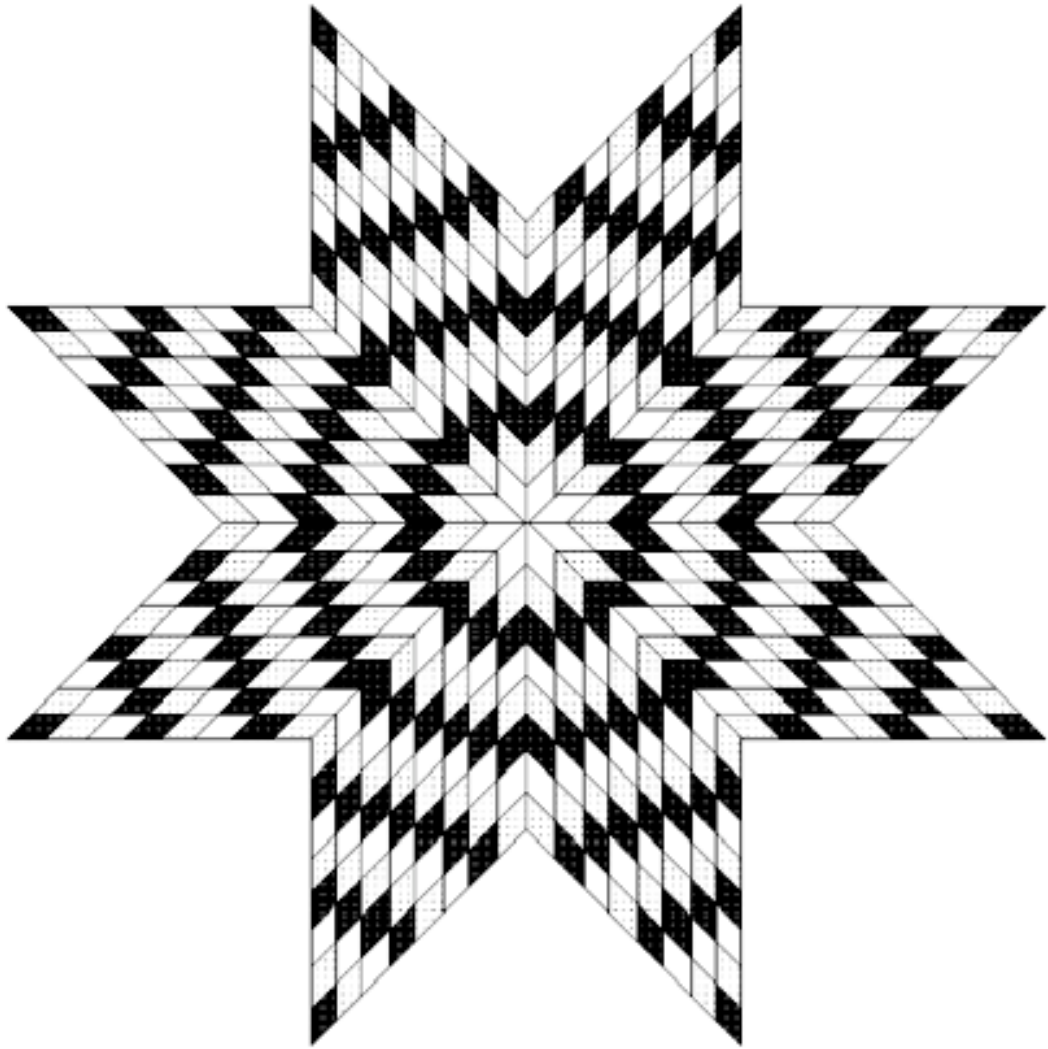
c.  $3/2$

# Template A

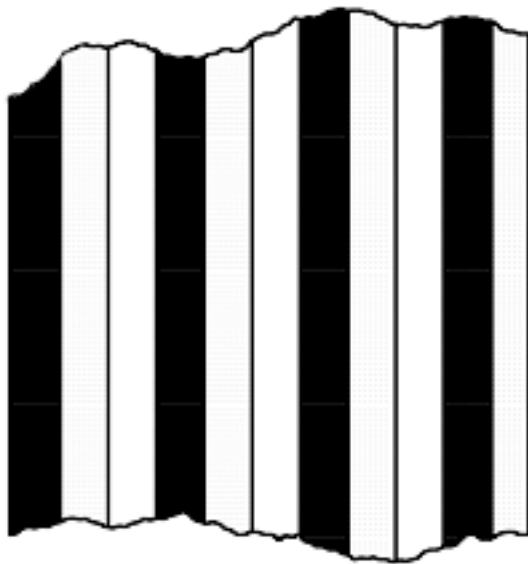
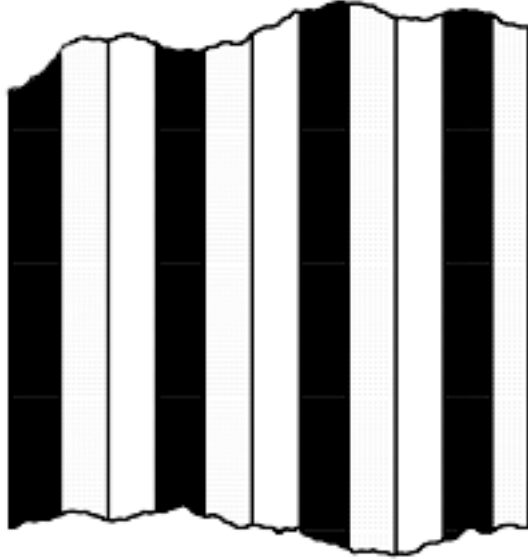




# Template B



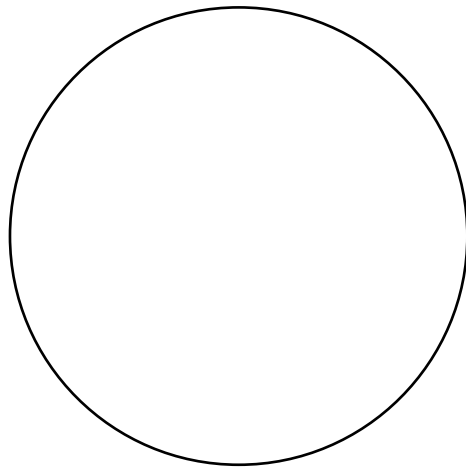
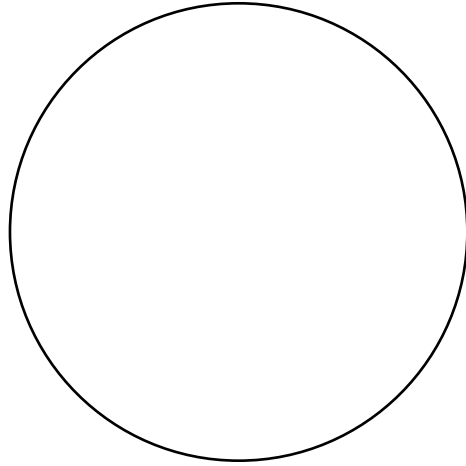
# Template C



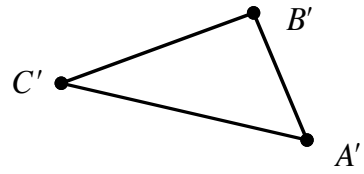
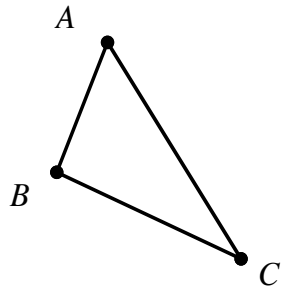
# Template D



# Template E



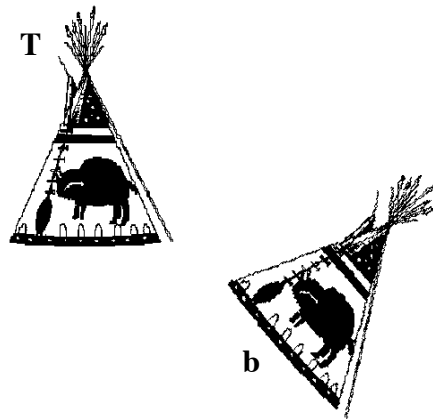
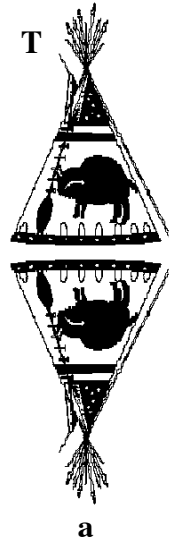
# Template F



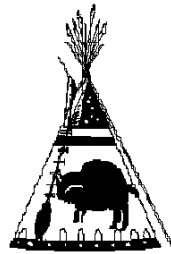
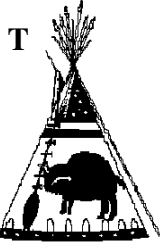
# Template G



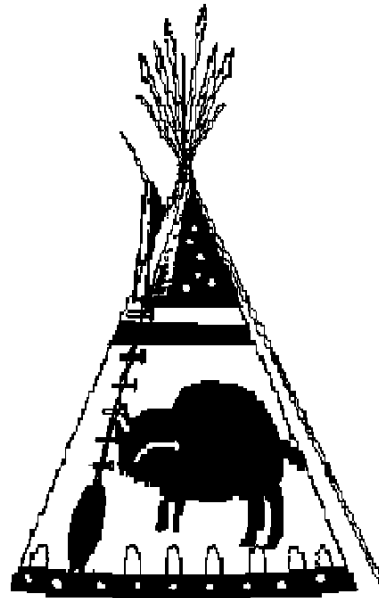
# Template H



# Template H (continued)



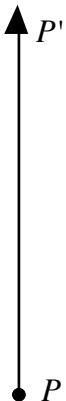
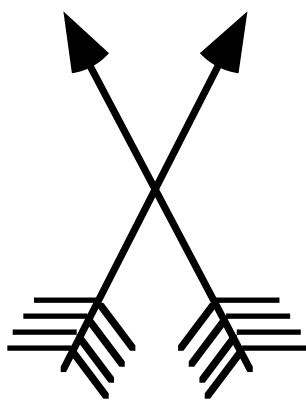
c



d



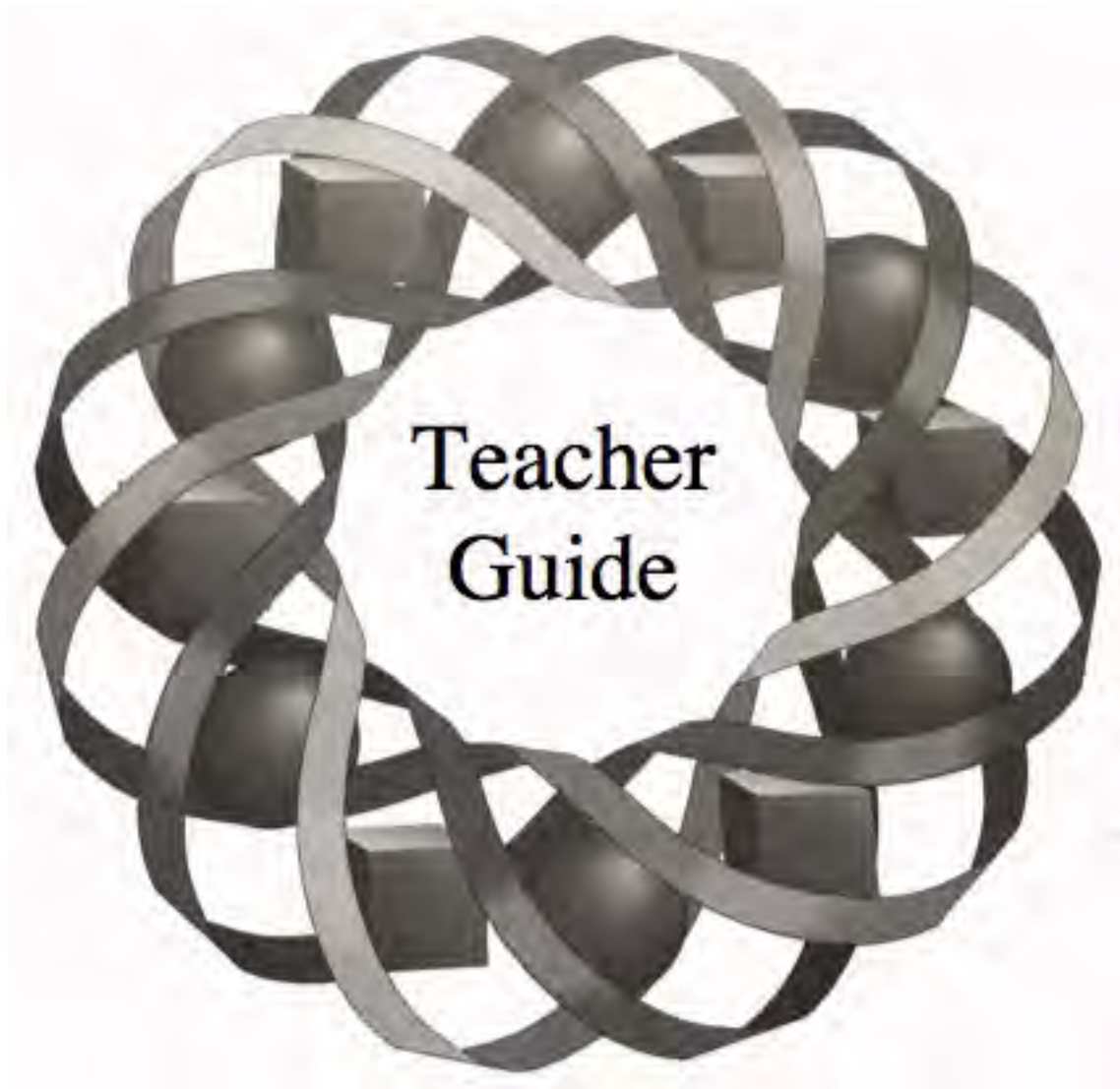
# Template I







# If the Shoe Fits . . .



How can you tell when two quantities—like age and heart rate—are related? In this module, you explore methods for finding and evaluating mathematical models of some important relationships.

*Byron Anderson • Ruth Brocklebank • Pete Stabio*



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## Teacher Edition

# If the Shoe Fits . . .

### Overview

In this module, students explore methods of determining linear models for sets of data. Using data from exercise physiology, students are introduced to median-median lines and regression lines. They also use residual plots to evaluate models.

### Objectives

In this module, students will:

- use the sum of the absolute values of the residuals to compare how well linear models fit a set of data
- model data using the median-median line
- use the least-squares method to find a linear regression equation
- graph and analyze residual plots.

### Prerequisites

For this module, students should know:

- how to identify graphs with positive or negative associations
- how to create scatterplots
- how to calculate residuals
- how to determine the equation of a line in the form  $y = mx + b$
- how to determine the median of a set of data.

### Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	3	2	1	8

## Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
graph paper	X	X	X	
straightedge	X	X	X	
meterstick or measuring tape		X		

## Technology

Software	Activity			
	1	2	3	Summary Assessment
graphing utility	X	X	X	X
spreadsheet	X	X	X	X

## If the Shoe Fits...

### Introduction

(page 351)

The module uses data from human physiology to introduce relationships that may be modeled mathematically.

### Discussion

(page 351)

- a. If the employee is paid by the hour, there is a positive association, since the more the employee works, the more the employee earns. If the employee is not paid by the hour, there may be no relationship, since earnings are the same regardless of the number of hours worked.
- b. Sample response: Since the taller players on the team typically play the forward positions and one of the main objectives of forwards is to score, there may be a positive association between height and the number of points scored. However, individual scoring depends on many factors other than height, so there may not always be a relationship.
- c. Unless students are subject to certain travel restrictions, no relationship is likely. However, there is probably a positive association between the distance a student travels and the amount of time it takes the student to arrive at school.
- d. The relationship between these two quantities is examined in later activities. Accidents are more likely for younger, novice drivers. They also become more likely for elderly drivers.
- e. Sample response: Since students with high GPAs usually study, there may be a positive association between the number of hours spent studying and a student's GPA. However, a student's GPA may depend on many factors other than studying, so there may not always be a relationship.

(page 351)

### *Activity 1*

In this activity, students find linear models for a set of data, determine the equations of their models, then calculate the sum of the absolute values of the residuals. **Note:** Students will examine the least-squares method in Activity 3. Both the sum of the absolute values of the residuals and the sum of the squares of the residuals are reasonable measures of how well a model fits the data. Developed by Carl Friedrich Gauss (1777–1855), the method of least squares is preferred by mathematicians.

## Materials List

- straightedge (one per student)
- graph paper (one sheet per student)

## Technology

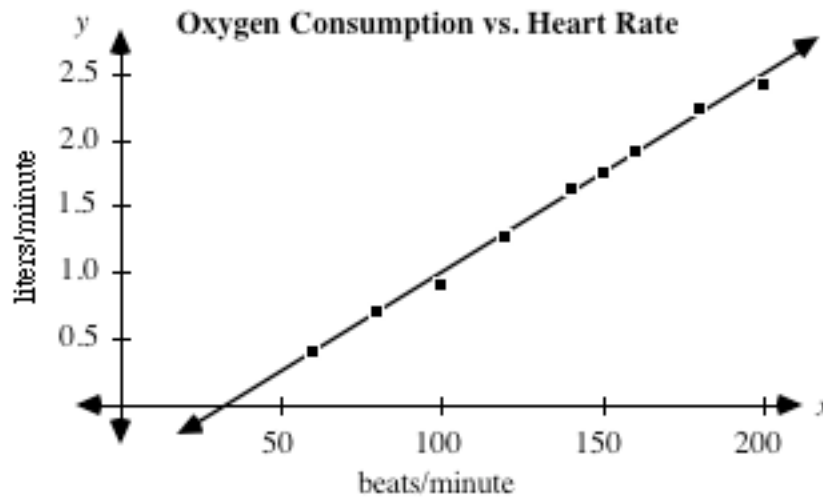
- spreadsheet
- graphing utility

## Exploration

(page 352)

Students fit lines to a set of data, then determine the equations of these lines. To evaluate their models, students calculate the sum of the absolute values of the residuals.

- a. Sample scatterplot and line:



- b. The equation of the sample line shown in Part a is  $y = 0.015x - 0.470$ .
- c. Answers will vary, depending on the line. The following table shows the sum of the squares of the residuals using the equation  $y = 0.015x - 0.470$ .

$x$	$y$	predicted $y$	residual
60	0.41	0.43	0.02
80	0.71	0.73	0.02
100	0.90	1.03	0.13
120	1.27	1.33	0.06
140	1.62	1.63	0.01
150	1.76	1.78	0.02
160	1.92	1.93	0.01
180	2.24	2.23	0.01
200	2.42	2.53	0.11
<b>Sum</b>			0.39



- d. Students try to reduce the sum of the absolute values of the residuals by adjusting the slope and  $y$ -intercept of the model equation. For example, the equation  $y = 0.015x - 0.49$  results in a lower sum of 0.29.

## Discussion

(page 353)

- a. Students should observe that an increase in heart rate is associated with an increase in oxygen consumption. The relationship appears to be linear.
- b. Answers may vary. Students may use residuals, sums of residuals, and visual criteria as justifications. Although a good model may not contain all the points of a scatterplot, it should approximate the general trend of the data.
- c. Sample response: Yes, the line comes very close to most of the data points.
- d. 1. The equation with the smallest sum of the absolute values of the residuals should have a slope of approximately 0.015 and a  $y$ -intercept of approximately  $-0.49$ .
2. Answers may vary. Some students may have adjusted either the slope or  $y$ -intercept, while others may have adjusted both.
- e. 1. Students may suggest determining an estimate from the graph of the line or using the equation to calculate the  $y$ -value when  $x = 168$ .
2. Students may suggest determining an estimate from the graph of the line or using the equation to calculate the  $x$ -value when  $y = 1.5$ .
- f. Answers will vary. Sample response: It is often dangerous to make predictions for values outside the observed data. For example, the domain and range of the model may be restricted in the real-world setting.

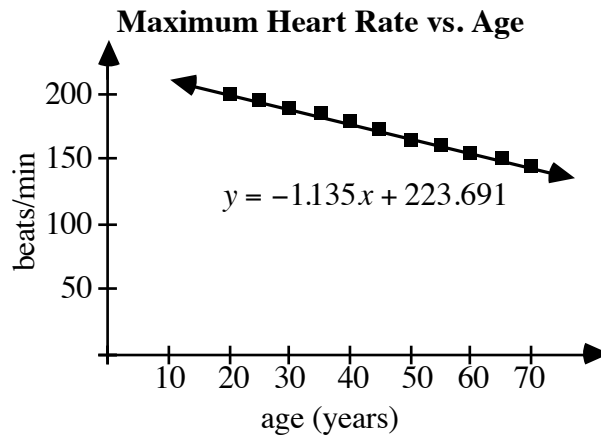
## Teacher Note

In the following assignment, students are asked to find linear models for data sets. For the sake of comparison, the sample responses given include the linear regression equation for each data set. Students will examine linear regression models in Activity 3. They are not expected to use these models in this assignment.

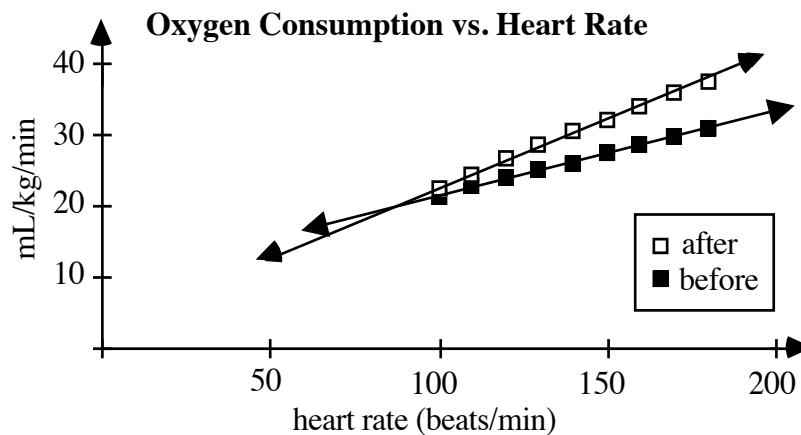
## Assignment

(page 354)

- 1.1 a. Sample scatterplot:



- b. Answers will vary. **Note:** The linear regression equation for this data is  $y = -1.135x + 223.691$ .
- c. Answers will vary, depending on the equation from Part b. Using the equation  $y = -1.135x + 223.691$ , the sum of the absolute values of the residuals is approximately 8.434.
- d. Using the equation  $y = -1.135x + 223.691$ , the suggested maximum heart rate for a person of age 15 would be about 207 beats/min.
- e. Sample response: Using the linear model, the suggested maximum heart rate at age 3 is about 220 beats/min. This is the upper limit for the human heart. Therefore, it would probably be wise to use a different model for small children.
- \*1.2 a. Sample scatterplot:



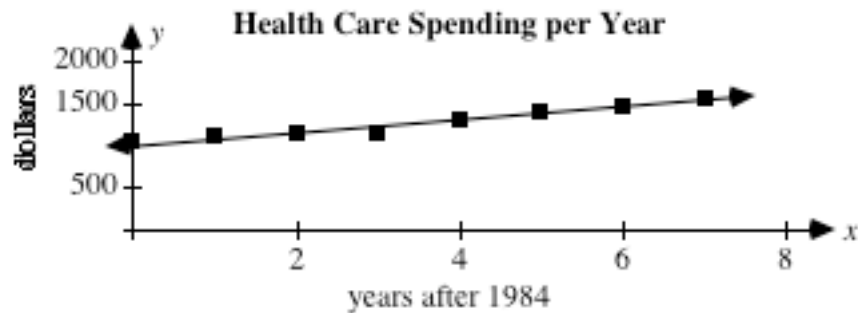
- b. Answers may vary. A possible equation for the “after” data is  $y = 0.186x + 4.128$ , while a possible equation for the “before” data is  $y = 0.12x + 9.403$ .
- c. Answers may vary. Using the equations from Part **b**, the model for the “before” data has a sum of the absolute values of the residuals of approximately 0.637, while the model for the “after” data has a sum of approximately 1.517. Therefore, the model for the “before” data appears to provide a better fit.
- d. Using the sample equations from Part **b**, the predicted heart rate before an aerobic conditioning program is about 255 beats/min, while the predicted heart rate after an aerobic conditioning program is about 193 beats/min.

\* \* \* \* \*

### Teacher Note

In Problems **1.3** and **1.4** (and again in Activities **2** and **3**), the sample responses use  $x = 0$  for the initial year. In Problem **1.3**, for example, the first data point is (0,1049) rather than (1984,1049). Students may choose to use the actual year. You may wish to point out that, for a linear model, the slope is the same in either case, although the  $y$ -intercept is different.

- \*1.3**    a. Sample scatterplot:



- b. Answers will vary. **Note:** The equation of the regression line for this data is  $y = 75.881x + 1005.167$ . A graph of this equation is shown on the scatterplot in Part **a**.
- c. Using the equation from Part **b**, the sum of the absolute values of the residuals is approximately 260.9.
- d. Using the equation from Part **b**, the predicted mean amount, per person, is approximately \$2219.

1.4 a. Sample scatterplot:



- b. Answers will vary. **Note:** The equation of the regression line for this data is  $y = 0.276x + 60.645$ .
- c. Using the equation from Part b, approximately 68.9% of the population will participate in the work force in the year 2000.
- d. Answers will vary. The sample prediction in Part c is only 0.2% higher than the prediction made by the Bureau of the Census.

\* \* \* \* \*

(page 357)

## Activity 2

Students explore the median-median line as a tool for modeling linear data.

### Materials List

- straightedge (one per student)
- graph paper (one sheet per student)
- meterstick or measuring tape (one per pair of students)

### Technology

- spreadsheet
- graphing utility

### Teacher Note

Some technology yields a median-median equation different from the one produced through the method described in the **Mathematics Note**.

## Exploration

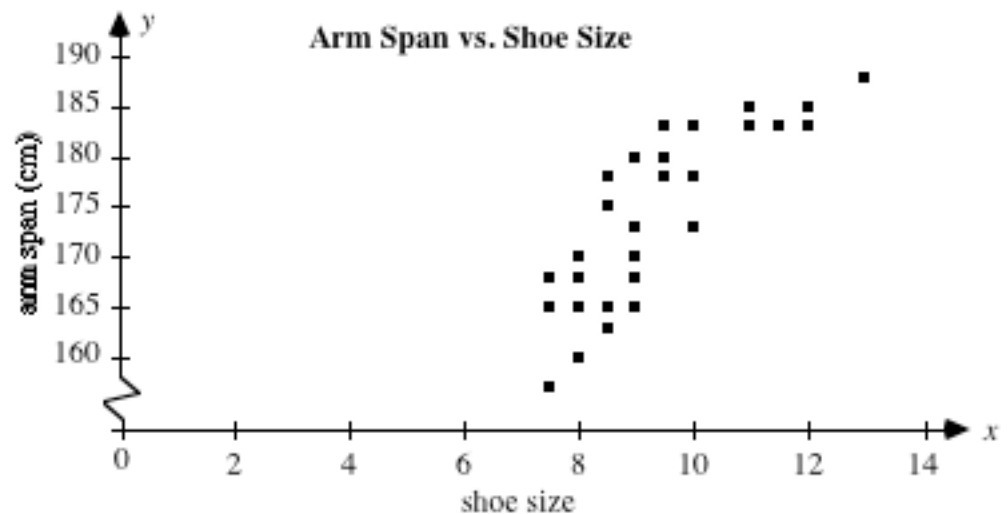
(page 358)

Students model the relationship between arm span and shoe size using a median-median line. **Note:** The differences between male and female shoe sizes appear to have little effect on the results of this exploration.

- a. The sample data shown in the following table was gathered from a class of 32 students consisting of 17 girls and 15 boys.

Shoe Size	Arm Span (cm)	Shoe Size	Arm Span (cm)
7.5	157	9	173
7.5	165	9	180
7.5	168	9.5	178
7.5	168	9.5	178
8	160	9.5	180
8	165	9.5	183
8	168	10	173
8	170	10	178
8.5	163	10	178
8.5	165	10	183
8.5	175	11	183
8.5	178	11	185
9	165	11.5	183
9	165	12	183
9	168	12	185
9	170	13	188

- b. A scatterplot of the sample data is shown below.



- c. The following table divides the sample data into three groups.

First Group		Middle Group		Third Group	
Shoe Size	Arm Span (cm)	Shoe Size	Arm Span (cm)	Shoe Size	Arm Span (cm)
7.5	157	9	165	10	173
7.5	165	9	165	10	178
7.5	168	9	168	10	178
7.5	168	9	170	10	183
8	160	9	173	11	183
8	165	9	180	11	185
8	168	9.5	178	11.5	183
8	170	9.5	178	12	183
8.5	163	9.5	180	12	185
8.5	165	9.5	183	13	188
8.5	175				
8.5	178				

- d. The coordinates of the summary points for the sample data are (8,166.5), (9,175.5), and (11,183).
- e.
- Using the sample data given in Part a, the equation of the line determined by the first and third summary points is  $y = 5.5x + 122.5$ .
  - Substituting the  $x$ -coordinate of the middle summary point into the equation from Step 1 gives the  $y$ -coordinate of the corresponding point on the line. For the sample data, the coordinates of this point are (9,172).
  - Since the  $y$ -coordinate of the middle summary point is 175.5, the vertical distance between these points is  $175.5 - 172 = 3.5$ . The coordinates of the point that is  $1/3$  the distance from the line are approximately (9,173.2).
  - Students should observe that this line has the same slope as the line in Step 1.
  - For the sample data, the equation of the median-median line is approximately  $y = 5.5x + 123.7$ .

## Discussion

(page 359)

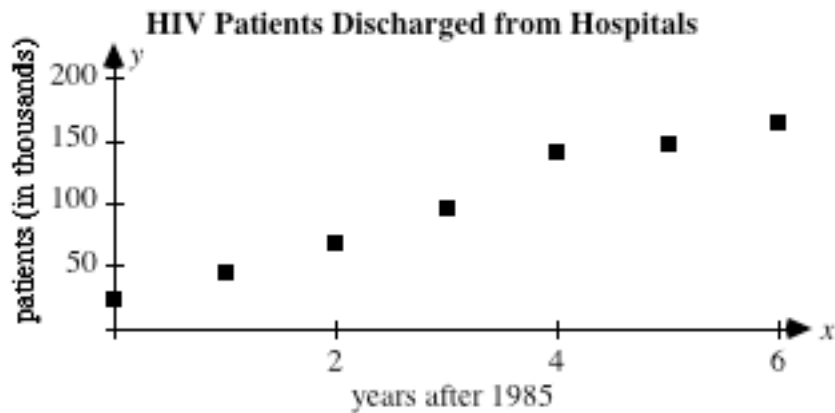
- The median is considered a good representative value for a set of data. It does not require complicated calculations, and is not as affected by outliers as other measures of central tendency.
- Sample response: The median-median line provides a reasonable model. Although it does not contain all the points of the scatterplot, it describes the general trend of the data.

- c. Sample response: The  $y$ -intercepts of the two lines differ by  $1/3$  of the vertical distance between the summary point of the middle group and the line containing the first and third summary points.
- d.
  1. Answers will vary. Using the median-median line for the sample data, the predicted arm span is 206 cm.
  2. Sample response: An arm span of 206 cm is about 6 ft 9 in. This seems reasonable for someone who wears size 15 shoes.

**Assignment**

(page 359)

- 2.1
  - a. Answers will vary. Using the sample equation given in the exploration, Wadlow’s predicted shoe size is 30.
  - b. The sample prediction is substantially low. Depending on student estimates, this question may illustrate the dangers of making predictions far outside the range of collected data.
- 2.2
  - a. Since students may have used estimated values during their calculations for the median-median line, their results may not exactly agree with those reported by technology.
  - b. Sample response: The two lines are very close. The equation of the median-median line found using technology does not take significant figures into account.
- 2.3
  - a. Sample scatterplot:



- b. Using 1985 as  $x = 0$ , the equation of the median-median line is approximately  $y = 24.4x + 21.5$ .
- c. Judging from the scatterplot, the number of HIV-positive patients discharged from hospitals appears to be increasing linearly.

\*\*\*\*\*

2.4 a. Sample scatterplot:

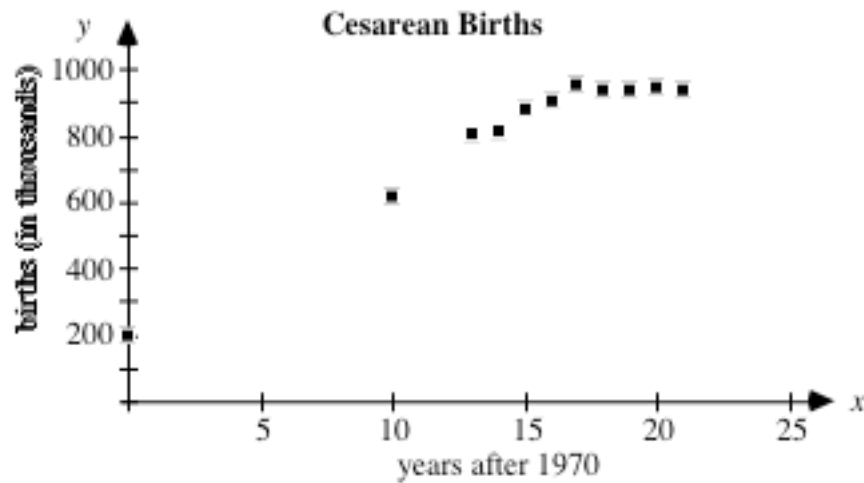


b. Using 1970 as  $x = 0$ , the equation of the median-median line is approximately  $y = 171x + 6663$ .

c. 1. Using the equation given in Part b, the estimated number of self-employed workers in 1988 is 9741.

2. The estimated value is less than 2% from the actual value.

2.5 a. Sample scatterplot:



b. Using 1970 as  $x = 0$ , the equation of the median-median line is approximately  $y = 27.8x + 417$ .

c. Sample response: No, the median-median line does not appear to be a very good model of the data. Many of the data points are relatively far from the line.

d. Sample response: No. After 1987, the scatterplot appears to level off. In fact, from 1988 to 1991, the increase in the annual number of cesarean births was 0.

\*\*\*\*\*



## Activity 3

In this activity, students determine linear regressions and evaluate linear models using residual plots. **Note:** The term *regression line* originated in statistical studies of the heights of children of tall parents because it described how those heights “regressed” toward the mean height of the population.

### Materials List

- straightedge (one per student)
- graph paper (one sheet per student)

### Technology

- spreadsheet
- graphing utility

### Teacher Note

The least-squares method is used to generate equations for regression lines. The equations that students determine in Part **d** of Exploration 1 using a spreadsheet should approximate the linear regression.

### Exploration 1

(page 361)

Students use the arm span and shoe size data from Activity 2 to examine a linear regression model. This equation is then compared to the median-median line found in Activity 2.

- a–b.** Students recreate a scatterplot of the data for arm span vs. shoe size and determine the equation of a line that models the data reasonably well. (See sample graph in given in Part **b** of the exploration in Activity 2.)
- c–d.** Students determine the sum of the squares of the residuals for their model. They then adjust the equation until they find the least sum.

For the sample data given in Activity 2, the sum of the squares of the residuals for the regression line is approximately 718.2. In comparison, the sum of the squares of the residuals for the median-median line is about 795.1.

- e.** Answers will vary. The linear regression equation for the sample data is  $y = 4.8x + 129.1$ .

## Discussion 1

(page 362)

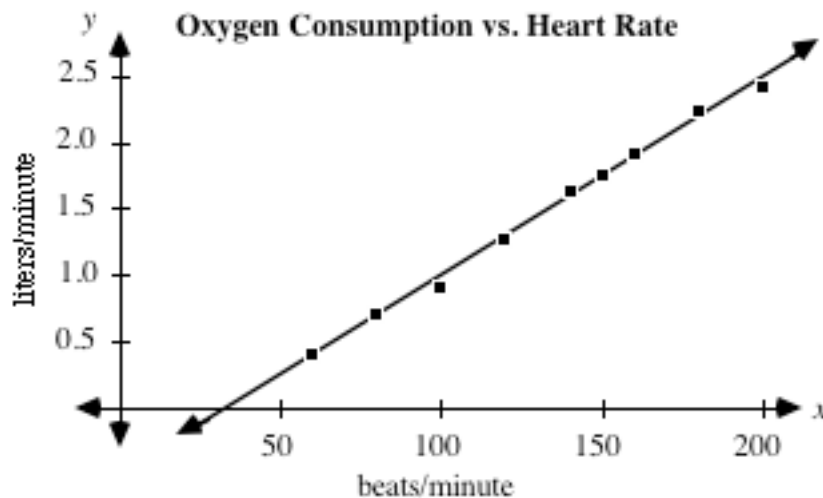
- a.
  1. Depending on the spread of the data, the median-median line may or may not be close to the linear regression.
  2. Students should find that the two equations are quite similar.
- b.
  1. Sample response: The two models would differ greatly when there are outliers in the data. The median-median line only takes into account the median of each group of data, while the linear regression considers the residuals of all data points. Therefore, the median-median line would tend to reflect only the pattern of the “middle” points.
  2. Sample response: The two models would be approximately the same when the data is grouped relatively tightly about the middle data points.
- c. Sample response: The linear regression would be better when you wish to consider all the data collected in your interpretation or predictions. **Note:** It should be pointed out to students that statisticians prefer the linear regression model.

## Exploration 2

(page 363)

Students investigate the use of residual plots as a tool for evaluating the appropriateness of a model.

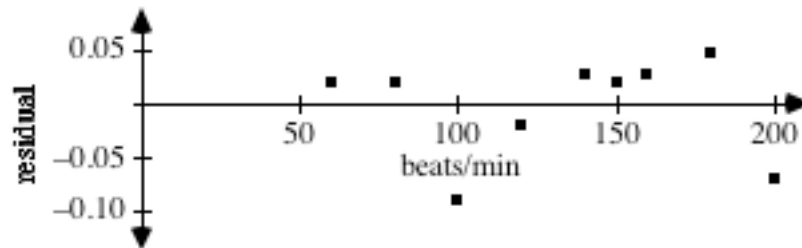
- a. Sample graph:



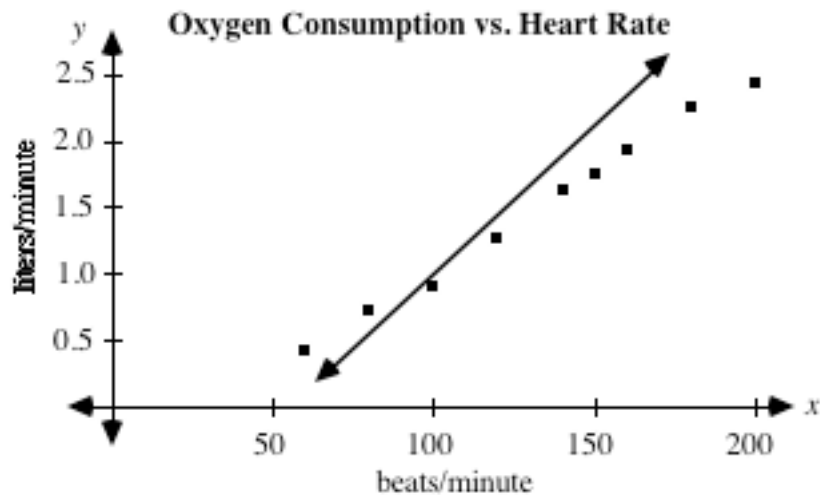
- b. A completed table is shown below.

Heart Rate (beats/min)	Oxygen Consumption (L/min)	Predicted Oxygen Consumption	Residual
60	0.41	0.391	0.019
80	0.71	0.691	0.019
100	0.90	0.991	-0.091
120	1.27	1.291	-0.021
140	1.62	1.591	0.029
150	1.76	1.741	0.019
160	1.92	1.891	0.029
180	2.24	2.191	0.049
200	2.42	2.491	-0.071

- c. The line  $y = 0.015x - 0.509$  models the data well, as shown by the graph in Part a.
- d. As shown in the following residual plot, the residuals appear to be scattered randomly above and below the  $x$ -axis.



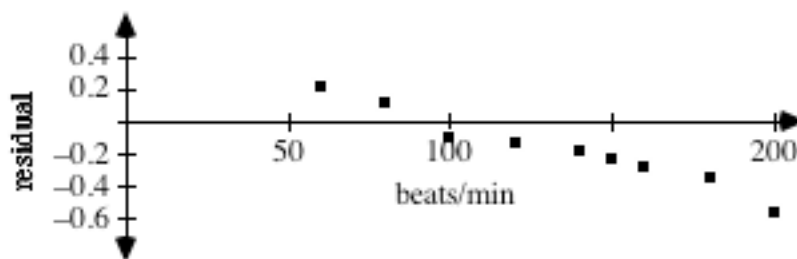
- e. The following graph shows a scatterplot of the data in Table 5 along with the line  $y = 0.02x - 1$ .



The table below shows the residuals that result using the model  $y = 0.02x - 1$ .

Heart Rate (beats/min)	Oxygen Consumption (L/min)	Predicted Oxygen Consumption	Residual
60	0.41	0.2	0.21
80	0.71	0.6	0.11
100	0.90	1	-0.1
120	1.27	1.4	-0.13
140	1.62	1.8	-0.18
150	1.76	2	-0.24
160	1.92	2.2	-0.28
180	2.24	2.6	-0.36
200	2.42	3	-0.58

The line  $y = 0.02x - 1$  does not model the data well. As shown in the following residual plot, the residuals form a roughly linear pattern with a negative slope.



- f. Students compare the residual plots for the two equations. In the residual plot for the better model,  $y = 0.015x - 0.509$ , the points are located closer to the  $x$ -axis and do not seem to form a recognizable pattern. **Note:** Although the equation  $y = 0.02x - 1$  does not model the data well, the shape of the residual plot is linear. This indicates that a linear model is probably a good one for the data.

## Discussion 2

(page 365)

- a. Sample response: The equation  $y = 0.015x - 0.509$  models the data better. A graph of this line is very close to most of the points on the scatterplot.
- b. Sample response: If the points of a residual plot have no definite pattern and are distributed fairly evenly above and below the  $x$ -axis, the linear model is probably appropriate for the data.

- c.
1. The points of this residual plot appear to have a linear pattern. Therefore, another linear model may provide a better fit for the data.
  2. The points of this residual plot form a curved pattern. In this case, a linear model is probably not appropriate. **Note:** Some students may recognize the shape of the residual plot as a parabola and therefore suggest a quadratic model.
  3. The points of this residual plot appear to have no definite pattern and are distributed fairly evenly above and below the  $x$ -axis.
  4. The points of this residual plot appear to form an exponential curve. This suggests that an exponential model may be more appropriate.

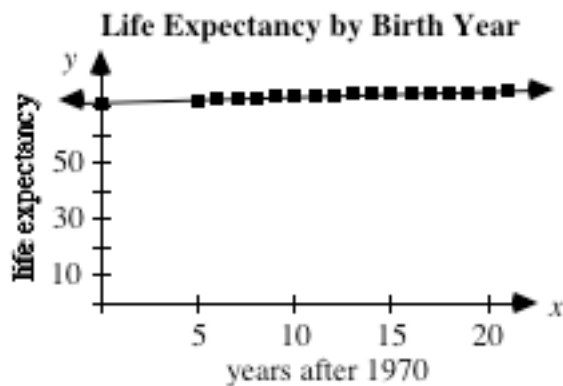
### Teacher Note

The use of some mathematical functions to model data involving time may be reasonable for relatively brief intervals, but may fail when applied to longer periods. For example, many relationships appear approximately linear only over short intervals. You may wish to point out the dangers of using such models to make predictions outside the range of the data.

### Assignment

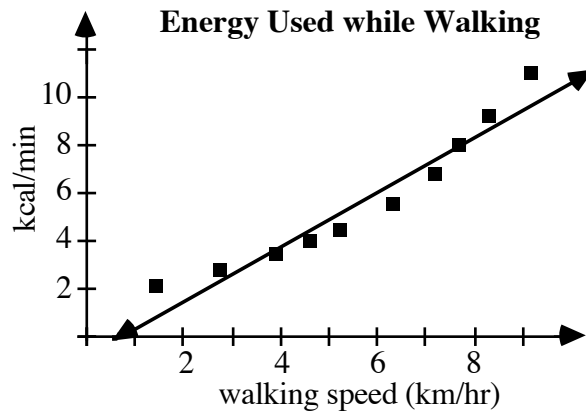
(page 366)

- 3.1 a. Sample scatterplot:

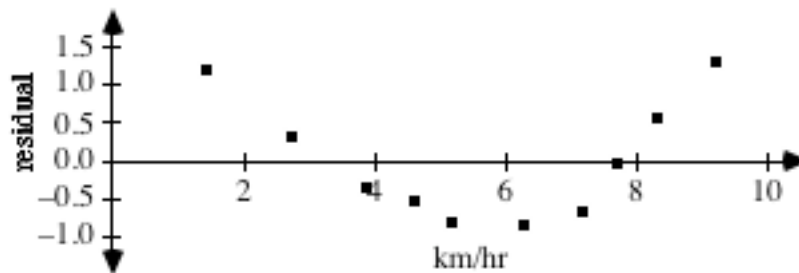


- b. Using 1970 as  $x = 0$ , the equation of the regression line is  $y = 0.197x + 71.698$ .
- c. Using the equation from Part b, the life expectancy of people born in 1973 is approximately 72.2 years; the life expectancy of people born in 2010 will be approximately 79.6 years.
- d. The life expectancy from Part c is 2 years more than that predicted by the U.S. Census Bureau.

- \*3.2** a. The following graph shows a scatterplot of the data as well as the regression line  $y = 1.129x - 0.655$ .



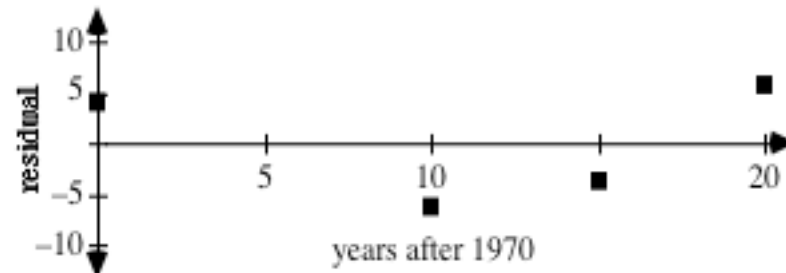
- b. Sample residual plot:



- c. Sample response: The linear regression equation does not appear to be an appropriate model for the data. The shape of the scatterplot appears to be curved, not linear. In addition, the residual plot shows a definite pattern. **Note:** Some students may recognize the shape of the parabola.

- \*3.3** a. Using 1970 as  $x = 0$ , the equation of the regression line is  $y = -0.939x + 83.937$ .

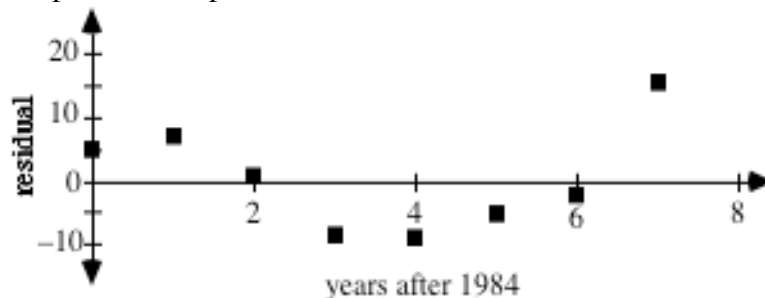
- b. Sample residual plot:



- c. Sample response: There is probably a better model for the data than a line since the points of the residual plot appear to form a curve.

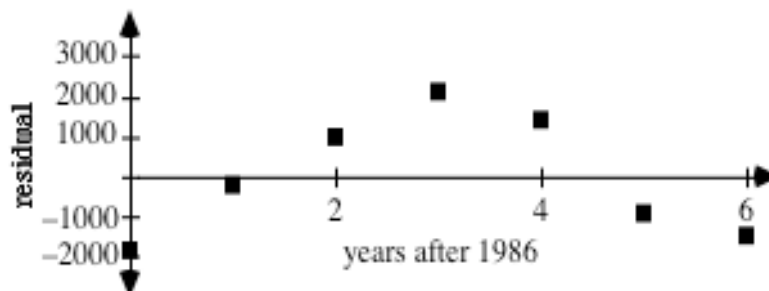
\*\*\*\*\*

- \*3.4**
- Using 1984 as  $x = 0$ , the equation of the regression line is  $y = 47.345x + 405.917$ .
  - Using the equation from Part a, the predicted cost is about \$1209.
  - Sample residual plot:

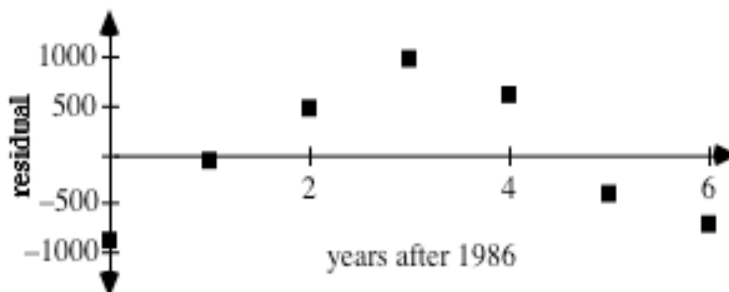


- Sample response: There is probably a better model for the data than a line since the points of the residual plot appear to form a curve.

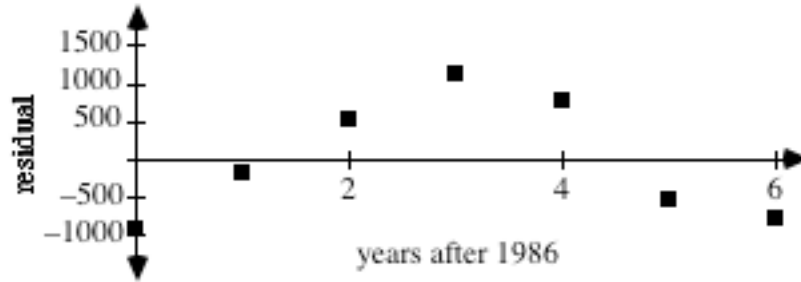
- 3.5**
- Using 1986 as  $x = 0$ , the equation of a regression line that models the data is  $y = 1279.179x - 111,410.893$ .
  - Using the equation in Part a, the average yearly increase in the American work force is about 1,279,000 men and women.
  - Sample residual plot:



- Sample response: Since the residual plot appears to show a pattern, a line is probably not an appropriate model for the data.
- The equation of the regression line for the data for females is  $y = 819.429x + 49587.286$ . According to this model, the average yearly increase in the number of females in the American work force is approximately 819,000. Since the residual plot appears to show a pattern, a line is probably not the best model for the data.



- f. The equation of the regression line that models the data for males is  $y = 459.75x + 61823.607$ . According to this model, the average yearly increase in the number of males in the American work force is approximately 460,000. Since the residual plot appears to show a pattern, a line is probably not an appropriate model for the data.



- g. Sample response: The average increase in the number of males is less than half as large as the average increase in the number of females. The average increase in total workers is the sum of the averages for males and females. **Note:** Students may also recognize that the regression equation for total workers is the sum of the two regression equations for males and females.

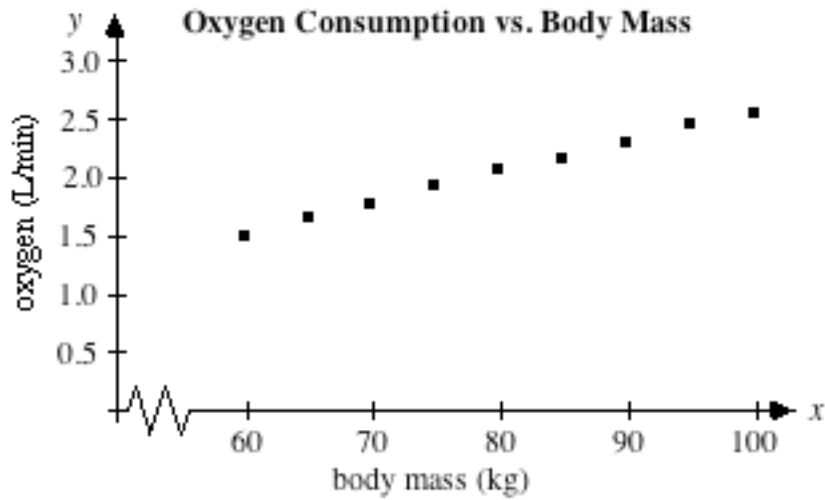
\* \* \* \* \*



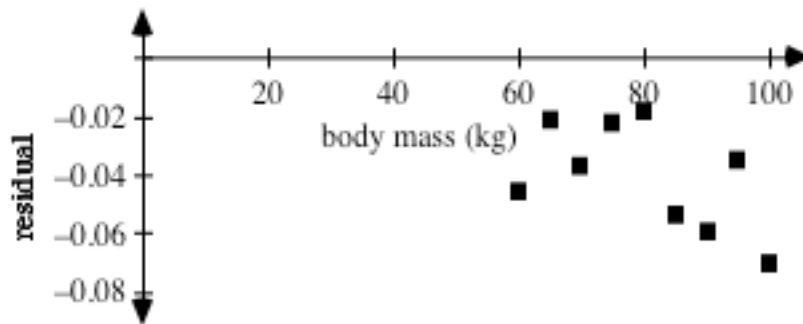
Answers to Summary Assessment

(page 369)

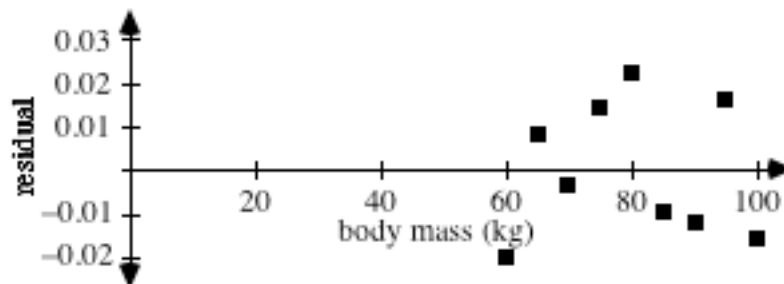
1. a. Sample scatterplot:



- b. 1. The equation of the linear regression is  $y = 0.0264x - 0.0642$ .  
 2. The equation of the median-median line is  $y = 0.027x - 0.07$ .  
 c. The sum of the squares of the residuals for the median-median line is 0.0042. The sum of the squares of the residuals for the regression line is 0.0019.  
 d. The following sample graph shows a residual plot for the median-median line.



The sample graph below shows a residual plot for the regression line.

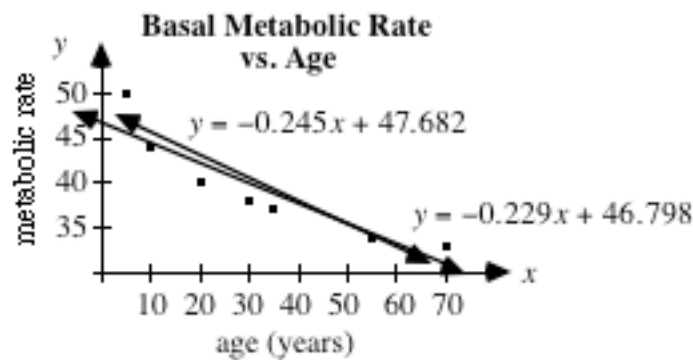


- e. Sample response: The equations for the median-median line and the regression line are extremely close. However, the regression line has a smaller sum of the squares of the residuals and a better residual plot. Therefore, the regression line appears to be a better model.
2. Student responses may vary. Some may argue for a linear model since the residuals for the regression line are relatively small. Others may argue against a linear model since the data points appear to follow a power curve and a residual plot reveals a noticeable pattern. In either case, students should support their arguments with graphs, equations, and residuals.

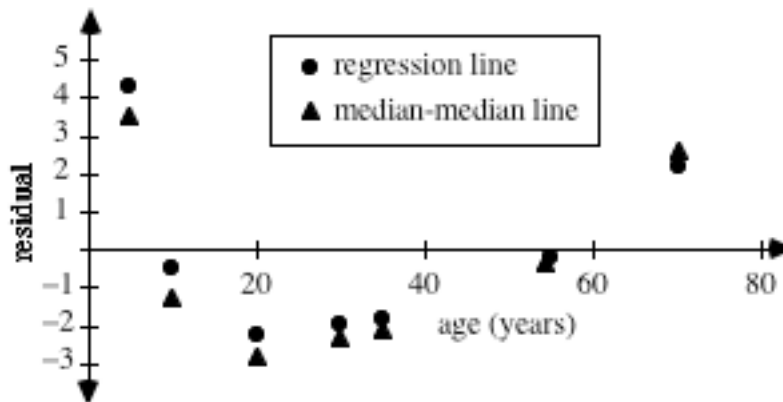
The equation for the regression line is  $y = -0.229x + 46.798$ . The sum of the squares of the residuals for this model is 35.993.

The equation for the median-median line is  $y = -0.245x + 47.682$ . The sum of the squares of the residuals for this model is 37.822.

The following graph shows a scatterplot of the data, along with both the regression line and the median-median line:



The graph below shows residual plots for these linear models:



## Module Assessment

1. Health professionals often recommend walking as a good way to stay fit. The table below shows some data for the amount of fat expended per minute of walking time.

Minutes of Walking	Fat Expended (kcal/min)
0	0.7
5	1.8
10	1.9
20	2.8
30	3.2
40	3.4
45	3.5

**Source:** Adapted from Thomas and Londeree, "Calories Expended, Walking vs. Jogging," *The Physician and Sports Medicine* 17 (May 1989): 98.

- a. Make a scatterplot of the data.
  - b. Find the regression line for the data.
  - c. Find the median-median line for the data.
  - d. Do the models from Parts **b** and **c** appear to be appropriate for the data? Support your response using graphs, equations, and residuals.
2. The table below shows the domestic motor fuel consumption in the United States for selected years from 1970 to 1991.

Year	Fuel Consumption (in billions of gallons)
1970	92.3
1975	109.0
1980	115.0
1981	114.5
1982	113.4
1983	116.1
1984	118.7
1985	121.3
1986	125.2
1987	127.5
1988	130.1
1989	131.8
1990	130.8
1991	128.6

**Source:** U.S. Bureau of the Census, 1995.

- a. Make a scatterplot of the data.
  - b. Find the regression line for the data.
  - c. Find the median-median line for the data.
  - d. Do the models from Parts **b** and **c** appear to be appropriate for the data? Support your response using graphs, equations, and residuals.
  - e.
    1. Use one of your models to predict the motor fuel consumption in 1993.
    2. According to the Bureau of the Census, the actual motor fuel consumption in 1993 was 137.2 billion gallons. How does your prediction compare with this value?
3. The table below shows the annual milk production in the United States for selected years from 1980 to 1992.

Year	Milk Production (n billions of pounds)
1980	128.4
1985	143.0
1988	145.2
1989	144.2
1990	148.3
1991	148.5
1992	151.8

**Source:** U.S. Bureau of the Census, 1995.

- a. Make a scatterplot of the data.
- b. Find the regression line for the data.
- c. Find the median-median line for the data.
- d. Do the models from Parts **b** and **c** appear to be appropriate for the data? Support your response using graphs, equations, and residuals.
- e.
  1. Use one of your models to predict the U.S. milk production in 1994.
  2. According to the Bureau of the Census, the actual milk production in 1994 was 153.6 billion pounds. How does your prediction compare with this value?

4. The table below shows the numbers of job-related accidental deaths in the United States for selected years since 1960.

Year	Deaths (in thousands)
1960	13.8
1965	14.1
1970	13.8
1975	13.0
1980	13.2
1984	11.5
1985	11.5
1986	11.1
1987	11.3
1988	11.0
1989	10.9
1990	10.6
1991	9.9

**Source:** U.S. Bureau of the Census, 1995.

- a. Make a scatterplot of the data.
- b. Find the regression line for the data.
- c. Find the median-median line for the data.
- d. Do the models from Parts **b** and **c** appear to be appropriate for the data? Support your response using graphs, equations, and residuals.
- e.
  1. Use one of your models to predict the number of job-related deaths in 1993.
  2. According to the Bureau of the Census, the actual number of job-related deaths in 1993 was 9100. How does your prediction compare with this value?

5. The cost of health services and supplies has risen dramatically for Americans over the past 30 years. The table below illustrates the increase in total expenditures for medical care since 1970.

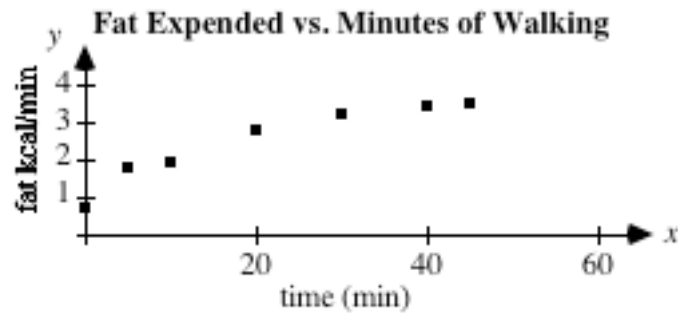
<b>Year</b>	<b>Total Expenditures for Health Services and Supplies (in billions of dollars)</b>
1970	69.1
1980	238.9
1984	374.2
1985	407.2
1986	438.9
1987	476.9
1988	526.2
1989	583.2
1990	652.4
1991	728.6

**Source:** U.S. Bureau of the Census, 1995.

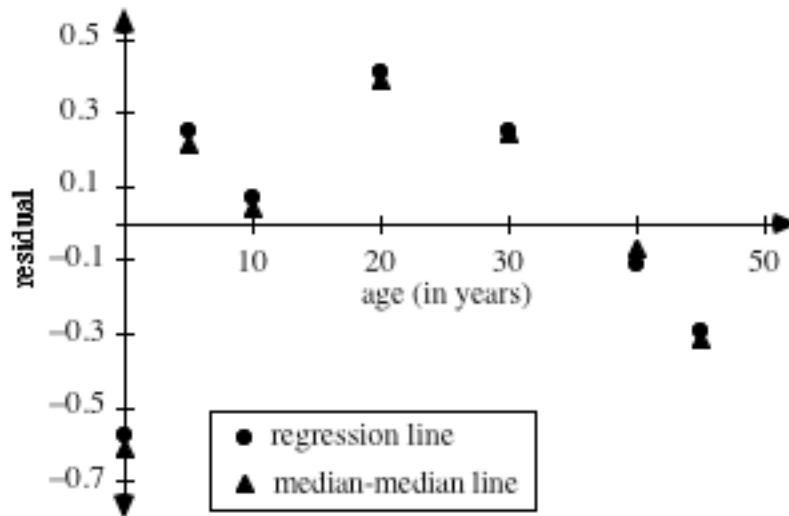
- Make a scatterplot of the data.
- Find the regression line for the data.
- Find the median-median line for the data.
- Do the models from Parts **b** and **c** appear to be appropriate for the data? Support your response using graphs, equations, and residuals.

## Answers to Module Assessment

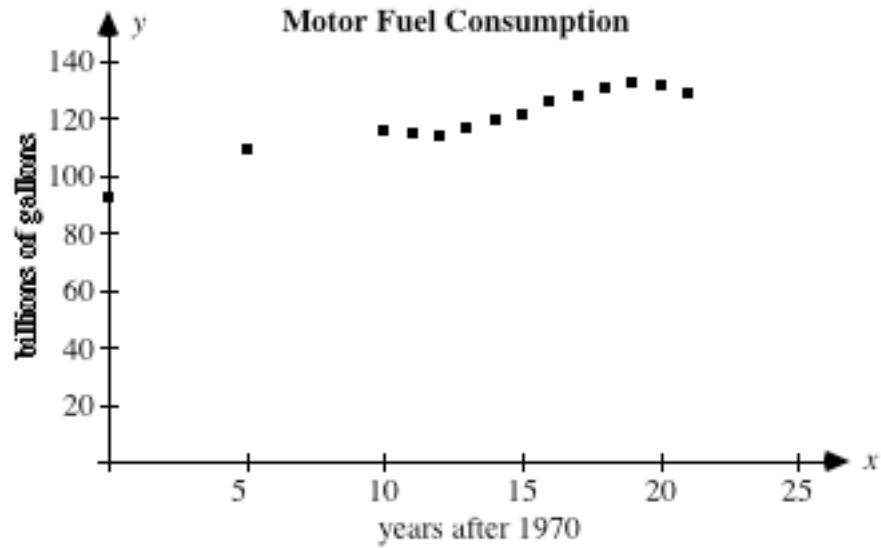
1. a. Sample scatterplot:



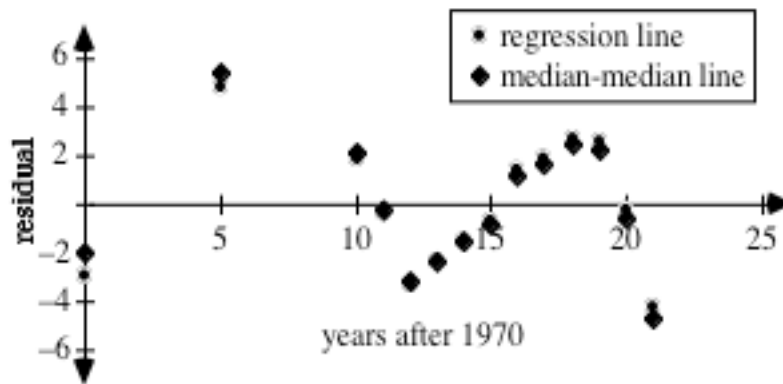
- b. The regression line is  $y = 0.056x + 1.272$ .
- c. The median-median line is  $y = 0.055x + 1.308$ .
- d. Both models appear to be reasonable for the data. The sum of the squares of the residuals for the regression line is 0.719, while the sum of the squares of the residuals for the median-median line is 0.722. As shown in the sample graph below, the residual plots also are very similar.



2. a. Sample scatterplot:



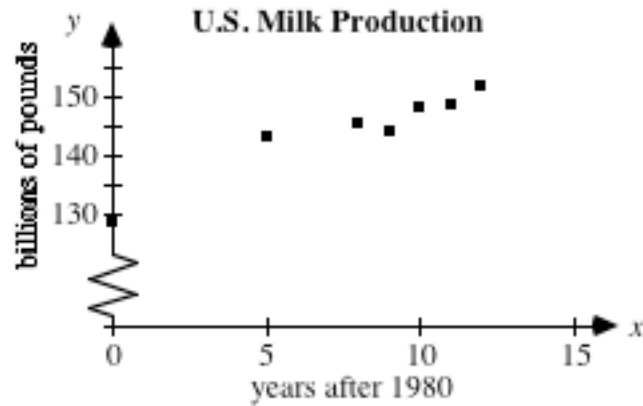
- b. Using 1970 as  $x = 0$ , the regression line is  $y = 1.790x + 95.171$ .
- c. Using 1970 as  $x = 0$ , the median-median line is  $y = 1.856x + 94.261$ .
- d. Neither line appears to be an appropriate model. The sum of the squares of the residuals for the regression line is 91.88, while the sum of the squares of the residuals for the median-median line is 93.825. As shown in the sample graph below, the residual plots show a definite pattern. (The three “turning points” in the plots indicate that a polynomial of higher degree may be a more appropriate model.)



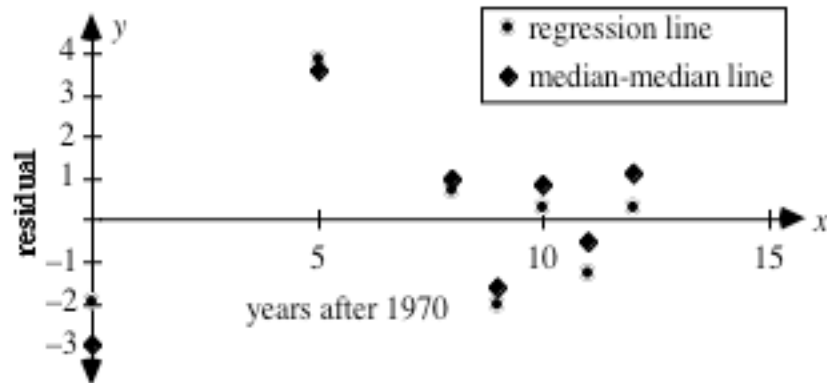
- e. 1. Using the linear regression equation, the predicted fuel consumption is approximately 136.3 billion gallons.
2. The prediction made above is approximately 900 million gallons less than the actual value.



3. a. Sample scatterplot:

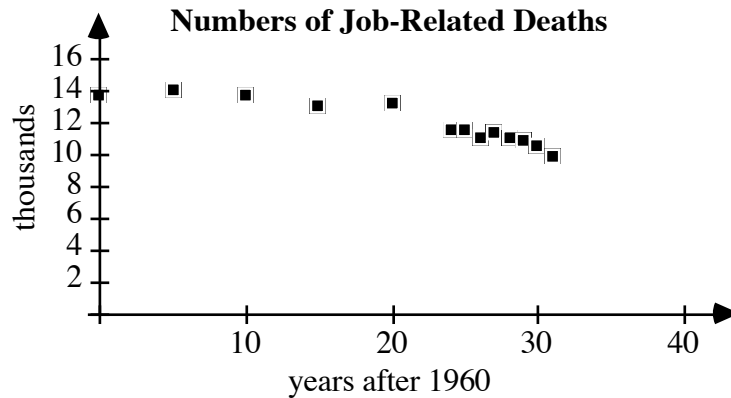


- b. Using 1980 as  $x = 0$ , the regression line is  $y = 1.765x + 130.335$ .
- c. Using 1980 as  $x = 0$ , the median-median line is  $y = 1.606x + 131.374$ .
- d. Both lines appear to be reasonable models. The sum of the squares of the residuals for the regression line is 24.868, while the sum of the squares of the residuals for the median-median line is 27.756. As shown below, the residual plots are similar.

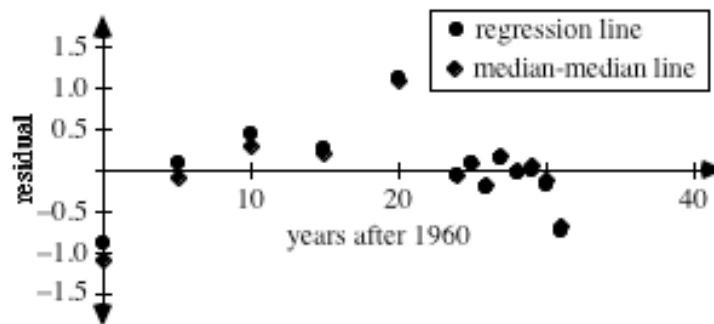


- e. 1. Using the linear regression equation, the predicted milk production is approximately 155 billion pounds.
2. The prediction made above is approximately 1.4 billion pounds more than the actual value.

4. a. Sample scatterplot:



- b. Using 1960 as  $x = 0$ , the regression line is  $y = -0.130x + 14.67$ .
- c. Using 1960 as  $x = 0$ , the median-median line is  $y = -0.139x + 14.882$ .
- d. Both lines appear to be reasonable models, although some students may observe that the number of deaths seems to be decreasing more rapidly in the 1990s. The sum of the squares of the residuals for the regression line is 8.054, while the sum of the squares of the residuals for the median-median line is 9.328. As shown in the sample graph below, the residual plots are similar.

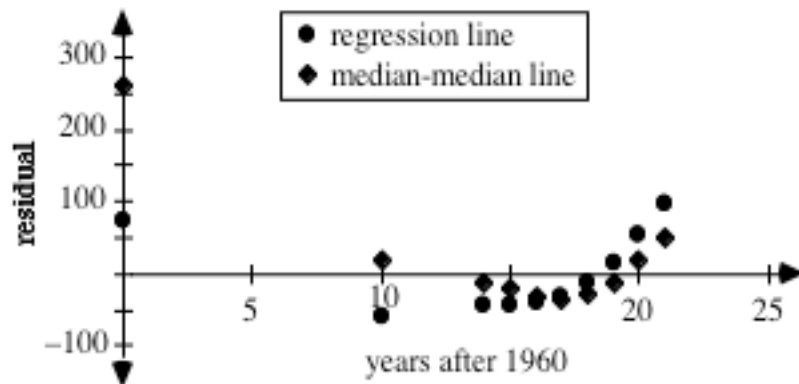


- e. 1. Using the linear regression equation, the predicted number of job-related deaths in 1993 is approximately 10,380.
2. The prediction made above is approximately 1280 more than the actual value.

5. a. Sample scatterplot:



- b. Using 1970 as  $x = 0$ , the regression line is  $y = 30.212x - 3.620$ .
- c. Using 1970 as  $x = 0$ , the median-median line is  $y = 41.350x - 191.192$ .
- d. Neither line appears to be an appropriate model. The sum of the squares of the residuals for the regression line is 28,037, while the sum of the squares of the residuals for the median-median line is 74,668. The residual plots both show a definite pattern. Students may observe that an exponential equation may be a more appropriate model.



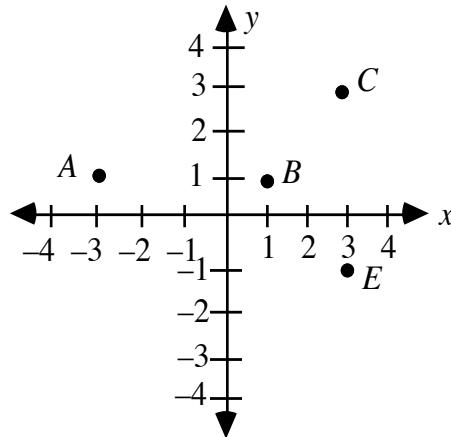
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- North Carolina School of Science and Mathematics, Department of Mathematics and Computer Science. *Contemporary Precalculus through Applications*. Dedham, MA: Janson Publications, 1992.
- Rodahl, K. *The Physiology of Work*. Bristol, PA: Taylor & Francis, 1989.
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- U.S. Bureau of the Census, *Statistical Abstract of the United States: 1995*. Washington, DC: U.S. Government Printing Office, 1995.

# Flashbacks

## Activity 1

- 1.1** Determine the equation of the line in the form  $y = mx + b$  that contains each of the following pairs of points:
- $A$  and  $C$
  - $B$  and  $E$



- 1.2** Given the equation  $y = 3x - 2$ , determine the value of  $x$  for each of the following values of  $y$ .
- $y = 13$
  - $y = -11$
- 1.3** Given the linear equation  $y = -4x + b$ , find the value of  $b$  when the line contains each of the following points:
- $(-2, 3)$
  - $(1, -2.5)$

## Activity 2

- 2.1** Find the median of the following set of numbers: {12, 45, 32, 81, 65, 24, 35, 26, 48, 50}.
- 2.2** Determine the equation of the line that passes through the points  $(-3, 4)$  and  $(6, -2)$ .

## Activity 3

- 3.1** Determine the slope of the line that passes through each of the following pairs of points:
- a.**  $(7, -4)$  and  $(2, -5)$
  - b.**  $(1, -2)$  and  $(4, -6)$ .
- 3.2** Find the equation of the line in the form  $y = mx + b$  that contains each of the following pairs of points:
- a.**  $(1, 1)$  and  $(5, 5)$
  - b.**  $(1, -2)$  and  $(4, -6)$ .

## Answers to Flashbacks

### Activity 1

1.1 a.  $y = \frac{1}{3}x + 2$

b.  $y = -1x + 2$

1.2 a.  $x = 5$

b.  $x = -3$

1.3 a.  $b = -5$

b.  $b = 1.5$

### Activity 2

2.1 The median is 40.

2.2  $y = -\frac{2}{3}x + 2$

### Activity 3

3.1 a.  $1/5$

b.  $-4/3$

3.2 a.  $y = x$

b.  $y = -\frac{4}{3}x - \frac{2}{3}$







# Take It to the Limit



What do hungry chickens, chain letters, and the perimeters of nested triangles have in common? This module introduces you to infinite sequences—and their limits.

*Todd Fife • Mark Lutz • Lisa Wood*



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## Teacher Edition

# Take It to the Limit

### Overview

In this module, students explore arithmetic and geometric sequences and series. They are introduced to limits through infinite geometric sequences and series.

### Objectives

In this module, students will:

- identify sequences that are arithmetic, geometric, or neither
- develop formulas for finite arithmetic and geometric series
- develop a formula for certain infinite geometric series
- explore limits graphically and geometrically.

### Prerequisites

For this module, students should know:

- how to interpret subscript notation
- how to find explicit and recursive formulas for arithmetic sequences
- how to find explicit and recursive formulas for geometric sequences.

### Time Line

Activity	1	2	3	4	Summary Assessment	Total
Days	3	2	2	2	1	10

### Materials Required

Materials	Activity				
	1	2	3	4	Summary Assessment
graph paper			X		

## Technology

Software	Activity				Summary Assessment
	1	2	3	4	
geometry utility				X	X
graphing utility				X	
spreadsheet	X	X	X	X	X
symbolic manipulator	X	X	X	X	X

# Take It to the Limit

## Introduction

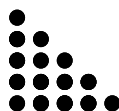
(page 375)

In this introduction, students review arithmetic and geometric sequences along with explicit and recursive formulas.

## Discussion

(page 375)

- a.
1. This is an arithmetic sequence. The first term is 7 and the common difference is 6. The next term is 31.
  2. This is a geometric sequence. The first term is 162 and the common ratio is  $1/3$ . The next term is 2.
  3. Sample response: This is a sequence of figures. Each figure is a set of dots forming a right triangle with  $n$  dots on each leg and on the hypotenuse, where  $n$  is the figure number. The next figure is:



4. This is a geometric sequence. The first term is  $-7$  and the common ratio is  $-7$ . The next term is  $-16,807$ .
5. This is a sequence of closed intervals. The left-hand value of the interval is always 2, while the right-hand value is  $2 + (1/n)$ , for  $n = 1, 2, 3, 4, \dots$ . The next interval is  $[2, 2.2]$ .
6. This is a Fibonacci sequence. The first two terms are 4 and 5. Each successive term is the sum of the previous two terms. The next term is 37.

- b. A sample formula for each sequence shown below.

1. 
$$\begin{cases} a_1 = 7 \\ a_n = a_{n-1} + 6 \text{ for } n > 1 \end{cases}$$

2. 
$$\begin{cases} a_1 = 162 \\ a_n = a_{n-1}(1/3) \text{ for } n > 1 \end{cases}$$

3. Although it is not possible to write a recursive formula for the sequence of figures, the number of dots in each figure can be described as follows:

$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + n, n > 1 \end{cases}$$

4. 
$$\begin{cases} a_1 = -7 \\ a_n = -7a_{n-1} \text{ for } n > 1 \end{cases}$$

5. A recursive formula for the right-hand value in each interval is shown below:

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} - \frac{1}{n(n-1)} \text{ for } n > 1 \end{cases}$$

6. 
$$\begin{cases} a_1 = 4 \\ a_2 = 5 \\ a_n = a_{n-2} + a_{n-1} \text{ for } n > 2 \end{cases}$$

- c. A sample formula for each sequence shown below.

1.  $a_n = 7 + 6(n - 1)$  for  $n \geq 1$

2.  $a_n = 162(1/3)^{n-1}$  for  $n \geq 1$

3. Although it is not possible to write an explicit formula for the sequence of figures, the number of dots in each figure can be described as follows:

$$a_n = \frac{n(n+1)}{2}$$

4.  $a_n = -7(-7)^{n-1}$  for  $n \geq 1$

5. An explicit formula for the right-hand value in each interval is shown below:

$$b_n = 2 + \frac{1}{n}, n > 1$$

6. Determining an explicit formula for this sequence is beyond the scope of this module.

- d. 1. The sequence in Step 1 is arithmetic.

2. The sequences in Steps 2 and 4 are geometric.

- e. Sample response: The common difference of an arithmetic sequence can be found by subtracting any term from the following term.

- f. Sample response: The common ratio of a geometric sequence can be found by dividing any term by the previous term.

## *Activity 1*

This activity continues the review of arithmetic and geometric sequences, as well as recursive and explicit formulas. **Note:** These concepts were introduced in the Level 1 module “From Rock Bands to Recursion.”

### Materials List

- none

### Technology

- spreadsheet (optional)
- symbolic manipulator (optional)

### Exploration

(page 377)

This exploration is designed for groups of four students. To ensure equal participation, each student should create a sequence, and each group should pass in a consistent clockwise or counterclockwise rotation.

The following responses describe a sample game.

- a. The first student creates an arithmetic sequence with a common difference of  $-2$  and a first term of  $20$ . The student writes down the first five terms  $-20, 18, 16, 14, 12$ —and passes the sheet of paper to the second student.
- b. The second student identifies the sequence as arithmetic and writes this below the first five terms. The student then writes the following rule: “To make this sequence, start with  $20$  and add  $-2$  (or subtract  $2$ ) every time to get the next term.”
- c. The third student uses this algorithm to write the recursive formula

$$\begin{cases} a_1 = 20 \\ a_n = a_{n-1} + (-2) \text{ for } n \geq 1 \end{cases}$$

and the explicit formula  $a_n = 20 + (-2)(n - 1)$  for  $n \geq 1$ .

- d. The fourth student uses these formulas to write the first five terms of the sequence:  $20, 18, 16, 14, 12$ .
- e. The first student compares the sequence written by the fourth student with the original sequence. (If the terms do not match, the group should examine the paper and identify the source of the mistake.)

## Discussion

(page 377)

- a. Responses will vary.
- b. Sample response: For arithmetic sequences, the difference between consecutive terms must be the same. For geometric sequences, the ratio of any two successive terms must be the same. A sequence that does not have a common ratio or a common difference between successive terms is neither arithmetic nor geometric.
- c.
  - 1. If the sequence is arithmetic with a common difference of 1, the next three terms will be 4, 5, 6.
  - 2. If the next term in the sequence is the sum of the two previous terms, then the next three terms will be 5, 8, 13.
  - 3. If the next term in the sequence is the sum of the previous three terms, then the next three terms will be 6, 11, 20.
- d. Sample response: Yes. Since the sequence is defined as arithmetic, then there must be a common difference and the next three terms would follow the same pattern.

## Assignment

(page 378)

- 1.1
  - a. The first five terms are 9, 6, 3, 0, -3.
  - b. The first five terms are -5, -6.5, -8.45, -10.985, -14.2805.
- 1.2
  - a.  $99 = 11 + 2(n - 1) \Rightarrow n = 45$
  - b.  $6.5536 = 20(0.8)^{n-1} \Rightarrow n = 6$
  - c.  $357 = 7 + 3.5(n - 1) \Rightarrow n = 101$
  - d.  $3072 = 3(2)^{n-1} \Rightarrow n = 11$
- \*1.3
  - a.
    - 1. Two possible terms are 63, 127.
    - 2. The explicit formula is  $a_n = 2^n - 1$  for  $n \geq 1$ . The recursive formula is:
$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 2^{n-1} \text{ for } n > 1 \end{cases}$$
    - 3. The sequence is neither arithmetic nor geometric.
  - b.
    - 1. Two possible terms are  $-2/25$ ,  $2/125$ .
    - 2. The explicit formula is  $g_n = -50(-1/5)^{n-1}$  for  $n \geq 1$ . The recursive formula is:
$$\begin{cases} g_1 = -50 \\ g_n = (-1/5)g_{n-1} \text{ for } n > 1 \end{cases}$$
    - 3. The sequence is geometric.



- c. 1. Two possible terms are 13, 16.  
 2. The explicit formula is  $a_n = 1 + 3(n - 1)$  for  $n \geq 1$ . The recursive formula is:

$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 3 \text{ for } n > 1 \end{cases}$$

3. This is an arithmetic sequence.  
 d. 1. Two possible terms are  $2\frac{1}{6}$ ,  $2\frac{1}{7}$ .  
 2. The explicit formula is:

$$a_n = 2 + \frac{1}{n+1} \text{ for } n \geq 1$$

The recursive formula is:

$$\begin{cases} a_1 = 2\frac{1}{2} \\ a_n = a_{n-1} - \frac{1}{n(n+1)} \text{ for } n > 1 \end{cases}$$

3. This sequence is neither arithmetic nor geometric.  
 e. 1. Two possible terms are 6, -6.  
 2. The explicit formula is  $a_n = 6(-1)^{n-1}$  for  $n \geq 1$ . The recursive formula is:

$$\begin{cases} a_1 = 6 \\ a_n = -1(a_{n-1}) \text{ for } n > 1 \end{cases}$$

3. This sequence is geometric.

- 1.4 a. Any geometric sequence with a common ratio of 1 is also an arithmetic sequence with a common difference of 0. For example, the following sequence is both arithmetic and geometric:

$$\pi, \pi, \pi, \pi, \pi, \dots$$

- b. An explicit formula for the sample sequence given in Part a is  $a_n = \pi + 0(n - 1)$  for  $n \geq 1$ .  
 c. An explicit formula for the sample sequence given in Part a is  $g_n = \pi \cdot (1)^{n-1}$  for  $n \geq 1$ .

- 1.5** a. The common difference can be found by solving the following equation:

$$93 = 13 + d(50 - 1)$$

$$d = 80/49$$

- b. The common ratio can be found by solving the equation below:

$$192 = 12 \cdot r^{5-1}$$

$$r = \pm 2$$

- c. The common difference is 3 and the number of terms is 18.

- d. The common ratio is 5 and the number of terms is 10.

- 1.6** Sample response: The dog's mass would be 43.2 kg. If the dog's loss of mass is actually an arithmetic sequence, then the dog would lose 0.9 kg per month for 12 months—a total of 10.8 kg. Depending on the size of the animal, this might not be reasonable. The veterinarian would not let the dog's mass fall below a healthy level.

- 1.7** a. Sample response: If the mass of a chick is 36 g when hatched, and it gains 10% of its mass each day for 8 weeks, then its daily mass is a geometric sequence with the following formula:

$$g_n = 36 \cdot 1.1^{n-1}. \text{ After 56 days, the chick's mass would be about 6.8 kg.}$$

- b. This is probably not reasonable. Students may know from experience that the mass of a broiler chicken is typically under 2 kg. According to most poultry growers, the mass of an 8-week-old chicken should be about 43 times its mass at birth, or about 1548 g.

- c. The advertiser should claim a 7% daily mass gain.

- 1.8** The following sample responses assume that the first term is positive.

- a. If the common ratio is between  $-1$  and  $0$ , then the sequence will alternate between positive and negative terms and will get progressively closer to  $0$ .
- b. If the common ratio is between  $0$  and  $1$ , then the sequence will get progressively closer to  $0$ .
- c. If the common ratio is less than  $-1$ , then the sequence will alternate between positive and negative terms. The absolute value of the terms will get progressively larger.
- d. If the common ratio is greater than  $1$ , the terms of the sequence will get progressively larger.

\* \* \* \* \*

- \*1.9 a–b.** Students generate sequences by paper folding.
- c.** The explicit formula for the area of the top surface (in  $\text{cm}^2$ ) is  $g_n = 128 \cdot (0.5)^{n-1}$  for  $n \geq 1$ . The recursive formula for the area of the top surface is:

$$\begin{cases} g_1 = 128 \\ g_n = g_{n-1} \cdot 0.5 \text{ for } n > 1 \end{cases}$$

- d.** The explicit formula for the number of layers is  $g_n = 2 \cdot 2^{n-1}$  for  $n \geq 1$ . The recursive formula for the number of layers is:

$$\begin{cases} g_1 = 2 \\ g_n = g_{n-1} \cdot 2 \text{ for } n > 1 \end{cases}$$

- e.** As the number of folds increases, the area of the top surface appears to approach  $0 \text{ cm}^2$ .
- f.** As the number of folds increases, the number of layers becomes larger and larger.
- 1.10 a.** The sheet of paper would have to have an area of about  $5.6 \cdot 10^{14} \text{ cm}^2$ . Its dimensions would be about  $2.4 \cdot 10^7 \text{ cm}$  (or 240 km) on each side.
- b.** Answers will vary, depending on the thickness of the paper. If a single layer were 0.1 mm thick, the height of the stack after 50 cuts would be about  $1.1 \cdot 10^8 \text{ km}$ .
- 1.11 a.** This is a Fibonacci sequence with the first two terms equal to 1. It is neither arithmetic nor geometric.
- b.** This is a geometric sequence with a common ratio of 1.05 and a first term of 2.
- c.** This is an arithmetic sequence with a common difference of  $-1000$  and a first term of 2000.
- d.** This sequence (0, 1, 3, 6, 10, 15, 21, ...) is neither arithmetic nor geometric. The pattern can be described by the recursive formula.
- e.** This sequence is neither arithmetic nor geometric. The first term is  $3/1$ . In each successive term, the numerator remains constant at 3 and the denominator increases by 1.
- 1.12 a.** The terms increase without bound.
- b.** The terms increase without bound.
- c.** The terms decrease without bound.
- d.** The terms increase without bound.
- e.** The terms approach 0.

\* \* \* \* \*

## Activity 2

This activity introduces students to finite arithmetic series and their expanded form.

### Materials List

- none

### Technology

- spreadsheet (optional)
- symbolic manipulator (optional)

### Exploration

(page 382)

- a. 1. Sample response:

$$\begin{aligned} S_{1000} &= 1 + 2 + \cdots + 999 + 1000 \\ S_{1000} &= 1000 + 999 + \cdots + 2 + 1 \\ \hline 2S_{1000} &= 1001 + 1001 + \cdots + 1001 + 1001 \end{aligned}$$

Since there are a total of 1000 terms of 1001:

$$S_{1000} = \frac{1000 \cdot 1001}{2} = 500,500$$

2. Sample response:

$$\begin{aligned} S_n &= n + (n-1) + \cdots + 2 + 1 \\ S_n &= 1 + 2 + \cdots + (n-1) + n \\ \hline 2S_n &= (n+1) + (n+1) + \cdots + (n+1) + (n+1) \end{aligned}$$

Since there are a total of  $n$  terms of  $(n+1)$ :

$$S_n = \frac{n \cdot (n+1)}{2}$$

- b. Sample response:

$$\begin{aligned} S_{50} &= 2 + 4 + \cdots + 998 + 100 \\ S_{50} &= 100 + 998 + \cdots + 4 + 2 \\ \hline 2S_{50} &= 102 + 102 + \cdots + 102 + 102 \end{aligned}$$

Since there are a total of 50 terms of 102:

$$S_{50} = \frac{50 \cdot (102)}{2} = 2550$$

- c. Using the explicit formula for an arithmetic sequence,  $a_{75} = 2 + (75 - 1) \cdot 4 = 298$ . Given this value, students should use a method like the one employed in Parts **a** and **b**:

$$\begin{aligned} S_{75} &= 2 + 6 + \cdots + 294 + 298 \\ S_{75} &= 298 + 294 + \cdots + 6 + 2 \\ \hline 2S_{75} &= 300 + 300 + \cdots + 300 + 300 \end{aligned}$$

Since there are a total of 75 terms of 300:

$$S_{75} = \frac{75 \cdot (300)}{2} = 11,250$$

- d. 1.  $a_2 = a_1 + d$ ,  $a_3 = a_1 + 2d$ ,  $a_4 = a_1 + 3d$   
 2.  $a_n = a_1 + (n - 1) \cdot d$ ,  $a_{n-1} = a_1 + (n - 2) \cdot d$ ,  $a_{n-2} = a_1 + (n - 3) \cdot d$ ,  
 $a_{n-3} = a_1 + (n - 4) \cdot d$   
 3. Using a method like the one employed in Parts **a** and **b**:

$$\begin{aligned} S_n &= a_1 + [a_1 + d] + \cdots + [a_1 + (n - 2)d] + [a_1 + (n - 1)d] \\ S_n &= [a_1 + (n - 1)d] + [a_1 + (n - 2)d] + \cdots + [a_1 + d] + a_1 \\ \hline 2S_n &= [2a_1 + (n - 1)d] + [2a_1 + (n - 1)d] + \cdots + [2a_1 + (n - 1)d] + [2a_1 + (n - 1)d] \end{aligned}$$

Since there are a total of  $n$  terms of  $[2a_1 + (n - 1)d]$ :

$$S_n = \frac{n \cdot [2a_1 + (n - 1)d]}{2} = na_1 + \frac{n \cdot [(n - 1)d]}{2}$$

## Discussion

(page 383)

- a. The equation below

$$S_n = \frac{n}{2}[2a_1 + (n - 1) \cdot d]$$

can be expanded as follows:

$$S_n = \frac{n}{2}[a_1 + a_1 + (n - 1) \cdot d]$$

Since  $a_n = a_1 + (n - 1) \cdot d$ , students can make the following substitution:

$$Sn = \frac{n}{2}(a_1 + a_n)$$

- b. The sequences in Parts **a** and **b** are arithmetic sequences because there is a common difference between consecutive terms.

- c. 1. Sample response: The sum of the first 100 even numbers should be twice the sum of the first 100 natural numbers because:
- $$2 + 4 + 6 + \cdots + 200 = 2(1 + 2 + 3 + \cdots + 100) = 2(5050) = 10,100$$
2. The sum of the first 100 multiples of  $k$  can be found as follows:
- $$k + 2k + 3k + \cdots + 99k + 100k = k(1 + 2 + 3 + \cdots + 99 + 100)$$
- $$= k \frac{100(101)}{2} = 5050k$$
- d. Sample response: If the common difference is positive, the sum of the infinite arithmetic series will increase without bound. If the common difference is negative, the sum of the infinite arithmetic series will decrease without bound.

## Assignment

(page 383)

- 2.1 The sum of the first  $n$  even numbers can be found as follows:

$$S_n = \frac{n}{2}(2 + 2n)$$

$$= \frac{n}{2} \cdot 2(1 + n)$$

$$= n(n + 1)$$

- 2.2 Student methods may vary. Using the formula from Part a of the exploration:

$$S_n = \frac{n(n + 1)}{2} = \frac{2000(2000 + 1)}{2} = 2,001,000$$

- 2.3 a. Sample response:

$$k + 2k + 3k + \cdots + (n - 1)k + nk = k(1 + 2 + 3 + \cdots + (n - 1) + n)$$

$$= k \frac{n(n + 1)}{2}$$

- b. The sum can be found as follows:

$$\frac{3(600)(600 + 1)}{2} = 540,900$$

- \*2.4 a. The first term of the sequence is 3.
- b. The common difference is 4.
- c. Using the explicit formula for an arithmetic sequence:

$$451 = 3 + 4(n - 1)$$

$$113 = n$$

- d. The sum is the finite arithmetic series  $3 + 7 + 11 + \cdots + 451$ . Using the formula for an arithmetic series:

$$S_{113} = \frac{113}{2}(3 + 451) = 25,561$$

- 2.5 Sample response: No, because the sum of each pair of numbers is not constant. For example, given the sequence 32, 16, 8, 4, 2, 1, the sums of the pairs are 33, 18, 12, 18, and 33.

- \*2.6 a. Sample response: The monthly payments may be considered an arithmetic sequence where the first term is \$106.26 and the common difference is \$106.26.
- b. Sample response: Yes. Since the payments themselves form an arithmetic sequence, their indicated sum forms an arithmetic series.
- c. The lessee pays  $\$150.00 + \$1,600.00 + (36 \cdot \$106.26)$  or \$5575.36, of which \$150.00 is refunded at the end of the lease.
- d. Sample response: The difference of \$3312.64 may be the cost to purchase the car at the end of the lease. Or it may be that the car costs an additional \$8,888.00 at the end of the lease.

\* \* \* \* \*

- 2.7 The number of newspapers delivered each week forms an arithmetic sequence where  $a_1 = 15,000$  and  $d = 50$ . Using the formula for an arithmetic series:

$$S_{52} = \frac{52}{2}(2 \cdot 15,000 + 51 \cdot 50) = 846,300$$

- 2.8 a. The first three terms are  $-2, 3, 8$ .

b.  $a_{150} = -2 + 5(149) = 743$

c.  $S_{150} = \frac{150}{2}(-2 + 743) = 55,575$

- 2.9 a.  $S_{41} = \frac{41}{2}(5 + 120) = 2562.5$

- b. Using the general formula for an arithmetic sequence, solve  $120 = 5 + 40d$  and the common difference is 2.875.

- c. The second and third terms of the sequence are 7.875 and 10.75.

\* \* \* \* \*

## Activity 3

In this activity, students develop an algorithm for finding the sum of the terms of a finite geometric sequence.

### Materials List

- graph paper (two sheets per student)

### Technology

- spreadsheet
- symbolic manipulator (optional)

### Exploration

(page 385)

- a. The number of e-mail warnings sent is 6.
- b. Since 6 people send 3 e-mail warnings each, the second group consists of  $6 \cdot 3 = 18$  people. If all decide to participate, the number of e-mail warnings sent is 18.
- c. The number of e-mail warnings sent at the third stage is 54, at the fourth stage is 162, and at the fifth stage is 486.
- d. The number of e-mail warnings sent at each stage forms the following geometric sequence: 6, 18, 54, ...,  $6 \cdot 3^{199}$ .
- e. The following geometric series (equation 1) represents the total number of warnings sent:

$$S_{200} = 6 + 18 + 54 + \cdots + 6 \cdot 3^{197} + 6 \cdot 3^{198} + 6 \cdot 3^{199}$$

- f. 1. Multiplying both sides of the equation above by 3 yields equation 2:

$$3S_{200} = 18 + 54 + 162 + \cdots + 6 \cdot 3^{198} + 6 \cdot 3^{199} + 6 \cdot 3^{200}$$

2. Subtracting equation 1 from equation 2 as shown below:

$$\begin{array}{r} 3S_{200} = 18 + 54 + \cdots + 6 \cdot 3^{197} + 6 \cdot 3^{198} + 6 \cdot 3^{199} + 6 \cdot 3^{200} \\ -(S_{200} = 6 + 18 + 54 + \cdots + 6 \cdot 3^{197} + 6 \cdot 3^{198} + 6 \cdot 3^{199}) \\ \hline 2S_{200} = 6 \cdot 3^{200} - 6 \end{array}$$

3. Solving for  $S_{200}$ , the number of e-mail warnings sent is:

$$S_{200} = \frac{6 \cdot 3^{200} - 6}{2} \approx 7.96842 \cdot 10^{95}$$



- g. Student answers may vary, but the general formula should appear as follows:

$$S = \frac{g_1 r^n - g_1}{r - 1} = \frac{g_1 (r^n - 1)}{r - 1}$$

## Discussion

(page 386)

- a. Sample response: If every recipient continues to e-mail warnings to three of their friends, we would expect to send approximately  $7.96842 \cdot 10^{95}$  e-mail warnings. This is not reasonable, since it far exceeds the earth's population.
- b. Sample response: If the virus is a hoax and people e-mail all their friends, the number of e-mail messages can quickly overload servers. This can stop the e-mail or at least slow it down. Whenever you receive an e-mail warning of a computer virus, check with web sources first to insure the warning is legitimate.
- c. Answers may vary, but the general formula should appear as follows:

$$S = \frac{g_1 r^n - g_1}{r - 1} = \frac{g_1 (r^n - 1)}{r - 1}$$

**Note:** Some students may have obtained the equivalent form shown below (found by multiplying both the numerator and the denominator by  $-1$ ):

$$S = \frac{g_1 (1 - r^n)}{1 - r}$$

- d. Sample response: If  $r$  equals 1, the denominator in the formula is 0. This makes the value of the fraction undefined.
- e. 1. Sample response:  $S_n = 4_1 + 4_2 + \dots + 4_{n-1} + 4_n$
2. Sample response: If  $r$  is 1, then each term is equal to the first term, and the sequence is arithmetic with a common difference of 0. The sum of  $n$  terms can be found as follows:  $S_n = n \cdot g_1$ .
- f. 1. These are equivalent by using the distributive property and factoring out  $g_1$ .
2. These are equivalent by factoring out  $r$  from  $g_1 r^n$  and substituting  $g_n$  for  $g_1 r^{n-1}$ .

g.

$$S_n = g_1 r^0 + g_1 r^1 + g_1 r^2 + \dots + g_1 r^{n-2} + g_1 r^{n-1}$$

$$rS_n = g_1 r^1 + g_1 r^2 + g_1 r^3 + \dots + g_1 r^{n-1} + g_1 r^n$$

$$-(S_n = g_1 r^0 + g_1 r^1 + g_1 r^2 + \dots + g_1 r^{n-2} + g_1 r^{n-1})$$

$$rS_n - S_n = g_1 r^n - g_1 r^0$$

$$S_n (r - 1) = g_1 r^n - g_1$$

$$S_n = \frac{g_1 r^n - g_1}{r - 1} = \frac{g_1 (r^n - 1)}{r - 1}$$

## Assignment

(page 387)

3.1 a.  $S_{17} = \frac{(3.5)(0.6)^{17} - 3.5}{0.6 - 1} \approx 8.75$

b.  $S_{10} = \frac{(10)(10)^{10} - 10}{10 - 1} = 11,111,111,110$

3.2 a.  $S = \frac{(6.5536)(0.8) - 20}{0.8 - 1} \approx 73.79$

b.  $S = \frac{(3072)2 - 3}{2 - 1} = 6141$

3.3 a. This is an arithmetic series.

$$S_{45} = \frac{45}{2}(11 + 99) = 2475$$

b. This is an arithmetic series.

$$S_7 = \frac{7}{2}(2 + (-28)) = -91$$

c. This series is neither arithmetic nor geometric.

$$S_6 = \frac{1903}{140} \approx 13.59$$

d. This is a geometric series.

$$S_5 = \frac{\frac{1}{2} \cdot \left(\frac{1}{2}\right)^5 - \frac{1}{2}}{\frac{1}{2} - 1} = \frac{31}{32} \approx 0.97$$

\*3.4 a. Sample response: When the tortoise has run 1.11111 m, six time intervals have elapsed since

$$1.11111 = 1 + 0.1 + 0.01 + 0.001 + 0.0001 + 0.00001$$

b. To determine the total distance Achilles covered, I could multiply the distance covered by the tortoise by 10, or use the formula for a finite geometric series:

$$S_6 = \left( \frac{10(0.1)^6 - 10}{0.1 - 1} \right) = 11.11111$$

c. Sample response: No, with its 10-m head start, the tortoise is 11.11111 m from Achilles' starting point. After six time intervals, Achilles has only traveled 11.1111 m. He is still 0.00001 m behind.

- d. Students may respond in a number of different ways. Some possible responses are given below:
- No, because the tortoise would always be some distance ahead of Achilles, even though that distance is getting smaller and smaller.
  - Yes, because the lead of the tortoise will become so small that it will be impossible to determine if the tortoise is ahead.
  - Yes, because if Achilles travels 10 m/sec, in 2 sec, he will cover 20 m while the tortoise—who travels 1/10 as fast, or 1 m/sec—will have moved only 12 m (including the 10-m head start). Therefore, at some time during the first 2 sec, Achilles must pass the tortoise.
  - Yes, because the total distance traveled by both Achilles and the tortoise appears to approach a maximum of  $11\frac{1}{9}$  m.

- \*3.5 a. Using the formula for the sum of a geometric series, the total number of wheat kernels in the reward is:

$$S_{64} = \frac{1 \cdot 2^{64} - 1}{2 - 1} \approx 1.8 \cdot 10^{19}$$

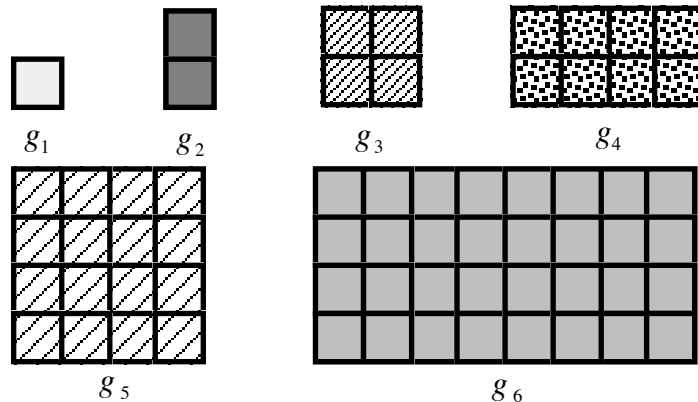
- b. The wheat would cover approximately  $2.4 \cdot 10^8$  km<sup>2</sup>.  
 c. The wheat would be 24 layers thick

- 3.6 Using the formula for the sum of a geometric series, the total number of wheat kernels in the reward is:

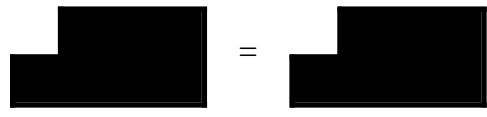
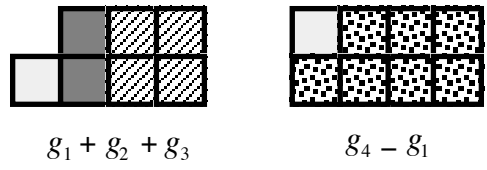
$$S_{64} = \left( \frac{1(3)^{64} - 1}{3 - 1} \right) \approx 1.72 \cdot 10^{30}$$

This is enough wheat to cover the United States with approximately 2.3 trillion layers. **Note:** This is a good opportunity to compare the magnitudes of numbers like  $2^{64}$  and  $3^{64}$ .

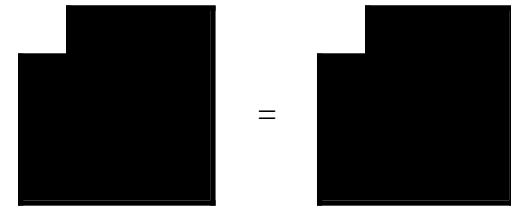
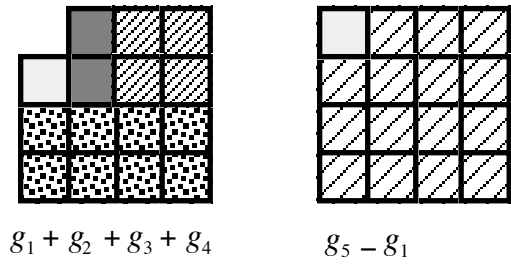
- \*3.7 a. The first six terms of the sequence are illustrated below:



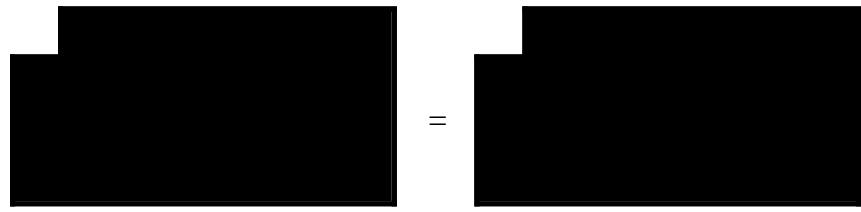
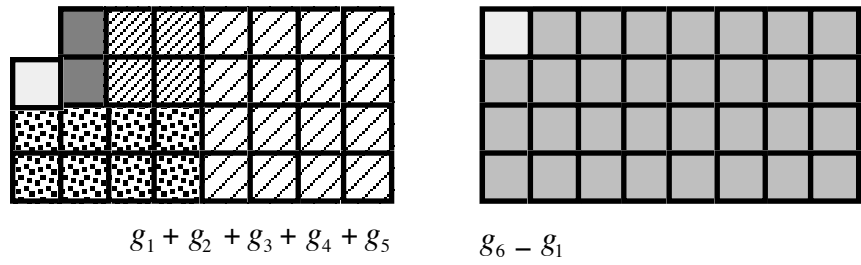
b. 1. Sample response:



2. Sample response:



3. Sample response:



c. Sample response: The sum of the terms is equal to the difference between the term following the last term of the series and the first term of the series, or  $g_1 + g_2 + g_3 + \dots + g_n = g_{n+1} - g_1$ .

- d. Sample response: The sum of a geometric series can be found using the formula:

$$S_n = \frac{g_n r - g_1}{r - 1}$$

In this series  $r$  is 2. By substituting 2 for  $r$ , the formula becomes:

$$S_n = \frac{g_n(2) - g_1}{2 - 1}$$

or  $S_n = g_n(2) - g_1$ . Since  $g_n r$  represents the term following  $g_n$ ,  $g_{n+1}$  can be substituted for  $g_n(2)$ . The result is  $S_n = g_{n+1} - g_1$ .

\* \* \* \* \*

- 3.8** a. The geometric sequence is: 6, 4.8, 3.84, ...,  $6(0.8)^{n-1}$ .

b.  $S_n = 6 + 4.8 + 3.84 + \dots + 6(0.8)^{n-1}$

c. 1.  $S_{10} = \frac{6(0.8)^{10} - 6}{0.8 - 1} \approx 26.78$  m

2.  $S_{20} = \frac{6(0.8)^{20} - 6}{0.8 - 1} \approx 29.65$  m

3.  $S_{30} = \frac{6(0.8)^{30} - 6}{0.8 - 1} \approx 29.96$  m

4.  $S_{40} = \frac{6(0.8)^{40} - 6}{0.8 - 1} \approx 30.00$  m

- d. 1. The geometric series appears to approach 30 m.  
2. Sample response: The number is the sum of all the heights that the ball reaches.

**3.9** a.  $S_5 = 9 + 6 + 4 + \frac{8}{3} + \frac{16}{9}$  or  $S_5 = 9 + 9\left(\frac{2}{3}\right)^1 + 9\left(\frac{2}{3}\right)^2 + 9\left(\frac{2}{3}\right)^3 + 9\left(\frac{2}{3}\right)^4$

- b. The common ratio is  $2/3$ .

- c. 1. The height of the tree is:

$$4 + S_{10} = 4 + \frac{9\left(\frac{2}{3}\right)^{10} - 9}{\frac{2}{3} - 1} \approx 30.53$$
 m

2. The height of the tree is:

$$4 + S_{20} = 4 + \frac{9\left(\frac{2}{3}\right)^{20} - 9}{\frac{2}{3} - 1} \approx 30.99 \text{ m}$$

3. The height of the tree is:

$$4 + S_{30} = 4 + \frac{9\left(\frac{2}{3}\right)^{30} - 9}{\frac{2}{3} - 1} \approx 31.00 \text{ m}$$

$$4. \quad 4 + S_{40} = 4 + \frac{9\left(\frac{2}{3}\right)^{40} - 1}{\frac{2}{3} - 1} \approx 31.00 \text{ m}$$

- d. 1. The height appears to approach 31 m.  
2. The value represents the maximum height of the tree.

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## Activity 4

In this activity, students are introduced to limits, sequences that approach limits, and sequences that do not approach limits.

### Materials Required

- none

### Technology

- graphing utility
- spreadsheet
- geometry utility
- symbolic manipulator (optional)

### Exploration

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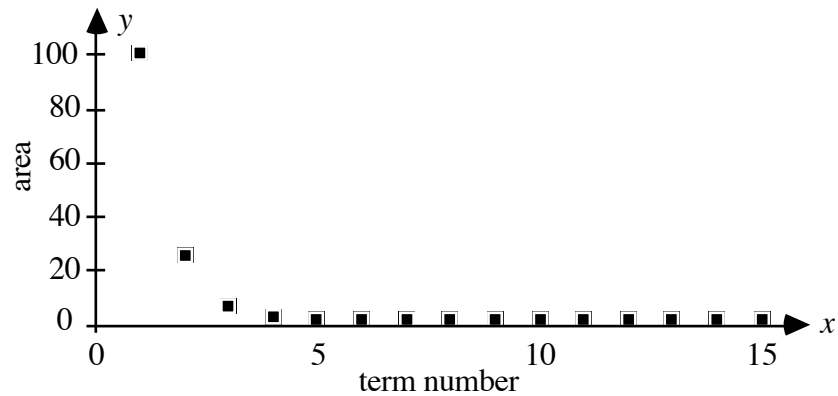
- a. Students create a sequence by connecting the midpoints of sides of successive triangles. See Figure 2 in student edition.

b. Sample table:

Term No.	Name of Triangle	Area (cm <sup>2</sup> )	Perimeter (cm)	Area of $\Delta$	Perimeter of $\Delta$
				Area of Previous $\Delta$	Perimeter of Previous $\Delta$
1	$\Delta ABC$	100.2	45.7	none	none
2	$\Delta DEF$	25.05	22.9	0.25	0.5
3	$\Delta GHJ$	6.3	11.4	0.25	0.5
4	$\Delta KLM$	1.6	5.7	0.25	0.5
5	⋮	0.4	2.9	0.25	0.5
6		9.8E-2	1.4	0.25	0.5
7		2.4E-2	7.1E-1	0.25	0.5
8		6.1E-3	3.6E-1	0.25	0.5
9		1.5E-3	1.8E-1	0.25	0.5
10		3.8E-4	8.9E-2	0.25	0.5
11		9.6E-5	4.5E-2	0.25	0.5
12		2.4E-5	2.2E-2	0.25	0.5
13		6.0E-6	1.1E-2	0.25	0.5
14		1.5E-6	5.6E-3	0.25	0.5
15		3.7E-7	2.8E-3	0.25	0.5

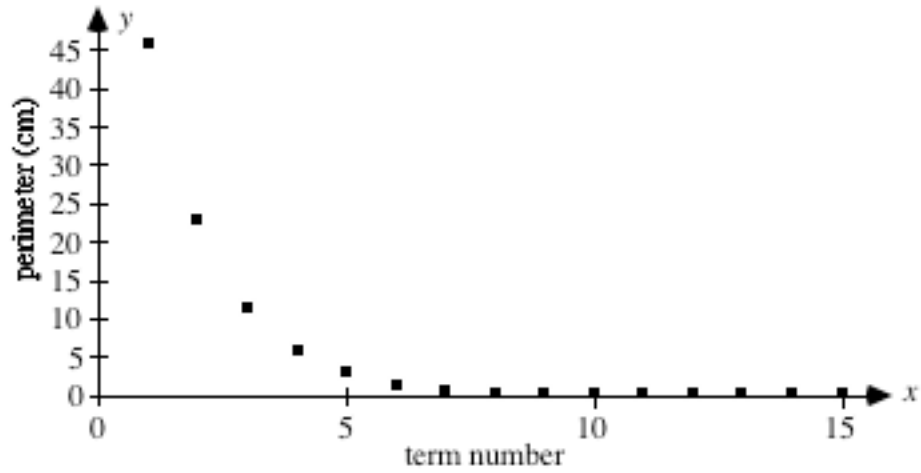
c. 1. See table given in Part b.

2. Sample scatterplot:



d. 1. See table given in Part b.

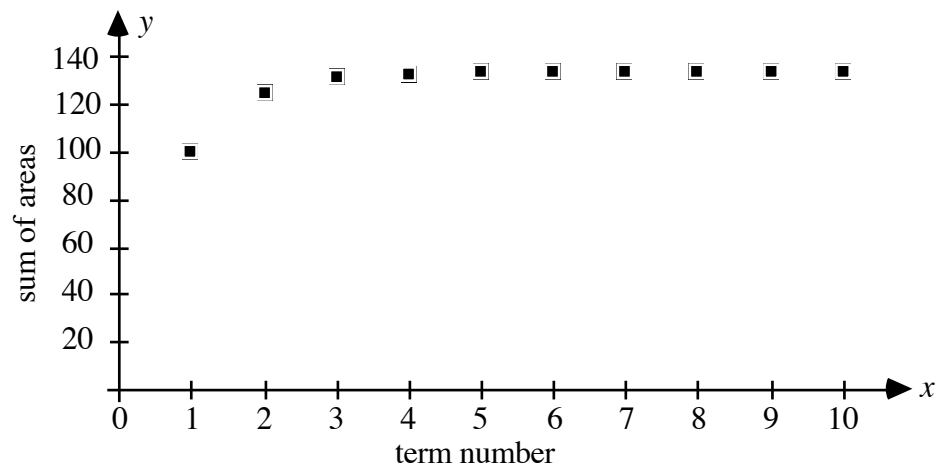
2. Sample scatterplot:



e. 1. The following table shows the sums for the sample triangle used in Parts **a** and **b** (rounded to the nearest hundredth).

Sum	Area (cm <sup>2</sup> )
$S_1$	100.20
$S_2$	125.25
$S_3$	131.51
$S_4$	133.07
$S_5$	133.47
$S_6$	133.57
$S_7$	133.59
$S_8$	133.60
$S_9$	133.60
$S_{10}$	133.60

2. Sample graph:





- f. The following table shows the sums for the sample triangle used in Parts a and b (rounded to the nearest hundredth).

Sum	Area (cm <sup>2</sup> )
$S_{20}$	133.60
$S_{30}$	133.60
$S_{50}$	133.60
$S_{100}$	133.60

## Discussion

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- a.
1. The area of each triangle is  $1/4$  the area of the previous triangle.
  2. The perimeter of each triangle is  $1/2$  the perimeter of the previous triangle.
- b.
1. Sample response: The area would approach 0 but never actually reach it.
  2. Sample response: The perimeter would approach 0 but never actually reach it.
- c.
- Sample response: When finding a limit, you can choose a prescribed accuracy however small you wish. This prescribed accuracy sets up boundaries above and below the conjectured limit. If all the terms after a certain term lie within the boundaries we picked, no matter how small, we know what the limit is as the number of terms increases indefinitely.
- d.
- Sample response: My answers are consistent with the graphs. The perimeters and areas appear to approach 0 after only a few terms.
- e.
1. Sample response: From the spreadsheet, the perimeter of each triangle after the 13th one is within  $1/100$  of 0.
  2. Sample response: From the spreadsheet, the area of each triangle after the 7th one is within  $1/100$  of 0.
- f.
1. As  $n$  increases indefinitely, the value of  $(1/2)^n$  approaches 0.
  2. As  $n$  increases indefinitely, the formula approaches the following expression:

$$S = \frac{g_1 \cdot (0 - 1)}{\frac{1}{2} - 1} = \frac{g_1 \cdot -1}{-\frac{1}{2}} = \frac{g_1}{\frac{1}{2}} = 2g_1$$

- g.
1. As  $n$  increases without bound, the value of  $r^n$  (for  $-1 < r < 1$ ) approaches 0.
  2. As  $n$  increases without bound, the formula approaches the following expression:

$$S = \frac{g_1 \cdot (0 - 1)}{r - 1} = \frac{g_1 \cdot -1}{r - 1} = \frac{g_1}{1 - r}$$

- h.
1. If  $r \leq -1$ ,  $|r^n|$  increases indefinitely as  $n$  increases, while  $r^n$  alternates between positive and negative values. If  $r > 1$ ,  $r^n$  increases indefinitely as  $n$  increases.
  2. Sample response: The formula is affected in the same manner as  $r^n$ .

### Assignment

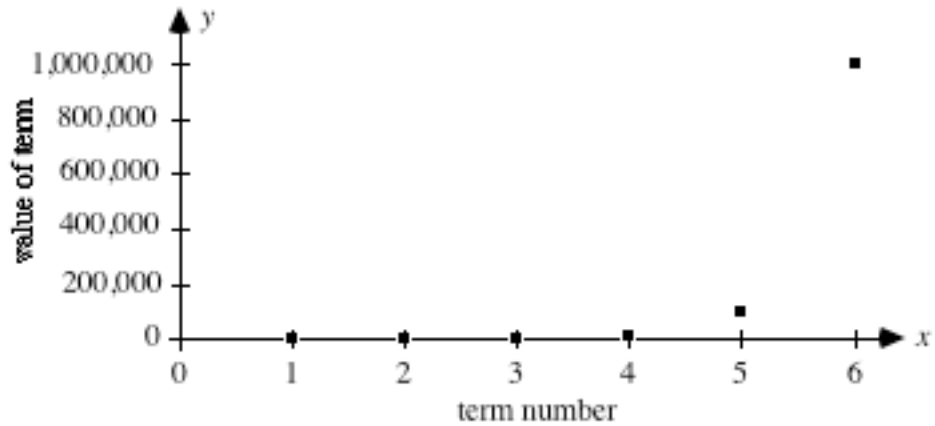
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- 4.1.
- a. Sample response: These terms do not appear to have a limit. As the term number increases, the value of the term appears to increase without bound.
  - b. Sample response: These terms appear to have a limit of 0. As the term number increases, the value of the term approaches 0.
  - c. Sample response: These terms appear to have a limit of 100 because all the terms are the same.

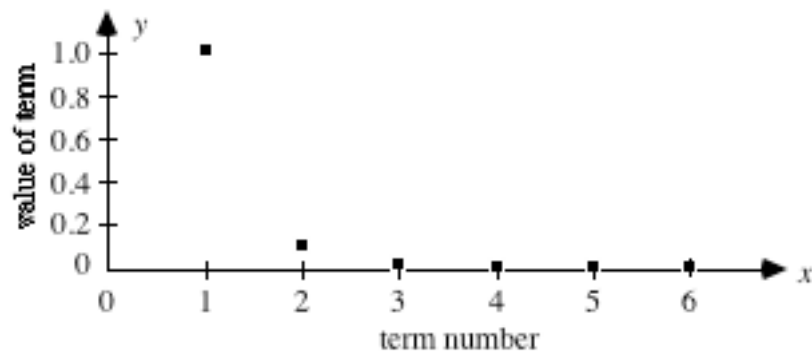
4.2 Sample spreadsheet:

A	B	C	D	E
1	1		1	
2	10	10	0.1	0.1
3	100	10	0.01	0.1
4	1000	10	0.001	0.1
5	10000	10	0.0001	0.1
6	100000	10	0.00001	0.1
7	1000000	10	0.000001	0.1
8	10000000	10	0.0000001	0.1
9	100000000	10	0.00000001	0.1
10	1000000000	10	1E-09	0.1
⋮	⋮	⋮	⋮	⋮
45	1E+44	10	1E-44	0.1
46	1E+45	10	1E-45	0.1
47	1E+46	10	1E-46	0.1
48	1E+47	10	1E-47	0.1
49	1E+48	10	1E-48	0.1
50	1E+49	10	1E-49	0.1

- a. The values in column B are increasing and do not appear to approach a limit. **Note:** Some students may respond that they approach infinity. You may wish to remind them that, although we often talk about positive and negative infinity, infinity is not an actual number.
- b. The ratio of consecutive terms is 10.
- c. The limit of the sequence is 10. After the first term, each successive term is within 0.01 of 10.
- d. See sample spreadsheet above.
- e. The terms appear to approach 0. After the fifth term, each successive term is within 0.005 of 0.
- f. The ratio of consecutive terms is 0.1.
- g. The limit of the sequence is 0.1. After the first term, each successive term is within 0.001 of 0.1.
- h. 1. Sample graph:



- 2. Sample graph:

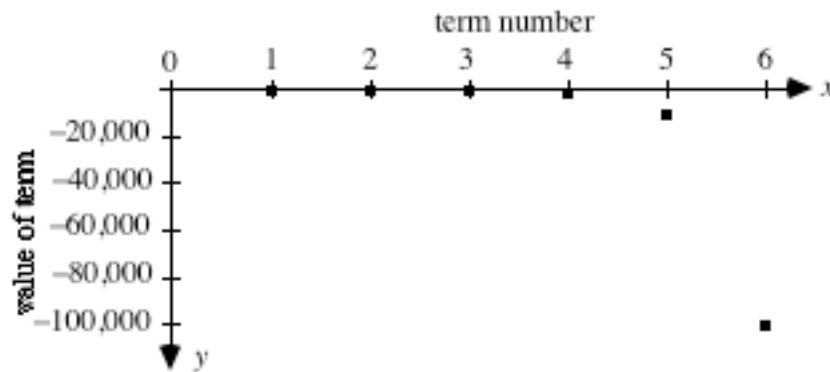


- 3. Sample response: As the number of terms increases, the scatterplot of column B is increasing and does not approach a limit. As the number of terms increases, the scatterplot of column D is decreasing and appears to approach a limit of 0.

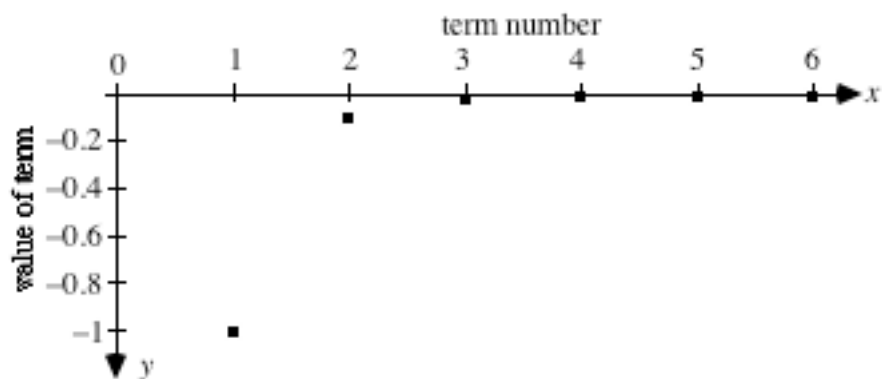
4.3 Sample spreadsheet:

A	B	C	D	E
1	-1		-1	
2	-10	10	-0.1	0.1
3	-100	10	-0.01	0.1
4	-1000	10	-0.001	0.1
5	-10000	10	-0.0001	0.1
6	-100000	10	-0.00001	0.1
7	-1000000	10	-0.000001	0.1
8	-10000000	10	-0.0000001	0.1
9	-100000000	10	-1E-08	0.1
⋮	⋮	⋮	⋮	⋮
48	-1E+47	10	-1E-47	0.1
49	-1E+48	10	-1E-48	0.1
50	-1E+49	10	-1E-49	0.1

- a. The terms in column B are decreasing negatively and do not appear to approach a limit.
- b. The ratio of consecutive terms is 10.
- c. The limit of the sequence is 10. After the first term, all terms are within 0.01 of 10.
- d. See sample spreadsheet above.
- e. The terms appear to approach 0. After the fourth term, all terms are within 0.005 of 0.
- f. The ratio of consecutive terms is 0.1.
- g. The limit of the sequence is 0.1. After the first term, all terms are within 0.001 of 0.1.
- h. 1. Sample graph:



2. Sample graph:

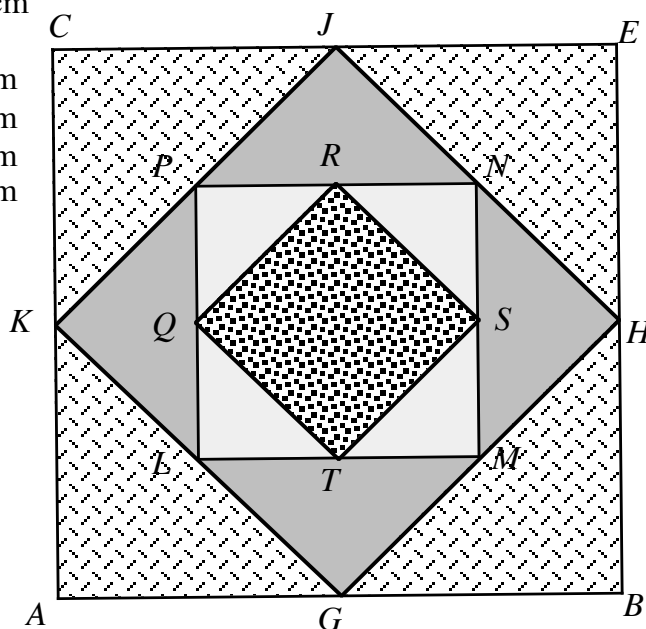


3. Sample response: As the number of terms increases, the scatterplot of column B is decreasing and does not approach a limit. As the number of terms increases, the scatterplot of column D is increasing and appears to approach a limit of 0.

\*4.4 In the following sample construction, polygon 1 is  $ABEC$ , polygon 2 is  $GHJK$ , polygon 3 is  $LMNP$  and polygon 4 is  $QRST$ .

Area(Polygon 1) = 50.28 square cm      Area(Polygon 4)/Area(Polygon 3) = 0.50  
 Area(Polygon 2) = 25.14 square cm      Area(Polygon 3)/Area(Polygon 2) = 0.50  
 Area(Polygon 3) = 12.57 square cm      Area(Polygon 2)/Area(Polygon 1) = 0.50  
 Area(Polygon 4) = 6.29 square cm

Perimeter(Polygon 1) = 28.36 cm  
 Perimeter(Polygon 2) = 20.06 cm  
 Perimeter(Polygon 3) = 14.18 cm  
 Perimeter(Polygon 4) = 10.03 cm



- Answers may vary. Students should show that the figure is equilateral and that all angles are right angles.
- Sample response: Each of the other three figures is a square. The four sides of each one are the same length and the four angles in each are right angles.

- c. Answers will vary, depending on the size of the original square.
- d. 1. Students should observe the ratio of consecutive areas is  $1/2$ .  
2. The ratio of the areas of consecutive squares is not the same as the ratios of the areas of the consecutive triangles.
- e. Sample response: The ratio of the perimeters is  $\sqrt{2}/2 \approx 0.707$ , which is not the same as the ratios of the perimeters for the triangles.
- f. 1. The sequence of areas approaches 0.  
2. The sequence of perimeters approaches 0.

4.5 Since  $r = 0.2$  and  $g_1 = 98$ , the sum can be found as follows:

$$S = \frac{98}{1 - 0.2} = 122.5$$

- 4.6 a. Sample response:  $[3, 4], [3, 3.5], [3, 3.\bar{3}], [3, 3.25]$ .
- b. Sample response: As  $n$  increases, the limit appears to be the interval  $[3, 3]$ , which is equivalent to 3.

\* \* \* \* \*

- 4.7 a.  $r = 1/10$
- b. The limit of the sequence appears to be 0.
- c. After the third term, all the following terms are within 0.001 of 0.
- d.  $\frac{6}{10}, \frac{66}{100}, \frac{666}{1000}, \frac{6666}{10,000}, \frac{66,666}{100,000}, \frac{666,666}{1,000,000}$
- e. The limit of the sequence of sums appears to be  $2/3$ . **Note:** You may wish to discuss other infinitely repeating decimals whose limits are exact values, such as  $0.3\bar{3} = 1/3$ .

- 4.8 Students may wish to use a spreadsheet to investigate this situation.
  - a.  $250, 250 + 0.10(250), 250 + 0.10(250 + 0.10(250)), 250 + 0.10(250 + 0.10(250 + 0.10(250)))$  or  $250, 275, 277.5, 277.75$
  - b. 277.5 mg

- c–d. Sample response: By the middle of the third day (after five tablets), the amount of medication in the body approaches a limit of approximately 277.8 mg.

Number of Tablets	Amount of Medication in Body (mg)
0	250
1	275
2	277.5
3	277.75
4	277.775
5	277.7775
6	277.778
7	277.778
8	277.778

4.9  $\frac{5}{11} = 0.\overline{45} = \frac{45}{100} + \frac{45}{10,000} + \frac{45}{1,000,000} + \dots$

4.10 a.  $g_1 = 0.27, g_2 = 0.0027, g_3 = 0.000027, g_4 = 0.00000027$

- b. The sum of the infinite geometric sequence can be found as follows:

$$S = \frac{0.27}{1 - 0.01} = \frac{0.27}{0.99} = \frac{27}{99} = \frac{3}{11}$$

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## Research Project

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- a. One good resource for students is Peitgen, et al., *Fractals for the Classroom*. The authors provide many references to Sierpinski and describes how to produce these triangles using graphing calculators or dot paper.
- b. When students construct these figures, they should observe that the first quadrilateral is a parallelogram (a rhombus), but not a rectangle. The second quadrilateral is a rectangle. This alternating pattern continues. The ratio of the areas of consecutive rectangles is a constant, while the ratio of the areas of consecutive parallelograms is a different constant.
-

## Answers to Summary Assessment

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1. Students may use different methods to analyze this situation. The area added to the snowflake at each stage after stage 1 can be described by the geometric sequence below, where  $n$  is the stage number minus 1:

$$\frac{3}{9}, \frac{3}{9}\left(\frac{4}{9}\right), \frac{3}{9}\left(\frac{4}{9}\right)^2, \dots, \frac{3}{9}\left(\frac{4}{9}\right)^{n-1}$$

Thus, the area of the snowflake at each stage is:

$$1 + S_n = 1 + \frac{3}{9} + \frac{3}{9}\left(\frac{4}{9}\right) + \frac{3}{9}\left(\frac{4}{9}\right)^2 + \dots + \frac{3}{9}\left(\frac{4}{9}\right)^{n-1}$$

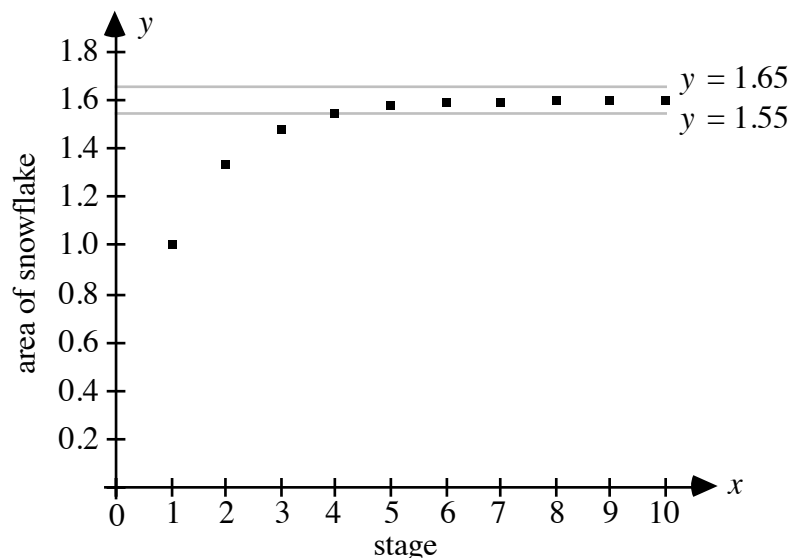
- a.  $1 \text{ unit}^2$
- b. Using the expression shown above, where  $n$  is the stage number minus 1:

$$1 + S_1 = 1 + \frac{3}{9} = 1\frac{1}{3} \text{ units}^2$$

- c. By combining the expression shown above, where  $n$  is the stage number minus 1, with the formula for an infinite geometric series in which  $-1 < r < 1$ :

$$1 + S_{14} = 1 + \frac{\left(\frac{3}{9}\right)\left(\frac{4}{9}\right)^{14} - \frac{3}{9}}{\frac{4}{9} - 1} \approx 1.60 \text{ units}^2$$

2. Sample response: The area of the snowflake approaches  $1.6 \text{ units}^2$ . The following graph shows how, after stage 5, all of the terms fall within 0.05 units of 1.6.





## *Module Assessment*

1.
  - a. Consider an arithmetic sequence in which the 1st term is  $-4$  and the 11th term is  $-38$ .
    1. What is the common difference?
    2. What are the intermediate 9 terms?
    3. What is the sum of the first 11 terms?
  - b. Consider a geometric sequence in which the 1st term is  $-4$  and the 6th term is  $128$ .
    1. What is the common ratio?
    2. What are the intermediate four terms?
    3. What is the sum of the first 20 terms?
2. In Problem 1.6, you examined the claims of a veterinary diet that promised to “help your pet lose 0.9 kg a month.”
  - a. If that claim is true, a pet’s continued loss of mass could be represented by a finite arithmetic sequence. How many terms could be in the sequence? Explain your response.
  - b. Imagine that the ad promised a pet could lose 1 kg the first month, 0.5 kg the second month, 0.25 kg the third month, and so on. If that claim were true, a pet’s continued loss of mass could be represented by a finite geometric sequence. What is the common ratio of this sequence? Explain your response.
  - c. Could the continued loss of the pet’s mass be represented by an infinite geometric sequence? Explain your response.
3. The sequence 1, 3, 4, 7, 11, 18, 29, 47, 76, ... is credited to the French mathematician Edouard Lucas.
  - a. Write the next five terms in the sequence.
  - b. Determine whether the sequence is arithmetic, geometric, or neither.
  - c. Write a recursive formula for the sequence.
4.
  - a. Write an explicit formula for a geometric sequence with a common ratio of 0.5 and a first term of 40.
  - b. Create a graph of the first 15 terms of the sequence.
  - c. If this sequence continued indefinitely, would it approach a limit?
  - d. Find the sum of the first 15 terms of the sequence.

## Answers to Module Assessment

1. a. 1. The common difference is  $-3.4$ . Solve for  $d$  in  $-38 = -4 + (11 - 1)d$ .
2. The intermediate terms are  $-7.4, -10.8, -14.2, -17.6, -21, -24.4, -27.8, -31.2$ , and  $-34.6$ .
3. The sum of the first 11 terms can be found as follows:

$$S_{11} = \frac{11}{2}(-4 + (-38)) = -231$$

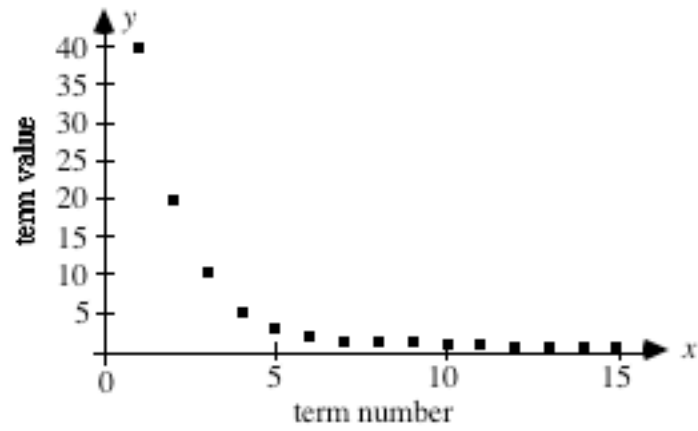
- b. 1. The common ratio is  $-2$ . Solve for  $r$  in  $128 = -4(r)^{6-1}$ .
2. The intermediate terms are  $8, -16, 32$ , and  $-64$ .
3. The sum of the first 20 terms can be found as follows:

$$S_{20} = \frac{-4(-2)^{20} - (-4)}{-2 - 1} = 1,398,100$$

2. a. Answers will vary. Sample response: If represented as a sequence, the pet's continued mass loss cannot have more than a few terms. Unless the pet is extremely overweight, it should not take more than a few months to reach a healthy level. Taken literally, losing mass continually at  $0.9$  kg per month means that a  $20$ -kg dog would have no mass at the end of 22 months.
- b. Sample response: If the claim is true, it's possible that a pet's mass loss could be represented by a finite geometric sequence with a first term of  $1$  and a common ratio of  $0.5$ .
- c. Sample response: The geometric sequence  $1, 0.5, 0.25, \dots$  appears to approach a limit of  $0$ . The corresponding series appears to approach a limit of  $2$ . It might be possible for a pet to lose mass according to this geometric sequence—whether finite or infinite. Continuing the pattern, the total mass loss would never be more than  $2$  kg.
3. a. The next five terms are  $123, 199, 322, 521, 843$ .
- b. The sequence is neither arithmetic nor geometric because it has neither a common difference nor a common ratio.
- c. Sample response:

$$\begin{cases} p_1 = 1 \\ p_2 = 3 \\ p_n = p_{n-1} + p_{n-2}, n > 2 \end{cases}$$

4. a. The explicit formula is  $g_n = 40 \cdot (0.5)^{n-1}$ .  
b. Sample graph:



- c. As  $n$  increases, the terms appear to approach 0.  
d. Using the formula for the sum of a geometric series:

$$S_n = \frac{g_1 r^n - g_1}{r - 1} = \frac{40(0.5)^{15} - 40}{0.5 - 1} \approx 80$$

### **Selected References**

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# Flashbacks

## Activity 1

**1.1** Find a possible next term for each of the following sequences.

**a.** 10, 5, 0, -5, ...

**b.**  $\frac{1}{2}, \frac{7}{6}, \frac{11}{6}, \frac{5}{2}, \dots$

**1.2** Consider the following arithmetic sequence:

$$-2, -5.4, -8.8, \dots, -53$$

**a.** What is the common difference?

**b.** How many terms are in the sequence?

**1.3** If  $a_n$  describes any term in a sequence, which terms are described by each of the following notations?

**a.**  $a_{n+1}$

**b.**  $a_{n-1}$

**1.4** Write the first three terms of the sequence defined by the formula below:

$$x_n = 2(n - 1)^{n+1}$$

## Activity 2

**2.1** What is the sum of the first 10 natural numbers?

**2.2** Write both an explicit and a recursive formula for each of the following sequences:

**a.** 8, 16, 24, 32, ...

**b.** 1, 4, 16, 64, ...

**2.3** Consider the following geometric sequence:

$$5, -15, 45, \dots, -10,935$$

**a.** What is the common ratio?

**b.** How many terms are in the sequence?

### Activity 3

3.1 Simplify each of the following expressions:

a.  $x \cdot x^n$

b.  $x \cdot x^{n-1}$

c.  $x \cdot x^{n-2}$

3.2 Multiply both sides of the equation below by  $x$ :

$$A = b + bx + bx^2 + bx^3 + \cdots + bx^{m-2} + bx^{m-1}$$

3.3 Solve the following equation for  $x$ :

$$ax + bx = 12$$

3.4 Find the sum of the arithmetic series below:

$$6, 8, 10, 12, \dots, 52$$

### Activity 4

4.1 Consider a geometric sequence in which  $g_1 = 8/3$  and  $r = 1/2$ . What is the sum of the first 5 terms of this sequence?

4.2 Consider an arithmetic sequence in which  $a_1 = 8/3$  and  $d = 1/2$ . What is the sum of the first 5 terms of this sequence?

4.3 Is there an integer  $n$  for which  $1/n = 0$ ? Explain your response.

# Answers to Flashbacks

## Activity 1

- 1.1 a. Answers may vary. Sample response:  $-10$ .  
b. Answers may vary. Sample response:  $19/6$ .

- 1.2 a. The common difference is  $-3.4$ .  
b. The number of terms can be found as follows:

$$-53 = -2 + (n - 1) \cdot -3.4$$

$$\begin{aligned} n &= \frac{-53 + 2}{-3.4} + 1 \\ &= 16 \end{aligned}$$

- 1.3 a. Sample response: The term  $a_{n+1}$  is the term following  $a_n$ .  
b. Sample response: The term  $a_{n-1}$  is the term before  $a_n$ .

- 1.4 0, 2, 32

## Activity 2

- 2.1 The sum of the first 10 natural numbers is 55.

- 2.2 a. One possible explicit formula is  $a_n = 8 + 8(n - 1)$  for  $n \geq 1$ . A possible recursive formula is shown below:

$$\begin{cases} a_1 = 8 \\ a_n = a_{n-1} + 8 \text{ for } n > 1 \end{cases}$$

- b. One possible explicit formula is  $g_n = 1(4)^{n-1}$  for  $n \geq 1$ . A possible recursive formula is shown below:

$$\begin{cases} g_1 = 1 \\ g_n = 4g_{n-1} \text{ for } n > 1 \end{cases}$$

- 2.3 a. The common ratio is  $-3$ .  
b. The number of terms can be found as follows:

$$-10,935 = 5(-3)^{n-1}$$

$$\frac{-10,935}{5} = (-3)^{n-1}$$

$$n = 8$$

### Activity 3

3.1 a.  $x \cdot x^n = x^{n+1}$

b.  $x \cdot x^{n-1} = x^n$

c.  $x \cdot x^{n-2} = x^{n-1}$

3.2  $xA = bx + bx^2 + bx^3 + bx^4 + \cdots + bx^{m-1} + bx^m$

3.3 Sample response:

$$x(a + b) = 12$$

$$x = \frac{12}{a + b}$$

3.4 The number of terms can be found as follows:

$$52 = 6 + (n - 1) \cdot 2$$

$$n = \frac{52 - 6}{2} + 1$$

$$= 24$$

Using the formula for a finite arithmetic series:

$$S_{24} = \frac{24}{2}(6 + 52) = 696$$

### Activity 4

4.1 Using the formula for a finite geometric series:

$$S_5 = \frac{\frac{8}{3} \left( \frac{1}{2}^5 - 1 \right)}{\frac{1}{2} - 1} = \frac{31}{6} \approx 5.16$$

4.2 Using the formula for a finite arithmetic series:

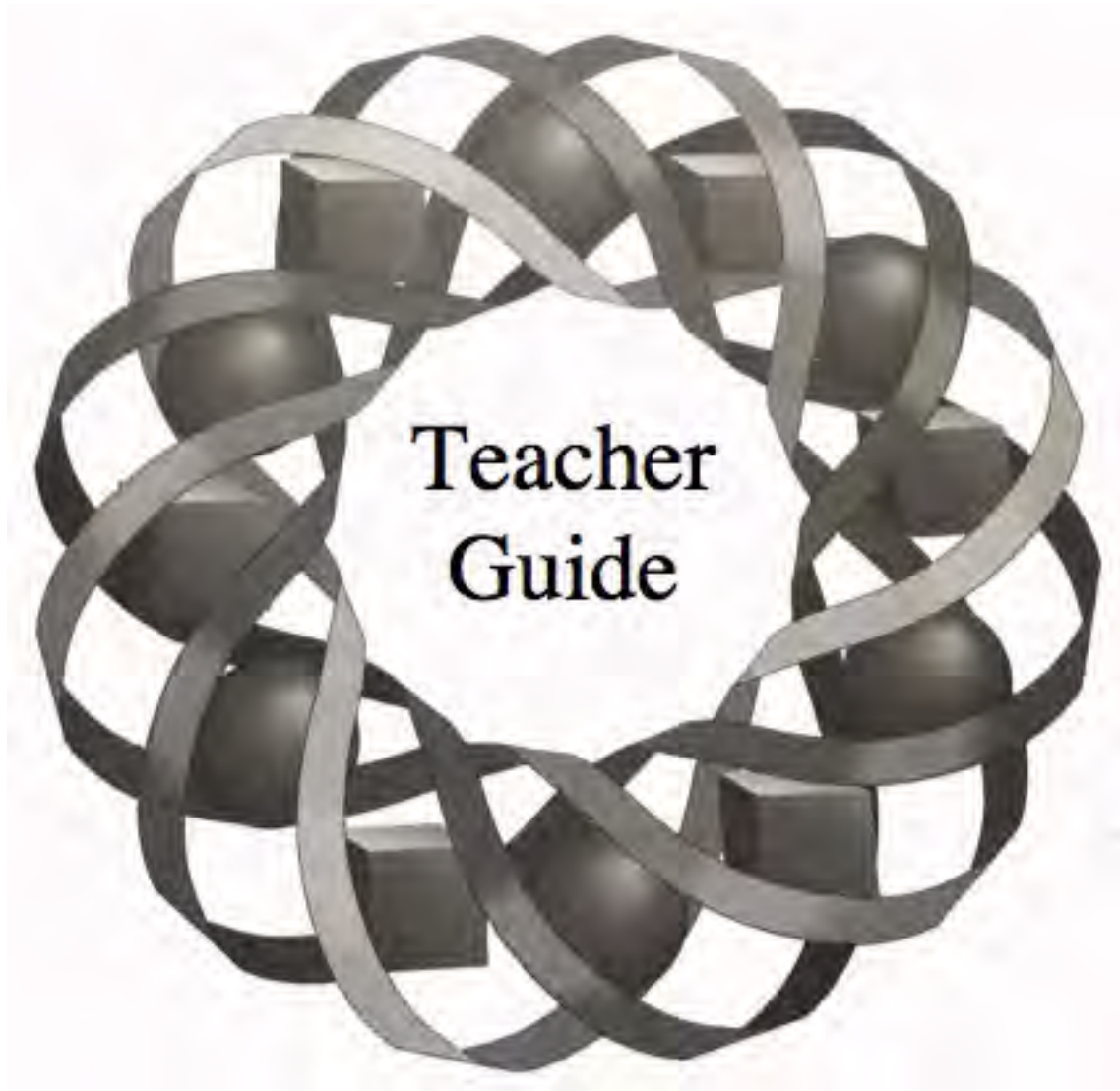
$$S_5 = \frac{5}{2} \left( 2 \cdot \frac{8}{3} + (5 - 1) \cdot \frac{1}{2} \right) = \frac{55}{3} \approx 18.3$$

4.3 Sample response: No. If  $1/n = 0$ , then  $1 = n \cdot 0$ . This cannot be true.





# Algorithmic Thinking



What do baking a cake, programming a VCR, and adding a pair of two-digit numbers have in common? In this module, you explore how algorithms affect everything from cooking to computers.

*Wendy Driscoll • Darlene Pugh • Todd Robins*



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## Teacher Edition

# Algorithmic Thinking

### Overview

In this module, students examine some classic algorithms, create flowcharts, and use recursion in algorithms.

### Objectives

In this module, students will:

- construct algorithms
- use algorithmic thinking
- examine the efficiency of algorithms
- organize algorithms into flowcharts
- develop algorithms that use recursion.

### Prerequisites

For this module, students should know:

- the definition of prime numbers
- how to write recursive formulas of sequences.

### Time Line

Activity	1	2	3	Summary Assessment	Total
Days	2	3	3	1	9

### Materials Required

Materials	Activity			
	1	2	3	Summary Assessment
compass	X			
ruler	X			
protractor	X			
flowchart template		X		
scissors		X		
tape or glue		X		

## Teacher Note

A blackline master of the template appears at the end of the teacher edition FOR THIS MODULE.

## Technology

Software	Activity			
	1	2	3	Summary Assessment
geometry utility			X	

# Algorithmic Thinking

## Introduction

(page 405)

Students may recall the definition of an algorithm from the Level 1 module “Going in Circuits.” The historical note describes the derivation of the word from the name of an Arabian mathematician, Mohammed ibn-Musa al-Khwarizmi. (The title of another of al-Khwarizmi’s influential works, *Al-jabr wa’l muqabalah*, is the source of the word *algebra*.)

(page 405)

## Activity 1

Students investigate some characteristics desirable in algorithms.

## Materials List

- compass (one per student)
- ruler (one per student)
- protractor (one per student)

## Exploration

(page 406)

Students experiment with writing clear and efficient algorithms.

- You may wish to encourage students to keep the pictures simple.
- Student instructions may appear in a list, as a paragraph, or in other forms.
- Each student should follow the partner’s instructions exactly as written, without additions or omissions.
- Students revise their algorithms based on the results of Part **d**.
- The revised algorithms may be more efficient.

## Discussion

(page 406)

- Answers will vary. In most cases, the two pictures are likely to look somewhat different.
- The simplest pictures—with the most concise instructions—should take the least amount of time to reproduce. A drawing of three concentric circles of specified radii, for example, should be fairly easy to duplicate.

- d. Answers will vary. Some students will consider only time when determining efficiency. (This is also the typical measure of efficiency in computer programming.) Others may consider the quality of the writing and the accuracy of the recreated picture as part of the “efficiency” of a set of instructions.
- e. Sample response: Yes, it is possible to have more than one set of instructions for the same picture. For example, the two instructions “draw an equilateral triangle with side length of 5 cm” and “draw an equiangular triangle whose sides are 5 cm long” both produce the same figure.
- f. Students should reflect on the characteristics of clear, easy-to-follow instructions. Eliminating trivial steps, ordering the steps numerically, and using vocabulary appropriate for the audience all may be factors in making instructions more clear.
- g. Sample response: The vocabulary may need to change. For example, an elementary student may not know the meaning of “regular heptagon.”

## Assignment

(page 407)

- 1.1 a. Sample response for washing clothes: Jillian packed the machine full of white cotton shirts, then added a red silk blouse. The load indicator was set on “small,” but she did not adjust it. She poured in half a box of soap, then turned on the machine. Complete disaster!

Sample response for preparing a bath: Jack set the water temperature just right, put the towels by the tub, and sat down to wait. Since he had neglected to put in the plug, however, the tub never filled.

- b. Sample response for washing clothes: The instructions do not represent a successful algorithm because they are not precise. The instructions should include directions for sorting clothes by color and fabric, and for setting water temperature, load size, and wash cycle (like permanent press, for example). They should also specify the amount of soap needed.

Sample response for preparing a bath: The instructions are for a task that will not terminate. As written, the instructions do not direct the person to close the drain—or tell when to turn off the water.

- c. Sample response:

### **Washing Clothes**

Sort the clothes into light or dark.

If the clothes are light, set the machine on hot or warm.

If the clothes are dark, set the machine on cool or cold.

Add one scoop of soap.

Add the clothes, but do not overfill the washer.

Select the load size and wash cycle.

Start the machine.

- 1.2**
- a. Sample response: This recipe will probably not always produce the same results. A difference in the amount of milk may change the consistency of the dough. And different cooks will have different ideas of what is “warm.” The variations in temperature may cause the yeast to act differently. The temperature of the hot grease will also affect the bread.
  - b. Sample response: There are many decision points in this recipe. For example, “Beat until smooth” demands a decision on the meaning of smoothness. Also “turning the bread until golden brown” creates another decision to be made. In the first case, if the dough is not smooth enough, the bread will have lumps. In the second case, what looks like golden brown to some may mean overcooked—or burned—to others.
- 1.3**
- a. The first 20 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, and 71.
  - b. Sample response: (1) Is the number 1? If yes, then it is not prime. (2) Is the number divisible by a natural number less than itself, but greater than 1? If yes, then it is not prime. (3) Otherwise, the number is not prime.
- \*1.4**
- a. Sample response:
    1. Subtract the cost of the item from 100 cents.
    2. Divide the difference in step 1 by 25 cents.
    3. The integer part of the quotient in step 2 is the number of quarters in change.
    4. Divide the remainder of the quotient in step 2 by 10 cents.
    5. The integer part of the quotient in step 4 is the number of dimes in change.
    6. Divide the remainder of the quotient in step 4 by 5 cents.
    7. The integer part of the quotient in step 6 is the number of nickels in change.
    8. The remainder of the quotient in step 6 is the number of pennies in change.
  - b. The following sample response uses an item which costs \$0.33.
    1.  $100 - 33 = 67$
    2.  $67/25 = 2$ , remainder 17
    3. There are two quarters in change.
    4.  $17/10 = 1$ , remainder 7
    5. There is one dime in change.
    6.  $7/5 = 1$ , remainder 2
    7. There is one nickel in change.
    8. There are two pennies in change.

- c. The sample algorithm given in Part a requires a knowledge of the arithmetic terms *product*, *quotient*, *remainder*, and *integer part*.

\* \* \* \* \*

- 1.5 a. Students should substitute and solve as shown below:

$$C = \frac{5}{9}(99 - 32) \approx 37^\circ$$

- b. The following algorithm will convert temperature in degrees Celsius to degrees Fahrenheit.
- Let  $F$  represent the temperature in degrees Fahrenheit and  $C$  the temperature in degrees Celsius.
  - Substitute the value of  $C$  into the following equation.

$$F = \frac{9}{5}C + 32$$

- Solve the equation for  $F$ .

- 1.6 a. Sample response: It would trace an equilateral triangle with sides 10 m long.

- b. Sample response:

- Forward( $x$ )
- Right( $90^\circ$ )
- Forward( $x$ )
- Right( $90^\circ$ )
- Forward( $x$ )
- Right( $90^\circ$ )
- Forward( $x$ )
- Right( $90^\circ$ )

- c. Answers will vary. Students create an algorithm that will trace a figure and ask another student to follow the algorithm.

\* \* \* \* \*

(page 409)

## Activity 2

In this activity, students create flowcharts for algorithms.

### Materials List

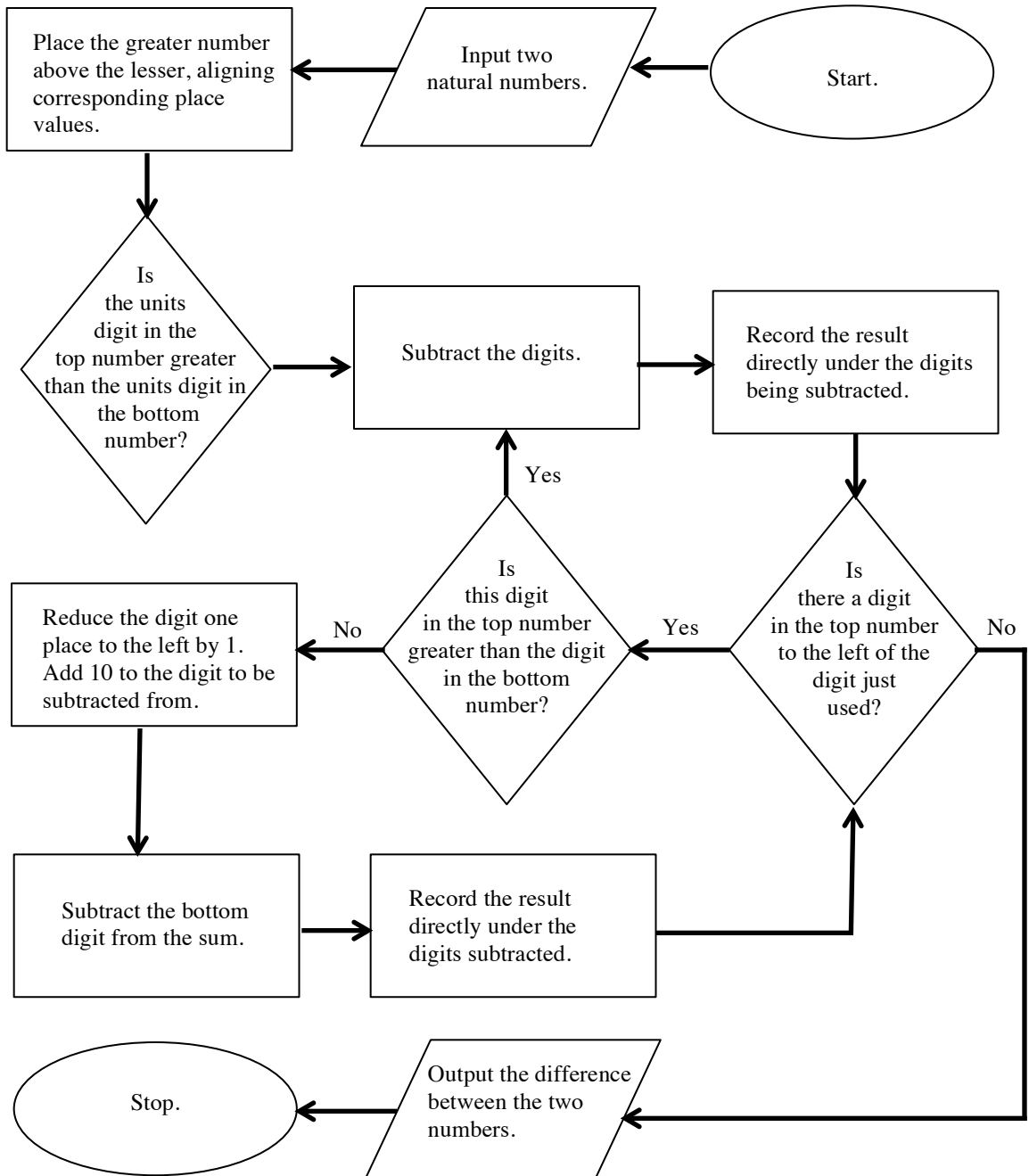
- flowchart template (one copy per student; a blackline master appears at the end of the teacher edition for this module)
- scissors (one pair per student)
- tape or glue



## Exploration 1

(page 411)

- a–b. Students organize the command boxes on the flowchart template into a usable order and draw the appropriate arrows. Sample response:



- c. After exchanging flowcharts, students should test the algorithm using two natural numbers.

## Discussion 1

(page 411)

- a. Sample response: No. Although the order is basically the same in all of the flowcharts, the arrangement of the command boxes on the page varies.
- b. Sample response: I placed the start command at the beginning and the input command next. Then I looked for a processing box that seemed to describe the first step in the subtraction algorithm. After that, I tested each command box to see if it contained the following step.
- c. Sample response: To make the flowchart easier to read, the command boxes should be lined up (when possible) and the arrows should be clearly drawn.

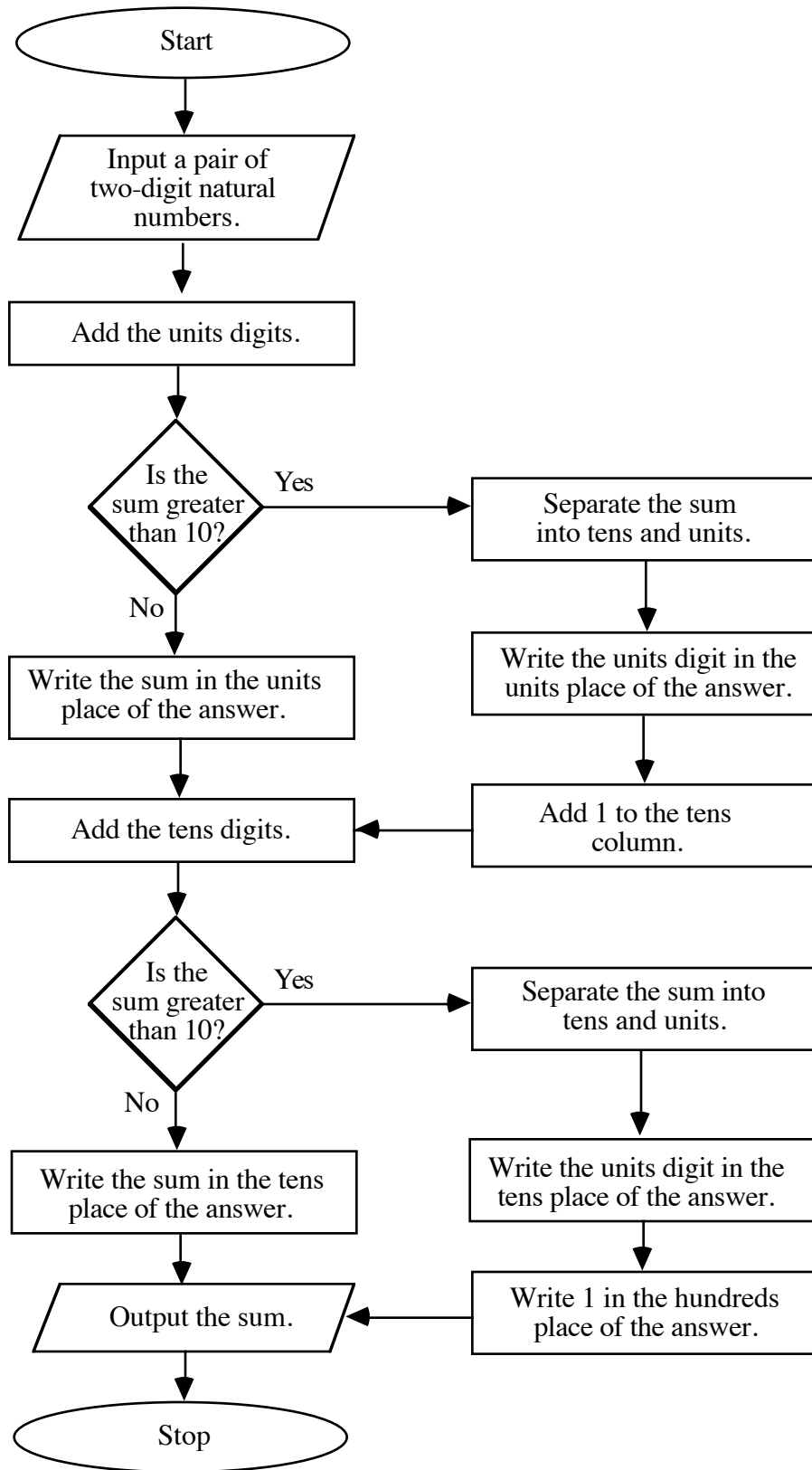
## Exploration 2

(page 411)

Students write an algorithm for adding a pair of two-digit numbers and draw the corresponding flowchart.

- a. Students may select any pair of two-digit numbers. The sum of 84 and 39 is 123.
- b. Sample response:
  1. Add the units digits of the pair of numbers. In this case,  $4 + 9 = 13$ .
  2. If the sum of the units digits is greater than 10, separate the sum into two parts, 10 and the rest. If not, the sum of the units digits is the units digit of the final sum. In this case, separate the sum into 10 and 3.
  3. Add the tens digits of the pair of numbers. In this case,  $8 + 3 = 11$ .
  4. Add the sum of the tens digits and the number of 10s from the sum of the units digits. If the number is less than 10, this is the tens digit of the final sum. In this case,  $11 + 1 = 12$ .
  5. If the number is more than 10, then separate this sum into 10 tens and the rest. In this case, separate the sum into 10 and 2. The number of 10 tens becomes the hundreds digit of the final sum; the remainder becomes the tens digit in the final sum. In this case, the hundreds digit is 1 and the tens digit is 2.
  6. The sum of the two original numbers has a hundreds digit (if any) from Step 5, a tens digit from Step 4 or Step 5, and a units digit from Step 2. In this case, the sum is 123.
- c. Students should use the flowchart symbols from Figure 1. See sample flowchart given in Part e.
- d. Students exchange flowcharts and test their partners' instructions.

e. Sample flowchart:



## Discussion 2

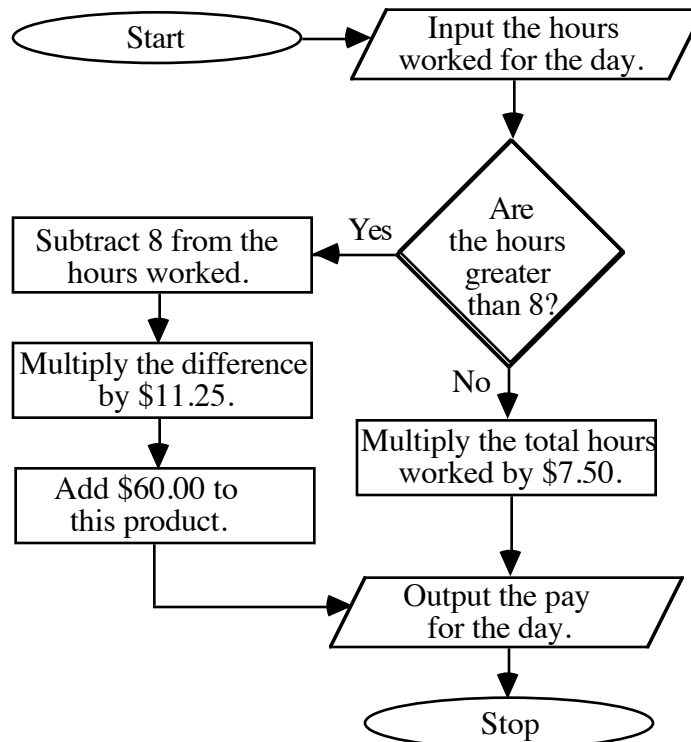
(page 411)

- a. Students should compare the sequence and organization of the steps in their algorithms, as well as the use of flowchart symbols.
- b. When evaluating flowcharts, students may consider precision, ease of interpretation, and time required to complete the algorithm.
- c. Sample response: The flowchart could be extended to find the sum of a pair of three-digit numbers by including boxes to find the sum of the hundreds digits and to determine whether or not the sum contains 10 hundreds to see how many thousands (if any) are in the sum.

## Assignment

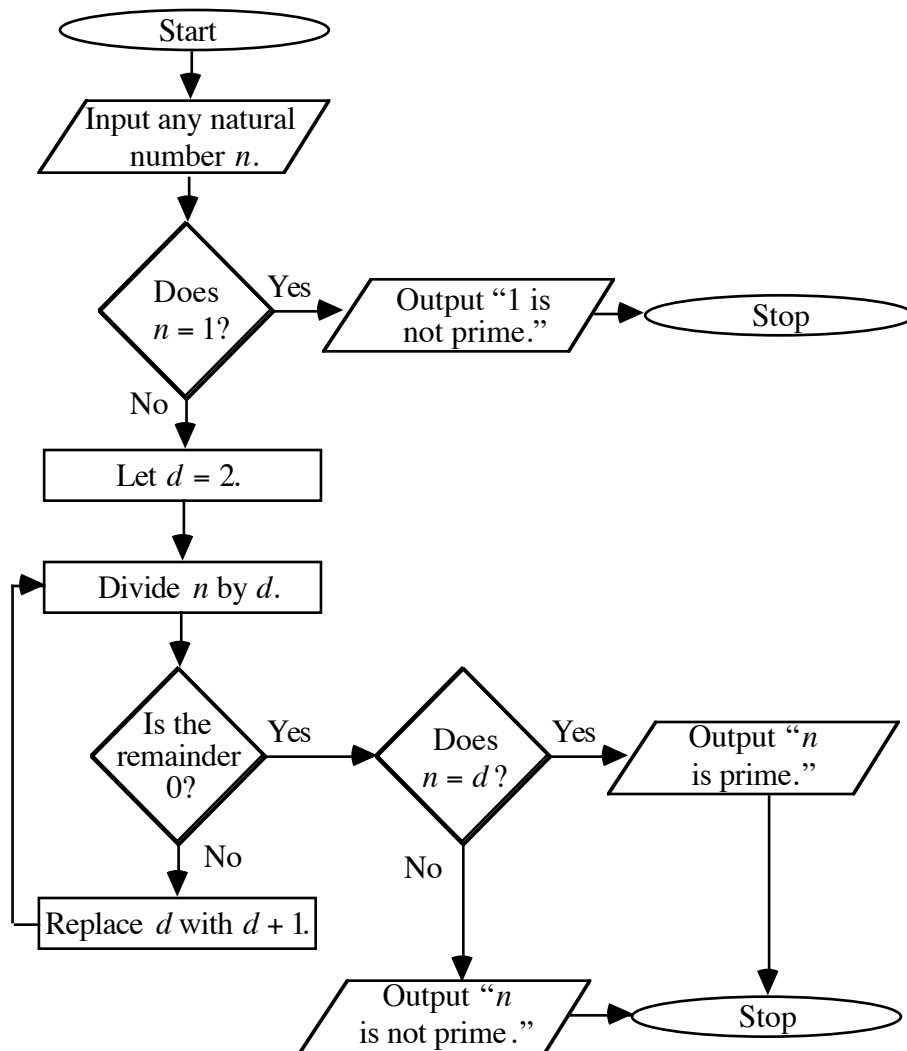
(page 412)

- 2.1 a. 1. \$367.50  
2. \$270.00
- b. Sample flowchart:



- c. Sample response: Company B would be better to work for because it pays overtime whenever an employee works more than 8 hours in 1 day. For example, an employee who worked 11 hours on 1 day of the week and 7 hours on each of the remaining 4 days would get paid \$303.75 if she worked for Company B, but only \$292.50 if she worked for Company A.

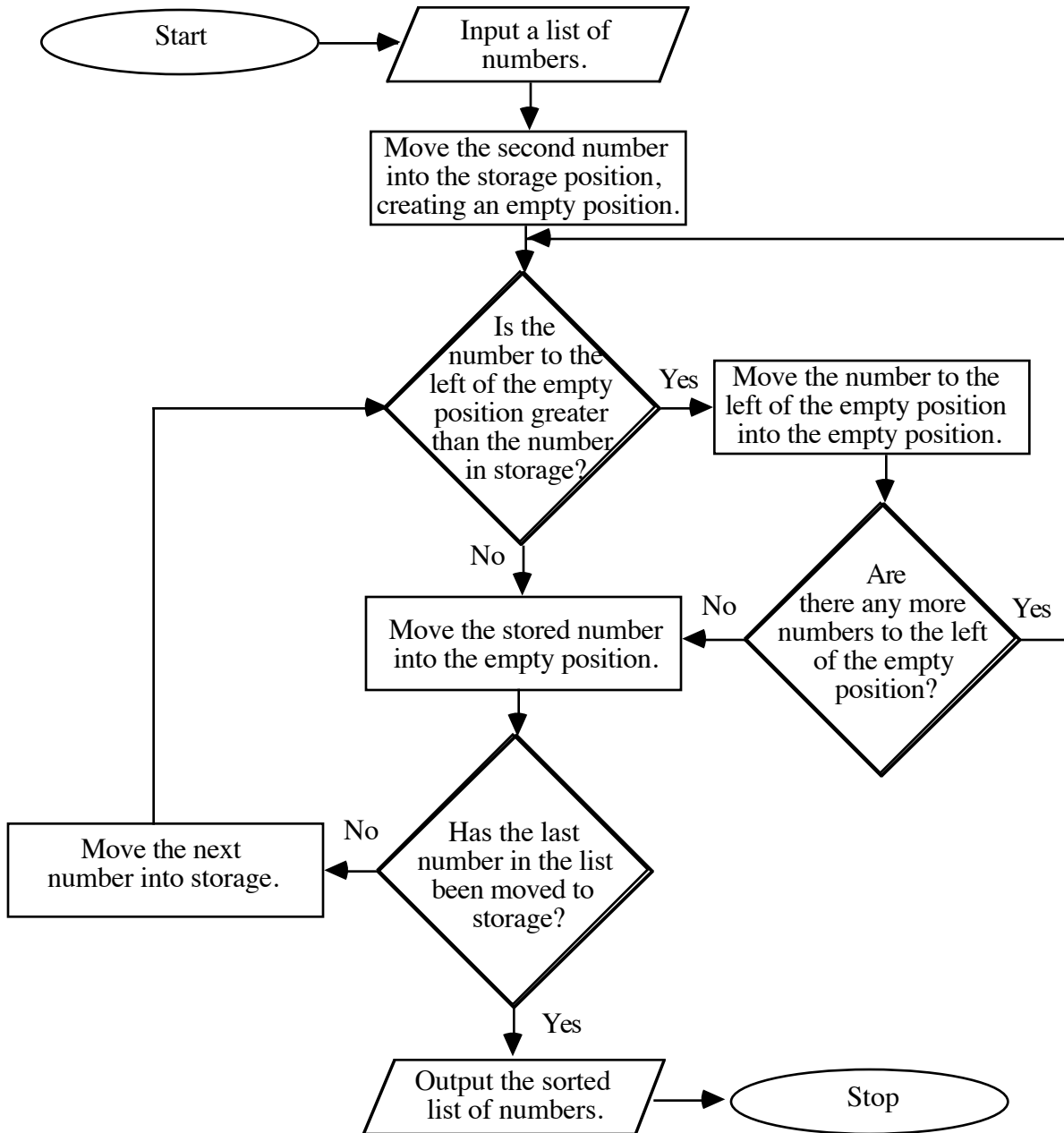
2.2 Sample flowchart:



- 2.3
- By playing the game several times, students should recognize that a winning algorithm exists.
  - Sample response: The winning algorithm is to allow your partner to play first and control the sum at the end of each round. If your partner plays 1, then you should play 2 to keep the total for the round at 3. If your partner plays 2, then you should play 1. By playing second and continuing to force the total to be a multiple of 3, the total after 7 rounds will be 21 and you will win.
  - Sample response: You could program a computer to win at this game if the human player always goes first. The program would have to instruct the computer to play a 2 every time its opponent played a 1, and a 1 every time its opponent played a 2.

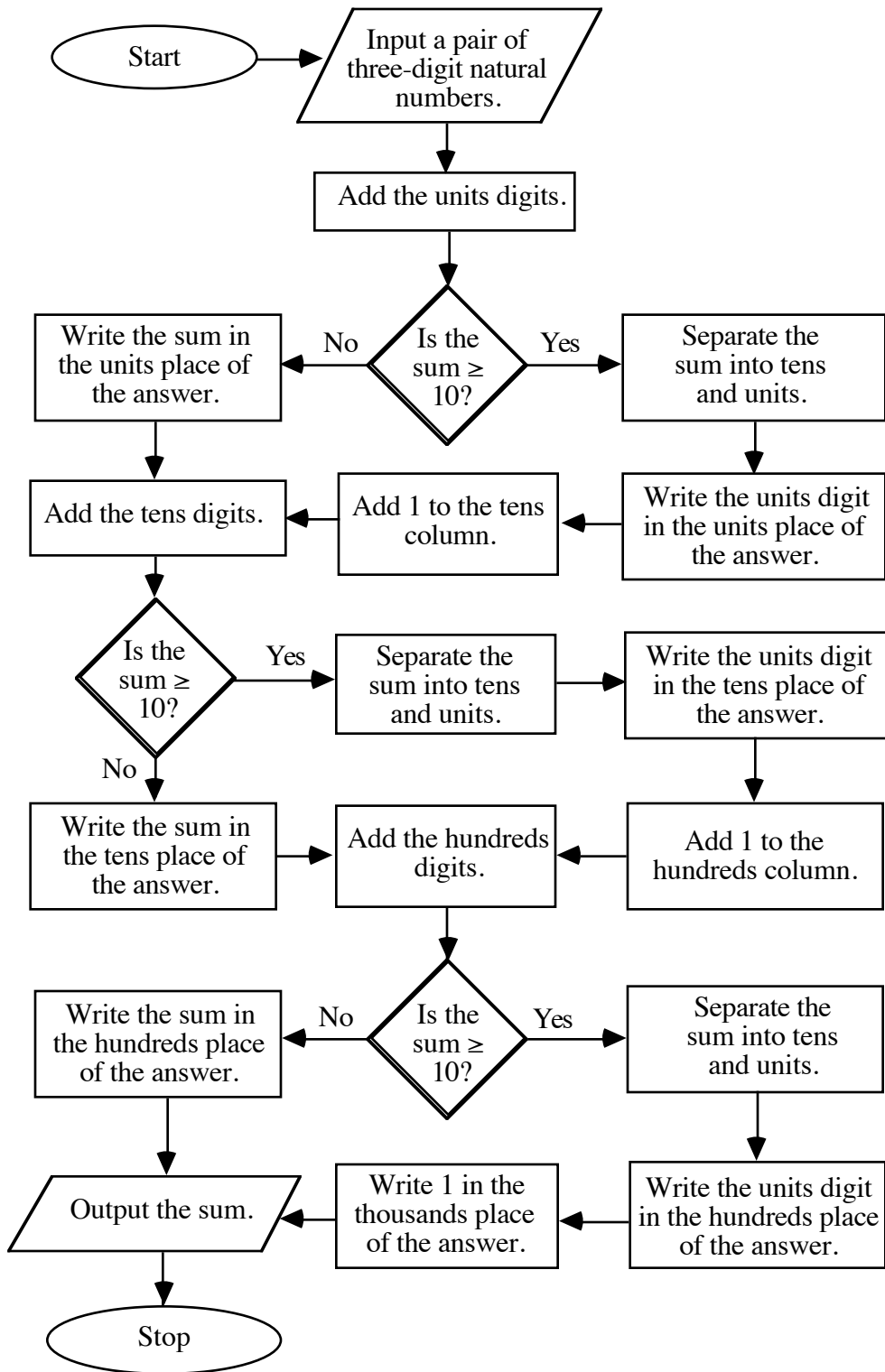
\*2.4 a. The insert sort algorithm requires 28 comparisons to sort this list of 10 numbers.

b. Sample flowchart:



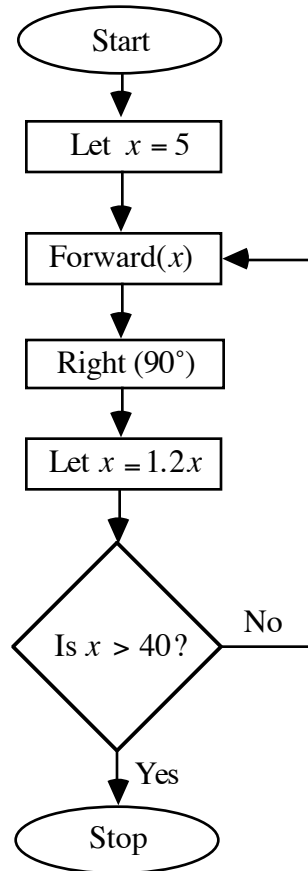
c. Sample response: The merge sort appears to be the most efficient, since it takes 4 fewer comparisons than the insertion sort and 21 fewer comparisons than the bubble sort.

2.5 Sample flowchart:



\* \* \* \* \*

2.6 Sample flowchart:



**Teacher Note**

The algorithm described in Problem 2.7 relies on the fact that the greatest common divisor of two natural numbers  $a$  and  $b$  is the same as the greatest common divisor of  $a$  and  $b - a$ . In other words,  $\gcd(a, b) = \gcd(a, b - a)$ . This may be proved as follows:

Let  $d = \gcd(a, b)$ . This means that  $d$  divides  $a$  or that there is an integer  $n$  such that  $dn = a$ . This also means that  $d$  divides  $b$  or that there is an integer  $m$  such that  $dm = b$ . Furthermore,  $d$  is the greatest integer that divides both  $a$  and  $b$ . Because  $dn = a$  and  $dm = b$ , then  $dm - dn = b - a$ . Thus  $d(m - n) = b - a$ , so  $d$  divides  $b - a$ .

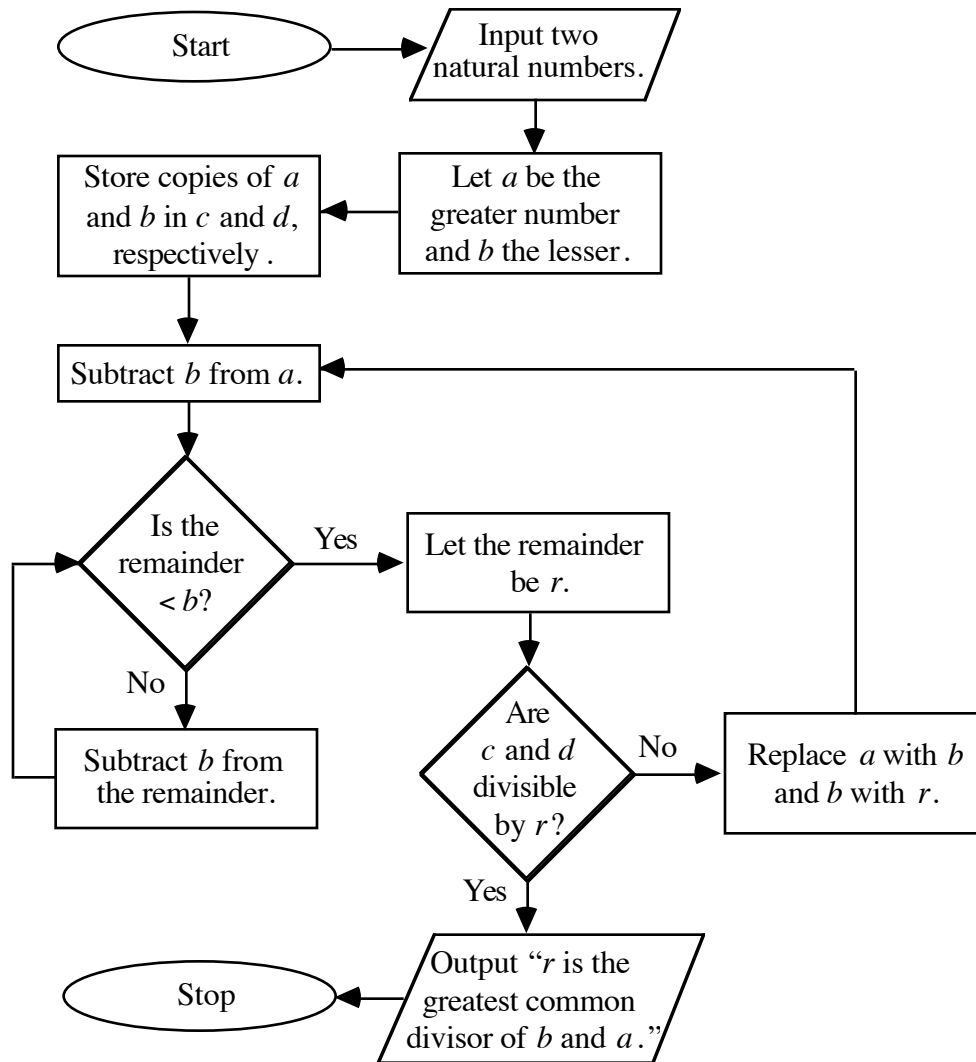
Suppose that an integer  $x = \gcd(a, b - a)$ . This means that  $x$  is the greatest integer that divides both  $a$  and  $b - a$ . This implies that  $xy = a$  and  $xz = b - a$  for some integers  $y$  and  $z$ . Thus,  $xz + xy = b - a + a$  and  $x(z + y) = b$ . Thus  $x$  divides  $b$ . Now  $x$  divides both  $a$  and  $b$ , but  $x$  must be less than  $d$ . This contradicts the assumption that  $x = \gcd(a, b - a)$ , since  $d$  is greater than  $x$  and divides both.

Therefore,  $d = \gcd(a, b - a)$  and  $\gcd(a, b) = \gcd(a, b - a)$ .

- 2.7 a. 1. 6  
2. 3  
3. 1



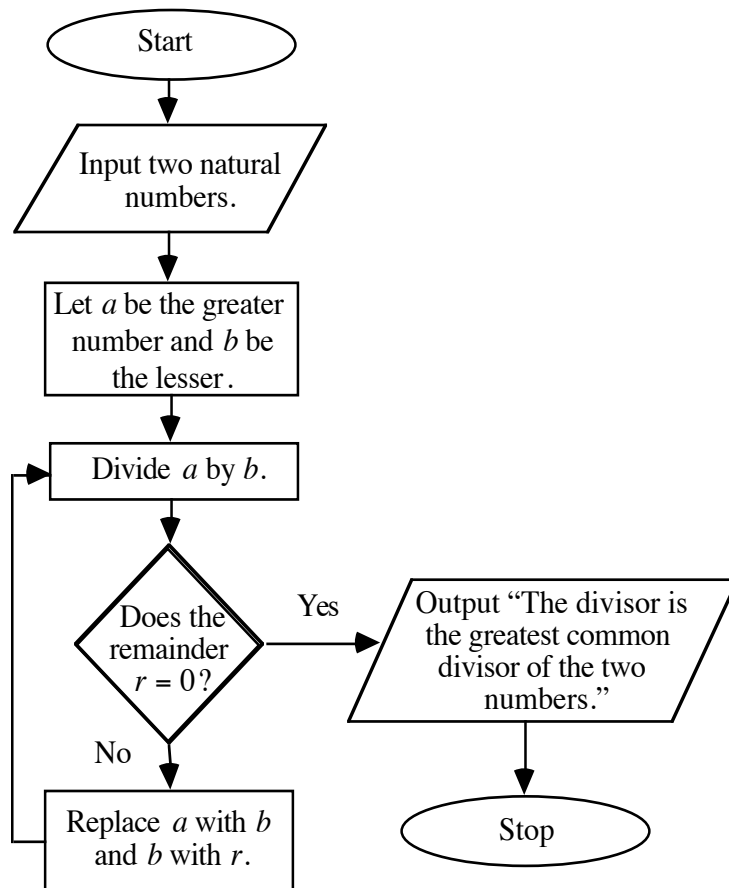
b. Sample flowchart:



c. Sample response: Yes, the Euclidean algorithm could be adapted to find the greatest common divisor of any two integers. This is because the greatest common divisor of  $a$  and  $b$ ,  $-a$  and  $b$ ,  $a$  and  $-b$ , and  $-a$  and  $-b$  are all equal.

- 2.8 a. Sample response: The Euclidean algorithm consists of a process for finding the greatest common divisor of two natural numbers. To do this using division, divide the greater number by the lesser and consider the remainder. If the remainder is 0, then the divisor is the greatest common divisor of the two numbers. If not, then divide the previous divisor by the remainder and consider the new remainder as described above. Repeat the process until the remainder is 0. The greatest common divisor of the two original numbers is the divisor when the remainder is 0.

b. Sample flowchart:



\*\*\*\*\*

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## Research Project

(page 416)

There are many variations of Nim. Using 1 and  $n$ , where  $n$  is a natural number, and a target number  $T$ , where  $T$  is a multiple of  $1 + n$ , the sum can be controlled by the second player as described in Problem 2.3.

If  $T$  is not a multiple of  $1 + n$ , then the first player can control the game by using the first turn to reduce the remaining sum to a multiple of  $1 + n$ , then continuing to keep the remaining sum a multiple of  $1 + n$ .

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### *Activity 3*

Students explore the use of recursion in algorithms involving geometric figures.

#### Materials List

- none

#### Technology

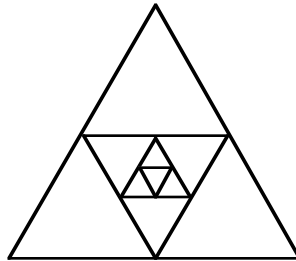
- geometry utility

#### Exploration

(page 416)

Students use recursive algorithms to create geometric shapes.

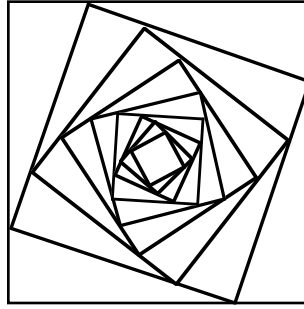
- a. Sample construction:



- b. Sample algorithm:

Given:	A square.
Procedure:	<ol style="list-style-type: none"> <li>1. Make a copy of the polygon.</li> <li>2. Locate the new polygon so that its center coincides with the center of the original polygon.</li> <li>3. Dilate the new polygon by a scale factor of 0.8 with center of dilation at the center of the polygon.</li> <li>4. Rotate the new polygon clockwise until it is inscribed in the previous polygon.</li> <li>5. Repeat the procedure on the new polygon.</li> </ol>

c. Sample construction:



### Discussion

(page 417)

- a. The procedure  $a_n = a_{n-1} + 7, n > 1$  creates recursion.
- b. In the algorithm described in Table 1, step 3 creates recursion.
- c. The limit of the perimeters of the inscribed polygons is 0.
- d.
  1. Sample response: Change step 3 to read: “Repeat the procedure 9 times.”
  2. Sample response: Change the given information to “a quadrilateral.”

### Assignment

(page 418)

- 3.1
- a. The first five terms are 3, 12, 48, 192, 768.
  - b. The given information is  $g_1 = 3$ .
  - c. The procedure is defined by  $g_n = 4g_{n-1}, n > 1$ .

\*3.2 Sample algorithm.

Given:	A pentagon.
Procedure:	<ol style="list-style-type: none"> <li>1. Make a copy of the polygon to create a new polygon.</li> <li>2. Locate the new polygon so that its center coincides with the center of the original polygon.</li> <li>3. Dilate the new polygon by a scale factor of 0.8 with center of dilation at the center of the original polygon.</li> <li>4. Rotate the new polygon clockwise until it is inscribed in the previous polygon.</li> <li>5. Repeat the procedure on the new polygon.</li> </ol>

**3.3** Sample algorithm.

Given:	A circle with a vertical diameter.
Procedure:	<ol style="list-style-type: none"> <li>1. Make a copy of the circle and its diameter to create a new circle with its corresponding diameter.</li> <li>2. Dilate the new circle by a scale factor of 0.5.</li> <li>3. Locate the new circle so that the lower endpoint of its diameter coincides with the upper endpoint of the diameter of the previous circle.</li> <li>4. Repeat the procedure on the new circle.</li> </ol>

**3.4** Answers will vary. Sample response: The geometric sequence 10, 5, 2.5, 1.25, 0.625, ... can be created by the following algorithm:

$$\begin{cases} g_1 = 10 \\ g_n = (0.5)g_{n-1}, n > 1 \end{cases}$$

\* \* \* \* \*

**\*3.5** a. The length of the first side is 5 m; the length of the second side is 4 m.

b. Sample response: The robot will go forward and turn indefinitely.

c. Sample response:

- Spiral( $x$ )  
Forward( $x$ )  
Right( $60^\circ$ )
- Spiral( $0.9x$ )

$$A_1 = 500 + 500 \cdot 0.08 = \$540$$

**3.6** a.  $A_2 = 540 + 540 \cdot 0.08 = \$583.20$

$$A_3 = 583.20 + 583.20 \cdot 0.08 = \$629.86$$

b. Sample response: Yes, the algorithm is recursive because the same procedure is first used on initial information and then repeatedly used on previously generated information. The initial information is the initial investment of \$500 and the annual interest rate of 8%. The previously generated information is the output from the machine, or the account balance. The process is the two steps in the function machine.

c. 1. \$2330.48

2. Since the initial investment was \$500, the total interest earned is:  $\$2330.48 - \$500.00 = \$1830.48$ .

\* \* \* \* \*

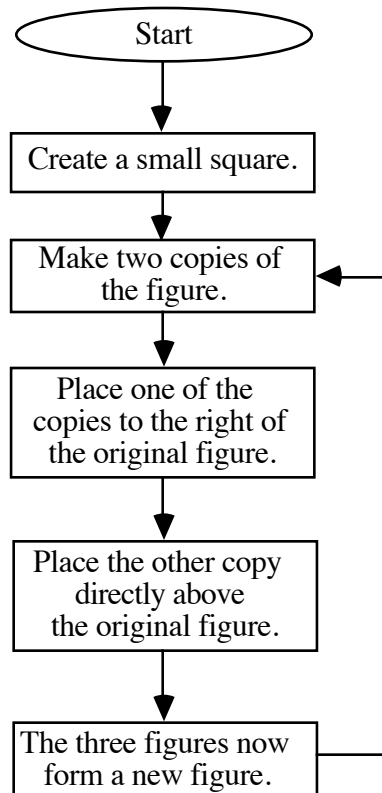
## Answers to Summary Assessment

(page 421)

1. a. Sample algorithm:

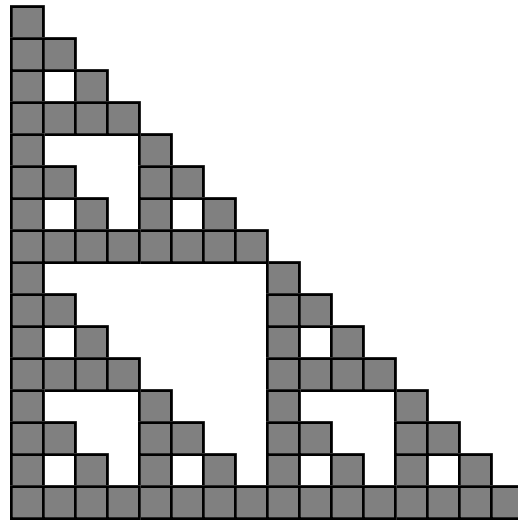
Given:	A square.
Procedure:	<ol style="list-style-type: none"><li>1. Make two copies of the figure.</li><li>2. Put one copy directly to the right of the original figure.</li><li>3. Put the other copy directly above the original figure.</li><li>4. The three figures now form a new figure.</li><li>5. Repeat the procedure on the new figure.</li></ol>

- b. The following flowchart illustrates the algorithm given in Part a.

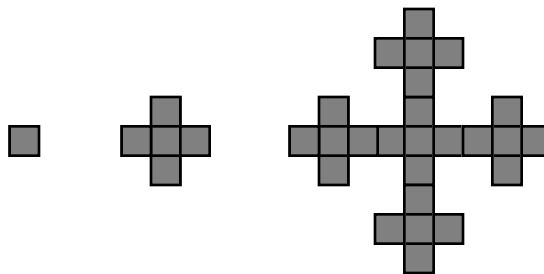


- c. Sample response: Yes, the algorithm is recursive because the same procedure is first used on given information and then used repeatedly on previously generated information.

d. The next figure in the sequence is shown below:



2. a. Answers will vary. Sample response:

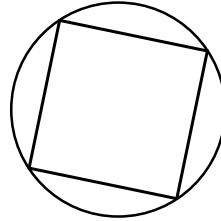


b. The following table describes an algorithm that could be used to create the sequence of figures given in Part a.

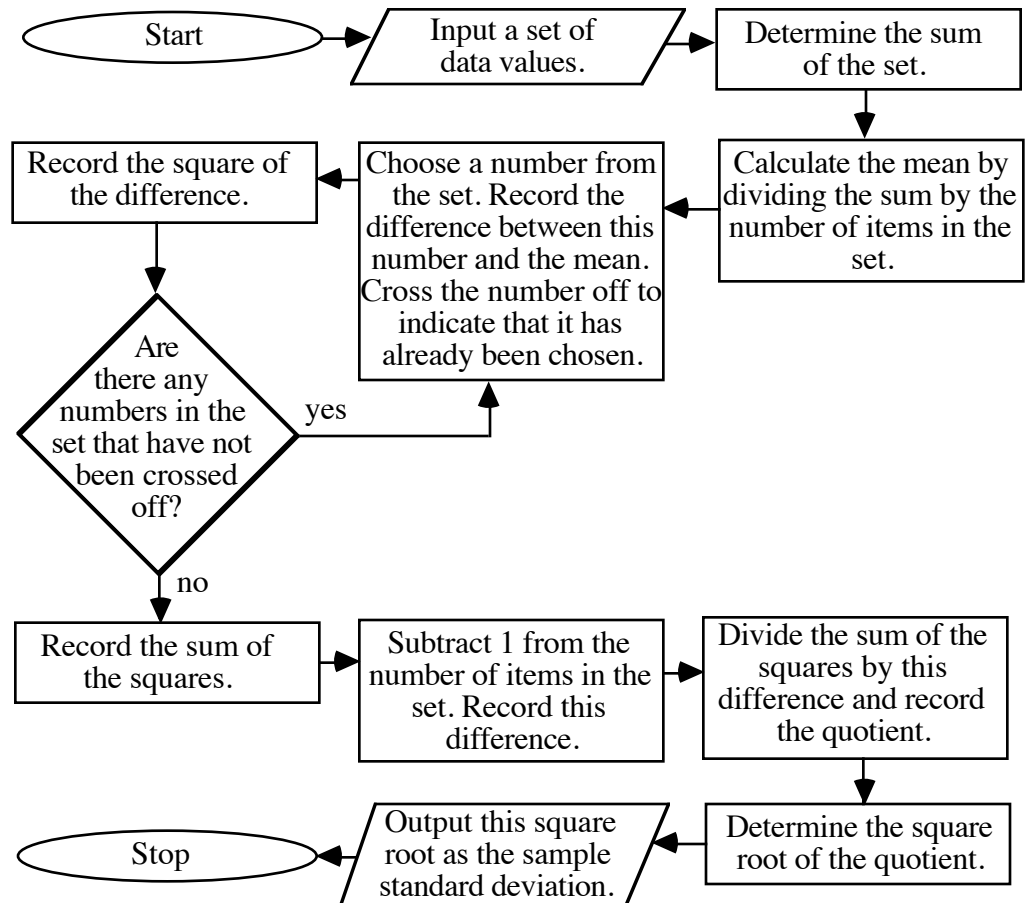
Given:	A square.
Procedure:	<ol style="list-style-type: none"> <li>1. Make four copies of the figure.</li> <li>2. Put one copy directly to the right of the original figure.</li> <li>3. Put one copy directly to the left of the original figure.</li> <li>4. Put one copy directly below the original figure.</li> <li>5. Put one copy directly above the original figure.</li> <li>6. The five figures now form a new figure.</li> <li>7. Repeat the procedure on the new figure.</li> </ol>

## Module Assessment

1. Write an algorithm for inscribing a square in a circle, as shown below.



2. The following algorithm can be used to calculate the sample standard deviation of a data set.



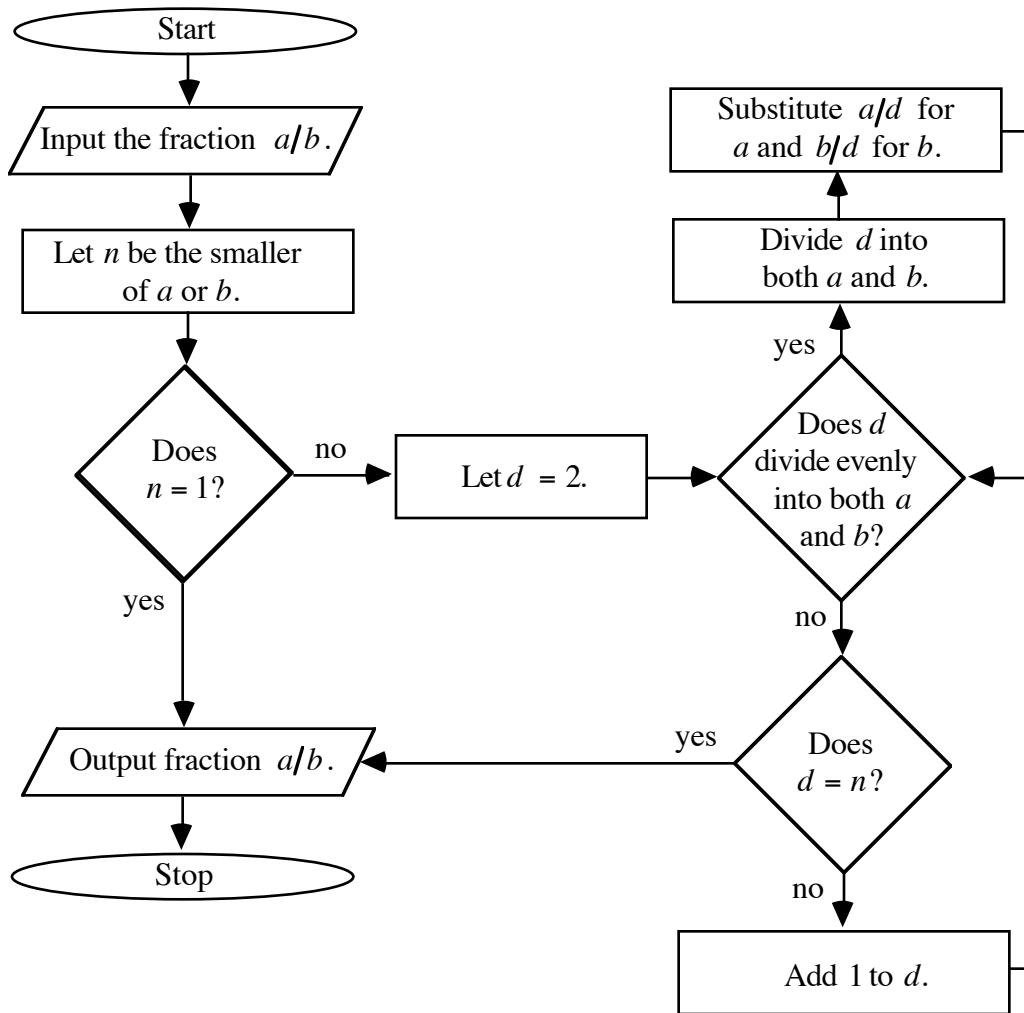
- a. Use the flowchart to calculate the sample standard deviation of the following set: {11, 13, 20, 25}.
- b. Use technology to confirm your response to Part a.
3. a. Create an algorithm for writing a fraction in lowest terms.
- b. Draw a flowchart for your algorithm in Part a.



## Answers to Module Assessment

1. Sample response: Draw a circle, construct a point on the circumference, define the center of the circle as the center of rotation and transform the point using a rotation of  $90^\circ$ . Repeat this transformation twice using the image of the previous transformation as the preimage. Connect the four points consecutively to make a square.
2.
  - a. Sample response: Input  $\{11, 13, 20, 25\}$ . The sum is 69 and the mean is 17.25. The differences are  $-6.25, -4.25, 2.75, \text{ and } 7.75$ . The squares of the differences are 39.0625, 18.0625, 7.5625, and 60.0625. The sum of the squares is 124.75. Subtracting 1 from the number of items in the set yields a difference of 3. The quotient of the sum and this difference is  $41.58\bar{3}$ . The square root of the quotient is approximately 6.4. This is the sample standard deviation of the set.
  - b. Technology will confirm that the sample standard deviation of this set is approximately 6.4.
3.
  - a. Sample algorithm:
    1. Let a trial divisor of both the numerator and denominator be  $d = 2$ .
    2. Check to see if  $d$  divides evenly into both the numerator and denominator. If it does, divide it into both, and find the new denominator and numerator. If it does not, go to Step 4.
    3. Using the new numerator and denominator, go back to Step 1.
    4. Let  $d$  be 1 more than it was before.
    5. If  $d$  is more than half of the original denominator or more than half of the original numerator, you have checked far enough. The last denominator and numerator you found are the lowest terms of the original fraction.
    6. If  $d$  is still too small to fulfill the requirements of Step 5, go back to Step 2.

b. The following flowchart describes another possible algorithm for writing a fraction in lowest terms:



**Note:** Stopping  $d$  at  $\sqrt{n}$  would make a more efficient algorithm. (Students must stop the divisors at some point, otherwise the algorithm will not terminate.)

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# Flashbacks

## Activity 1

- 1.1** Find a rule that describes the following arithmetic sequence.

7, 21, 35, 49, 63, ...

- 1.2** Identify the quotient and remainder in each of the following divisions:

a.  $37/25$

b.  $83/5$

- 1.3** Given that  $a = 1$ ,  $b = -8$ , and  $c = 7$ , use substitution to find the value of  $x$  in each of the following expressions.

a.  $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

b.  $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

## Activity 2

- 2.1** Find the greatest common divisor of each of the following pairs of numbers:

a. 59 and 24

b. 42 and 66

- 2.2** Determine whether each of the following statements is true or false:

a.  $\frac{16}{4} \leq \left(-2 + \frac{36}{6}\right)$

b. If  $x$  is 9, then  $x + 1 < 10$ .

- 2.3** A company pays its employees \$10 per hour, plus “time and a half” for overtime. Only those hours worked in excess of 40 hours per week qualify for overtime pay. Determine the weekly wage of an employee who works each of the following numbers of hours:

a. 30 hours

b. 50 hours

### Activity 3

- 3.1** Write the first three terms of the sequence described by the following recursive formula:

$$\begin{cases} a_1 = 5 \\ a_n = 2 + a_{n-1}, n > 1 \end{cases}$$

- 3.2** Consider the sequence 8, 14, 20, 26, ... .
- Write a recursive formula for the sequence.
  - Write an explicit formula for the sequence.

## Answers to Flashbacks

### Activity 1

- 1.1 Sample response: This sequence can be described by the explicit formula  $a_n = 7 + 14(n - 1)$ ,  $n \geq 1$ .
- 1.2 a. quotient 1, remainder 12  
b. quotient 16, remainder 3
- 1.3 a.  $x = 7$   
b.  $x = 1$

### Activity 2

- 2.1 a. 1  
b. 6
- 2.2 a. true  
b. false
- 2.3 a. \$300  
b. \$550

### Activity 3

- 3.1 5, 7, 9
- 3.2 a. Sample response:  
$$\begin{cases} a_1 = 8 \\ a_n = 6 + a_{n-1}, n > 1 \end{cases}$$
- b.  $a_n = 8 + 6(n - 1)$ ,  $n \geq 1$

# Flowchart Template

